Nonlinear Functional Analysis and Applications
Vol. 26, No. 2 (2021), pp. 237-272
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.02.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

SINGULAR SOLUTIONS OF AN INHOMOGENEOUS ELLIPTIC EQUATION

Arij Bouzelmate[1] and Abdelilah Gmira[2]

[1]Department of Mathematics, Faculty of Sciences
Abdelmalek Essaadi University, B.P. 2121, Tetouan, Morocco
e-mail: abouzelmate@uae.ac.ma

[2]Department of Mathematics, Faculty of Sciences
Abdelmalek Essaadi University, B.P. 2121, Tetouan, Morocco
e-mail: agmira@uae.ac.ma

Abstract. The main purpose of the present paper is to study the asymptotic behavior near the origin of radial solutions of the equation

$$\Delta_p u(x) + u^q(x) + f(x) = 0 \quad \text{in } \mathbb{R}^N \backslash \{0\},$$

where $p > 2$, $q > 1$, $N \geq 1$ and f is a continuous radial function on $\mathbb{R}^N \backslash \{0\}$. The study depends strongly of the sign of the function f and the asymptotic behavior near the origin of the function $|x|^\lambda f(|x|)$ with suitable conditions on $\lambda > 0$.

1. Introduction

This paper deals with the following radial equation

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) + u^q(r) + f(r) = 0, \quad r > 0 \tag{1.1}$$

where $p > 2$, $q > 1$, $N \geq 1$ and f is a continuous radial function on $]0, +\infty[$ and strictly positive near the origin.

The difficulty in studying the equation (1.1) lies in the presence of the inhomogeneous term $f(r)$ that can be singular at the origin and influences on the existence and the asymptotic behavior of solutions of equation (1.1).

---

[0]Received August 18, 2020. Revised February 4, 2021. Accepted February 6, 2021.
[0]2010 Mathematics Subject Classification: 35A01, 35A02, 35D05, 35K51.
[0]Keywords: Inhomogeneous radial equation, singular solutions, energy function, asymptotic behavior near the origin.
[0]Corresponding author: A. Bouzelmate(abouzelmate@uae.ac.ma).

Note that if $f \equiv 0$ and $p = 2$, the equation (1.1) becomes the Emden-Fowler equation

$$u''(r) + \frac{N-1}{r} u'(r) + u^q(r) = 0, \tag{1.2}$$

that has been the subject of much literature. The first study in the radial case is due to Emden-Fowler (see for example [8], [9] and [10]). He proved the existence results and gave a classification of entire radial solutions of equation (1.2) on $\mathbb{R}^N$ and $\mathbb{R}^N \backslash \{0\}$. In the case $N > 2$, it has been shown the existence of two critical values $\frac{N}{N-2}$ and $\frac{N+2}{N-2}$. Regarding the non-radial case, the study was made by Lions [13] when $q < \frac{N}{N-2}$, Aviles [1] when $q = \frac{N}{N-2}$ and Gidas-Spruck [11] when $q < \frac{N+2}{N-2}$. Caffarelli, Gidas and Spruck [7] extended the study to the case $q = \frac{N+2}{N-2}$.

In the case $f \equiv 0$ and $p > 2$, the first results are due to Ni and Serrin [14]. They have shown the existence of two critical values $\frac{N(p-1)}{N-p}$ and $\frac{N(p-1)+p}{N-p}$. Guedda and Véron [12] studied the existence of entire solutions and asymptotic behavior near 0 of radial solutions in the case $q < \frac{N(p-1)}{N-p}$.

The non-radial case was investigated by Bidaut-Véron and Pohozaev (see [4]) and also by Guedda and Véron in the case $q < \frac{N(p-1)}{N-p}$.

In the case where f is not identically zero and $p = 2$, Bae [2] studied the equation

$$\Delta\, u(x) + u^q(x) + f(x) = 0 \qquad \text{in } \mathbb{R}^N \backslash \{0\} \tag{1.3}$$

and gave the asymptotic behavior near zero and near infinity of positive radial solutions of (1.3), that is solutions that satisfy the equation

$$u''(r) + \frac{N-1}{r} u'(r) + u^q(r) + f(r) = 0, \quad r > 0. \tag{1.4}$$

The first step to understand the effect of the function f on the equation (1.1), is to deal with the type of $f(r) = Lr^{-\lambda}$ with $\lambda > 0$, that is, we consider the equation

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) + u^q(r) + Lr^{-\lambda} = 0, \quad r > 0. \tag{1.5}$$

In the case $N > p$ and $q > \frac{N(p-1)}{N-p}$, the equation (1.5) possesses a positive solution if

$$\lambda = \frac{pq}{q+1-p} \quad \text{and} \quad L = \frac{q+1-p}{p-1}\left(\frac{p-1}{q}\right)^{q/(q+1-p)} \Lambda^q, \tag{1.6}$$

where

$$\Lambda = \left(\left(\frac{p}{q+1-p}\right)^{p-1}\left(N - \frac{pq}{q+1-p}\right)\right)^{1/(q+1-p)}. \tag{1.7}$$

This solution is given by

$$\widetilde{u}(r) = \left(\frac{p-1}{q}\right)^{1/(q+1-p)} \Lambda r^{-p/(q+1-p)}. \tag{1.8}$$

In this work, we present a complete study of singular solutions of equation (1.1), that is to say solutions u such that $\lim_{r\to 0} u(r) = +\infty$. We prove existence and nonexistence results and we study the asymptotic behavior of solutions of equation (1.1) near the origin. The study depends strongly on the limit of the function $r^{pq/(q+1-p)} f(r)$ when r tends to 0 and the comparison of q with the two critical values $\frac{N(p-1)}{N-p}$ and $\frac{N(p-1)+p}{N-p}$.

This paper is organized as follows. In section 2 we present some preliminary results that will be useful to study equation (1.1) under some conditions of the inhomogeneous term f. In section 3, we study the asymptotic behavior near the origin of solutions of equation (1.1). Finally, existence and nonexistence results are respectively established in sections 4 and 5.

2. Preliminary results and other formulations of the problem

In this section, we give some properties of solutions of equation

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) + u^q(r) + f(r) = 0, \quad r > 0 \tag{2.1}$$

where $p > 2$, $q > 1$, $N \geq 1$ and f is a continuous radial function on $]0, +\infty[$ and strictly positive near the origin.

We start with the following result that gives an information of monotonicity of solutions of equation (2.1) near the origin. For this, we use some ideas introduced in [6].

Proposition 2.1. *Let u be a solution of equation* (2.1). *Then, u is strictly monotone near* 0 *and* $\lim_{r\to 0} u(r) \in [0, +\infty]$. *Moreover, if $N \geq p$, then u is strictly decreasing near* 0 *and* $\lim_{r\to 0} u(r) \in]0, +\infty]$.

Proof. According to equation (2.1), we have for any $r > 0$,

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) = -r^{N-1}u^q(r) - r^{N-1}f(r). \tag{2.2}$$

Since f is strictly positive near the origin, $r^{N-1}|u'|^{p-2}u'$ is strictly decreasing near the origin. Hence, u is strictly monotone near 0 and therefore we have

$$\lim_{r\to 0} u(r) \in [0, +\infty].$$

Suppose that $u'(r) > 0$ near 0. Then, $\lim_{r\to 0} u(r)$ exists and finite. On the other hand, using again the fact that $\left(r^{N-1}|u'|^{p-2}u'\right)'(r) < 0$ for small r, we obtain

$$\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) \in]0,+\infty].$$

Hence, there exists a small R and a constant $C>0$ such that

$$u'(r) > Cr^{(1-N)/(p-1)}, \quad \text{for any } r \in (0,R).$$

This cannot take place because $u' \in L^1(0,R)$ and $r^{(1-N)/(p-1)} \notin L^1(0,R)$ when $N \geq p$. Consequently, $u'(r) < 0$ for small r and $\lim_{r\to 0} u(r) \in]0,+\infty]$. □

Remark 2.2. Assume that $N \geq p$. Let u be a solution of equation (2.1) such that $\lim_{r\to 0} u(r)$ is finite and strictly positive. Then, $\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0$. Indeed, since the function $r^{N-1}|u'|^{p-2}u'(r)$ is strictly negative and strictly decreasing near the origin, then $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) \in]-\infty,0]$. Using the fact that u' is integrable near 0 and $N \geq p$, then necessarily $\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0$.

Now, if f is positive on $]0,+\infty[$, we have the following results.

Proposition 2.3. *Assume that $N \geq p$ and f is positive. Let u be a solution of equation* (2.1)*. Then $u'(r) < 0$ and $u(r) > 0$, for any $r > 0$.*

Proof. Since f is positive on $]0,+\infty[$, then by (2.2), $r^{N-1}|u'|^{p-2}u'$ is decreasing on $]0,+\infty[$. According to Proposition 2.1, we have $u'(r) < 0$ near 0, hence $u'(r) < 0$ for any $r > 0$.

We show now that u is strictly positive on $]0,+\infty[$. According to Proposition 2.1, we have $\lim_{r\to 0} u(r) \in]0,+\infty]$. Let r_0 be the first zero of u. Then necessarily $u'(r_0) \leq 0$. If $u'(r_0) < 0$, then u changes sign, which is impossible. If $u'(r_0) = 0$, then integrating (2.2) on (r,r_0) for $r > 0$ and using the fact that $u^q + f$ is strictly positive on (r,r_0) and $u'(r_0) = 0$, we obtain $u'(r) > 0$ on $(0,r_0)$. This contradicts the fact that $u'(r) < 0$ for any $r > 0$. Consequently, $u(r) > 0$ for any $r > 0$. □

Proposition 2.4. *Assume that $N \geq p$, $q > p-1$ and f is positive. Let u be a solution of equation* (2.1)*. Then, for any $r > 0$, we have*

$$u(r) \leq C(N,p,q)\, r^{-p/(q+1-p)}, \tag{2.3}$$

where

$$C(N,p,q) = \left(\frac{p}{q+1-p}\right)^{(p-1)/(q+1-p)} \left(\frac{N}{1-2^{-N}}\right)^{1/(q+1-p)}. \tag{2.4}$$

Proof. Since f is positive, we have

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) \le -r^{N-1}u^q(r).$$

Integrating this last inequality on $\left(\frac{r}{2}, r\right)$ for $r > 0$ and using the fact that u is strictly decreasing on $]0, +\infty[$, we get

$$r^{N-1}|u'|^{p-2}u'(r) < -\frac{1-2^{-N}}{N} r^N u^q(r).$$

Using the fact that u is strictly positive and strictly decreasing, we obtain

$$u'(r)u^{-q/(p-1)}(r) < -\left(\frac{1-2^{-N}}{N}\right)^{1/(p-1)} r^{1/(p-1)}.$$

Since $q > p - 1$, we arrive at

$$\left(u^{(p-1-q)/(p-1)}\right)'(r) > \frac{q+1-p}{p-1}\left(\frac{1-2^{-N}}{N}\right)^{1/(p-1)} r^{1/(p-1)}.$$

Integrating this last inequality on $(0, r)$ for $r > 0$ and using the fact that $\lim_{r\to 0} u(r) \in]0, +\infty]$ and $q > p - 1$, we get the desired estimate (2.3). This completes the proof. □

Remark 2.5. Assume that $N \ge p$ and $q > p - 1$. Let u be a solution of equation (2.1). Then $r^{p/(q+1-p)}u(r)$ is bounded for small r.

Indeed, we use the same proof of Proposition 2.4 since f is strictly positive and u is strictly decreasing near the origin.

Now, we focus on the study of solutions $u(r)$ of equation (2.1) which tend to $+\infty$ as r tends to 0.

Definition 2.6. The solution u of equation (2.1) is called singular if it can not be extended by continuity at zero, that is, $\lim_{r\to 0} u(r) = +\infty$.

Proposition 2.7. *Assume that* $N > p$ *and* $q > \dfrac{N(p-1)}{N-p}$. *Let* u *be a singular solution of equation* (2.1). *Then*

$$\lim_{r\to 0} r^{(N-p)/(p-1)}u(r) = 0$$

and

$$\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0.$$

Proof. We know from Remark 2.5, that the function $r^{p/(q+1-p)}u(r)$ is bounded near the origin. Therefore, for $q > \frac{N(p-1)}{N-p}$, that is $\frac{N-p}{p-1} > \frac{p}{q+1-p}$, we have

$$\lim_{r\to 0} r^{(N-p)/(p-1)}u(r) = 0.$$

On the other hand, the function $r^{N-1}|u'|^{p-2}u'$ is strictly decreasing and strictly negative near the origin, therefore $\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) \in]-\infty, 0]$. Hence, using Hpital's rule (because $\lim_{r\to 0} u(r) = +\infty$ and $N > p$), we obtain

$$\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0.$$

□

Remark 2.8. When $N < p$, the singular solution u of equation (2.1) is also strictly decreasing near the origin. Indeed, since u is strictly monotone near 0 by Proposition 2.1 and $\lim_{r\to 0} u(r) = +\infty$, necessarily $u' < 0$ near 0.

Before proving other results, we introduce the following change of variable that will play an important role in the proofs of theorems.

For any real c, we set

$$\upsilon_c(t) = r^c u(r) \quad \text{where } c \neq 0 \text{ and } t = -\ln r. \tag{2.5}$$

Then υ_c satisfies the following equation

$$\omega_c'(t) - \Gamma_c\,\omega_c(t) + e^{-(p-c(q+1-p))t}\upsilon_c^q(t) + g_c(t) = 0, \tag{2.6}$$

where

$$g_c(t) = e^{-(p+c(p-1))t} f(e^{-t}), \tag{2.7}$$

$$\omega_c(t) = |h_c|^{p-2}h_c(t), \tag{2.8}$$

$$h_c(t) = \upsilon_c'(t) + c\upsilon_c(t) \tag{2.9}$$

and

$$\Gamma_c = N - p - c(p-1). \tag{2.10}$$

Note that

$$h_c(t) = -r^{c+1}u'(r). \tag{2.11}$$

Hence, the study of monotonicity of u depends of the sign of $h_c(t)$ which is that of $\omega_c(t)$. On the other hand, it's easy to see that

$$(r^c u)' = r^{c-1}E_c(r), \tag{2.12}$$

where

$$E_c(r) = cu(r) + ru'(r). \tag{2.13}$$

Moreover, we have

$$\upsilon_c'(t) = -r^c E_c(r). \tag{2.14}$$

Hence, the monotonicity of $r^c u$ or υ_c can be obtained by the sign of E_c. In fact, observe that for any $r>0$ such that $u'(r)\neq 0$ we have

$$(p-1)\left|u'\right|^{p-2}(r)E_c'(r)=(p-1)\left(c-\frac{N-p}{p-1}\right)\left|u'\right|^{p-2}u'(r)-ru^q(r)-rf(r). \tag{2.15}$$

Consequently, if $E_c(r_0)=0$ for some $r_0>0$, equation (2.1) gives

$$\begin{aligned}(p-1)\,r_0^{p-1}\left|u'\right|^{p-2}(r_0)E_c'(r_0) \;=\; &-(p-1)\left(c-\frac{N-p}{p-1}\right)|c|^{p-2}c\,u^{p-1}(r_0)\\ &-r_0^p u^q(r_0)-r_0^p f(r_0),\end{aligned} \tag{2.16}$$

from which we can study the sign of $E_c(r)$.

The following results give some properties of singular solutions of equation (2.1).

Proposition 2.9. *Assume that* $N>p$ *and* $q>\frac{N(p-1)}{N-p}$. *Let* u *be a singular solution of equation* (2.1). *Then,*

(i) $E_{(N-p)/(p-1)}(r)>0$ *for small* r.

(ii) *if* $\liminf\limits_{r\to 0} r^{pq/(q+1-p)}f(r)>L$, *then we have* $E_{p/(q+1-p)}(r)\neq 0$ *for small* r *where* L *is given by* (1.6).

Proof. (i) According to (2.15), we have $E'_{(N-p)/(p-1)}(r)<0$ for small r (because $f>0$ and $u'<0$, near the origin). Therefore, $E_{(N-p)/(p-1)}(r)\neq 0$ for small r. Since $\lim\limits_{r\to 0} r^{(N-p)/(p-1)}u(r)=0$ by Proposition 2.7, necessarily $E_{(N-p)/(p-1)}(r)>0$ for small r.

(ii) Suppose that there exists a small r such that $E_{p/(q+1-p)}(r)=0$. Taking $c=\frac{p}{q+1-p}$ in (2.16) and multiplying by $r^{pq/(q+1-p)-1}$, we obtain

$$\begin{aligned}&(p-1)\,r^{pq/(q+1-p)-1}\left|u'\right|^{p-2}(r)E'_{p/(q+1-p)}(r)\\ &=\Lambda^{q+1-p}\,r^{p(p-1)/(q+1-p)}u^{p-1}(r)-r^{pq/(q+1-p)}u^q(r)-r^{pq/(q+1-p)}f(r),\end{aligned} \tag{2.17}$$

where Λ is given by (1.7). Using the change of variable (2.5), the last equation is equivalent to

$$(p-1)\,r^{pq/(q+1-p)-1}\left|u'\right|^{p-2}(r)E'_{p/(q+1-p)}(r)=\phi(\upsilon_{p/(q+1-p)}(t))-g_{p/(q+1-p)}(t), \tag{2.18}$$

where

$$\phi(s)=\Lambda^{q+1-p}s^{p-1}-s^q,\quad s\geq 0. \tag{2.19}$$

A simple calculation gives

$$\max_{s\geq 0}\phi(s)=\phi\left(\left(\frac{p-1}{q}\right)^{1/(q+1-p)}\Lambda\right)=L, \tag{2.20}$$

where L is given by (1.6). But $\liminf_{r\to 0} r^{pq/(q+1-p)} f(r) > L$, there exists $\varepsilon > 0$ such that $g_{p/(q+1-p)}(t) \geq L + \varepsilon$ for large t. Hence, $E'_{p/(q+1-p)}(r) < 0$ and so $E_{p/(q+1-p)}(r) \neq 0$ for small r. The proof is complete. □

Proposition 2.10. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation (2.1) satisfying $\lim_{r\to 0} r^{p/(q+1-p)} u(r) = 0$. Suppose that $r^{\sigma(p-1)+p} f(r)$ is bounded for small r and $\lim_{r\to 0} r^{\sigma} u(r) = +\infty$ for some $0 < \sigma < \frac{p}{q+1-p}$. Then $E_{p/(q+1-p)}(r) > 0$ and $E_\sigma(r) < 0$, for small r.*

Proof. The idea is to show that $E_{p/(q+1-p)}(r) \neq 0$ and $E_\sigma(r) \neq 0$, for small r. Thereafter, taking into account (2.12) and the fact that $\lim_{r\to 0} r^{p/(q+1-p)} u(r) = 0$ and $\lim_{r\to 0} r^{\sigma} u(r) = +\infty$, we deduce easily that $E_{p/(q+1-p)}(r) > 0$ and $E_\sigma(r) < 0$ for small r. The proof will be done in two steps.

Step 1. $E_{p/(q+1-p)}(r) \neq 0$ for small r.

Suppose that there exists a small r such that $E_{p/(q+1-p)}(r) = 0$. Taking $c = \frac{p}{q+1-p}$ in (2.16) and multiplying by $r^{\sigma(p-1)}$, we obtain

$$\begin{aligned} &(p-1)\, r^{(\sigma+1)(p-1)} \left|u'\right|^{p-2} E'_{p/(q+1-p)}(r) \\ &= r^{\sigma(p-1)} u^{p-1} \Big[\Lambda^{q+1-p} - r^p u^{q+1-p} - r^{p+\sigma(p-1)} f(r) \left(r^\sigma u(r)\right)^{1-p}\Big]. \end{aligned} \tag{2.21}$$

Since $\lim_{r\to 0} r^{p/(q+1-p)} u(r) = 0$, $r^{\sigma(p-1)+p} f(r)$ is bounded for small r and $\lim_{r\to 0} r^{\sigma} u(r) = +\infty$, we have $E'_{p/(q+1-p)}(r) > 0$. Hence, $E_{p/(q+1-p)}(r) \neq 0$ for small r.

Step 2. $E_\sigma(r) \neq 0$ for small r.

In the same way as the Step 1, assume that there exists a small r such that $E_\sigma(r) = 0$. Using (2.16), we have

$$\begin{aligned} &(p-1)\, r^{(\sigma+1)(p-1)} \left|u'\right|^{p-2} (r) E'_\sigma(r) \\ &= r^{\sigma(p-1)} u^{p-1}(r) \Big[\Gamma_\sigma \sigma^{p-1} - r^p u^{q+1-p}(r) - r^{p+\sigma(p-1)} f(r) \left(r^\sigma u(r)\right)^{1-p}\Big], \end{aligned} \tag{2.22}$$

where Γ_σ is given by (2.10). Taking into account our hypothesis and the fact that $\Gamma_\sigma > 0$ (because $0 < \sigma < \frac{p}{q+1-p} < \frac{N-p}{p-1}$), we deduce that $E'_\sigma(r) > 0$. Hence, $E_\sigma(r) \neq 0$ for small r. This completes the proof. □

3. Asymptotic behavior near the origin

In this section, we study the asymptotic behavior near the origin of solutions of equation (2.1) under given conditions on function f. The study requires some ideas of papers [2] and [3].

Lemma 3.1. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation (2.1). Then the function $r^{p/(q+1-p)+1}u'(r)$ is bounded near the origin.*

Proof. According to Proposition 2.1, Remark 2.5 and Proposition 2.9, we have u is strictly decreasing, $r^{p/(q+1-p)}u(r)$ is bounded and $E_{(N-p)/(p-1)} > 0$, for small r. Therefore, using (2.13), it's easy to see that $r^{p/(q+1-p)+1}u'(r)$ is bounded near the origin. □

Lemma 3.2. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation (2.1). If the functions $r^{p/(q+1-p)}u(r)$ and $r^{pq/(q+1-p)}f(r)$ converge when r tends to 0, then the function $r^{p/(q+1-p)+1}u'(r)$ converges also when r tends to 0.*

Proof. We use the change (2.5) with $c = \frac{p}{q+1-p}$. Then the function $h_{p/(q+1-p)}(t)$ is strictly positive and bounded for large t. We show that $h_{p/(q+1-p)}(t)$ is monotone for large t. In fact, suppose by contradiction that there exist two sequences $\{s_i\}$ and $\{k_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{s_i\}$ and $\{k_i\}$ are local minimum and local maximum of $h_{p/(q+1-p)}$, respectively, satisfying $s_i < k_i < s_{i+1}$ and

$$0 \le \liminf_{t\to+\infty} h_{p/(q+1-p)}(t) < \limsup_{t\to+\infty} h_{p/(q+1-p)}(t) < +\infty. \tag{3.1}$$

That is

$$0 \le \lim_{i\to+\infty} h_{p/(q+1-p)}(s_i) < \lim_{i\to+\infty} h_{p/(q+1-p)}(k_i) < +\infty, \tag{3.2}$$

which in turn implies that

$$0 \le \lim_{i\to+\infty} \omega_{p/(q+1-p)}(s_i) < \lim_{i\to+\infty} \omega_{p/(q+1-p)}(k_i) < +\infty. \tag{3.3}$$

On the other hand, equation (2.6) gives

$$\omega'_{p/(q+1-p)}(t) - \left(N - \frac{pq}{q+1-p}\right)\omega_{p/(q+1-p)}(t) + \upsilon^q_{p/(q+1-p)}(t) + g_{p/(q+1-p)}(t) = 0. \tag{3.4}$$

Since $\omega'_{p/(q+1-p)}(s_i) = \omega'_{p/(q+1-p)}(k_i) = 0$, equation (3.4) gives

$$
\begin{aligned}
&-\left(N - \frac{pq}{q+1-p}\right)\omega_{p/(q+1-p)}(s_i) + \upsilon^q_{p/(q+1-p)}(s_i) + g_{p/(q+1-p)}(s_i)\\
&= -\left(N - \frac{pq}{q+1-p}\right)\omega_{p/(q+1-p)}(k_i) + \upsilon^q_{p/(q+1-p)}(k_i) + g_{p/(q+1-p)}(k_i)\\
&= 0.
\end{aligned}
$$

But $\upsilon_{p/(q+1-p)}$ and $g_{p/(q+1-p)}$ converge when t tends to $+\infty$ and $N > \frac{pq}{q+1-p}$, hence we have

$$\lim_{i\to+\infty} \omega_{p/(q+1-p)}(s_i) = \lim_{i\to+\infty} \omega_{p/(q+1-p)}(k_i),$$

which contradicts (3.3). Consequently, $h_{p/(q+1-p)}$ converges when t tends to $+\infty$, that is, $r^{p/(q+1-p)+1}u'(r)$ converges when r tends to 0. □

For any $l \geq 0$, assume in the following that z_1 and z_2 are two roots of the equation

$$z^q - \Lambda^{q+1-p} z^{p-1} + l = l - \phi(z) = 0, \tag{3.5}$$

where ϕ is given by (2.19).

If $l > 0$, it's easy to see that $0 < z_1 \leq z_2$. In particular, if $l = L$, then $z_1 = z_2 = \left(\frac{p-1}{q}\right)^{1/(q+1-p)}\Lambda$, where L and Λ are given respectively by (1.6) and (1.7).

If $l = 0$, then $z_1 = 0$ and $z_2 = \Lambda$.

Lemma 3.3. *Assume that $N > p$, $q > \frac{N(p-1)}{N-p}$ and $\lim_{r\to 0} r^{pq/(q+1-p)} f(r) = l \geq 0$. Let u be a singular solution of equation (2.1) such that*

$$\lim_{r\to 0} r^{p/(q+1-p)} u(r) = d.$$

Then d is a root of equation (3.5). In particular, if $l = L$, then $d = \left(\frac{p-1}{q}\right)^{1/(q+1-p)}\Lambda$.

Proof. By hypothesis, we know that $\upsilon_{p/(q+1-p)}(t)$ converges when $t \to +\infty$, therefore according to Lemma 3.2, $h_{p/(q+1-p)}$ converges also, which in turn implies by relation (2.9) that $\upsilon'_{p/(q+1-p)}(t)$ converges necessarily to 0 when $t \to +\infty$, hence $\lim_{t\to+\infty} h_{p/(q+1-p)}(t) = \dfrac{p}{q+1-p}\, d$ and by (2.8),

$$\lim_{t\to+\infty} \omega_{p/(q+1-p)}(t) = \left(\frac{p}{q+1-p}\right)^{p-1} d^{p-1}.$$

Since $\lim_{t\to+\infty} g_{p/(q+1-p)}(t) = l$, by equation (3.4), $\omega'_{p/(q+1-p)}(t)$ converge necessarily to 0. By letting $t \to +\infty$ in equation (3.4), we obtain $l - \phi(d) = 0$. Finally, it's easy to see that if $l = L$, then $d = \left(\frac{p-1}{q}\right)^{1/(q+1-p)} \Lambda$. □

Theorem 3.4. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation* (2.1)*. If $\lim_{r\to 0} r^{pq/(q+1-p)} f(r) = L$, then we have*

$$\liminf_{r\to 0} r^{p/(q+1-p)} u(r) = \left(\frac{p-1}{q}\right)^{1/(q+1-p)} \Lambda.$$

Proof. First of all, if $\upsilon_{p/(q+1-p)}$ converges, the theorem is a direct result of Lemma 3.3. Since $\upsilon_{p/(q+1-p)}$ is bounded, it remains to handle the case where it oscillates. Suppose that there exists a sequence $\{\eta_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\upsilon_{p/(q+1-p)}$ has a local minimum in η_i. Hence, $\upsilon'_{p/(q+1-p)}(\eta_i) = 0$ and $\upsilon''_{p/(q+1-p)}(\eta_i) \geq 0$ (note that $\upsilon''_{p/(q+1-p)}$ exists because $u' < 0$ near 0). Therefore, using (2.9), we obtain

$$h_{p/(q+1-p)}(\eta_i) = \frac{p}{q+1-p}\, \upsilon_{p/(q+1-p)}(\eta_i)$$

and

$$h'_{p/(q+1-p)}(\eta_i) = \upsilon''_{p/(q+1-p)}(\eta_i) \geq 0.$$

This implies that

$$\omega_{p/(q+1-p)}(\eta_i) = \left(\frac{p}{q+1-p}\right)^{p-1} \upsilon^{p-1}_{p/(q+1-p)}(\eta_i)$$

and

$$\omega'_{p/(q+1-p)}(\eta_i) = (p-1)\left|h_{p/(q+1-p)}(\eta_i)\right|^{p-2} h'_{p/(q+1-p)}(\eta_i) \geq 0.$$

Taking $t = \eta_i$ in equation (3.4) and using (2.19), we obtain

$$\begin{aligned}
0 &\leq \omega'_{p/(q+1-p)}(\eta_i) \\
&= \Lambda^{q+1-p} \upsilon^{p-1}_{p/(q+1-p)}(\eta_i) - \upsilon^{q}_{p/(q+1-p)}(\eta_i) - g_{p/(q+1-p)}(\eta_i) \\
&= \phi\left(\upsilon_{p/(q+1-p)}(\eta_i)\right) - g_{p/(q+1-p)}(\eta_i) \\
&\leq L - g_{p/(q+1-p)}(\eta_i).
\end{aligned}$$

Since $\lim_{i\to+\infty} g_{p/(q+1-p)}(\eta_i) = L$, $\lim_{i\to+\infty} \phi\left(\upsilon_{p/(q+1-p)}(\eta_i)\right) = L$. Consequently, according to (2.20),

$$\lim_{i\to+\infty} \upsilon_{p/(q+1-p)}(\eta_i) = \liminf_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = \left(\frac{p-1}{q}\right)^{1/(q+1-p)} \Lambda.$$

The proof is complete. □

It is obvious that the previous theorem gives only an information of $\liminf_{r\to 0} r^{p/(q+1-p)}u(r)$. Next, we examine the convergence of $r^{p/(q+1-p)}u(r)$ at 0. For this reason, assume that f is differentiable and satisfies the following conditions:

(H_1) $\displaystyle\int_0^1 \left(r^{pq/(q+1-p)}f\right)_r^+ dr < \infty,$

(H_2) $\displaystyle\int_0^1 \left(r^{pq/(q+1-p)}f\right)_r^- dr < \infty.$

In addition to the above assumptions, the study of asymptotic behavior of u near the origin will depend on the convergence of $r^{pq/(q+1-p)}f(r)$ near 0 and the comparison of q with $\frac{N(p-1)}{N-p}$ and $\frac{N(p-1)+p}{N-p}$.

Lemma 3.5. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation* (2.1)*. Suppose that $\lim_{r\to 0} r^{pq/(q+1-p)}f(r) = l > 0$. Then we have*

$$\liminf_{r\to 0} r^{p/(q+1-p)}u(r) > 0$$

and

$$\limsup_{r\to 0} r^{p/(q+1-p)+1}u'(r) < 0.$$

Proof. The proof will be done in two steps.

Step 1. $\liminf_{r\to 0} r^{p/(q+1-p)}u(r) > 0$.

Assume by contradiction that $\liminf_{r\to 0} r^{p/(q+1-p)}u(r) = 0$. This means that $\liminf_{t\to +\infty} \upsilon_{p/(q+1-p)}(t) = 0$. Since $\upsilon_{p/(q+1-p)}(t)$ is bounded for large t, we distinguish two cases.

Case 1. Let $\upsilon_{p/(q+1-p)}(t)$ be monotone for large t. Then $\upsilon_{p/(q+1-p)}(t)$ converges to 0 when $t \to +\infty$. On the other hand, using the fact that u is strictly decreasing and $E_{(N-p)/(p-1)}(r) > 0$ for small r (by Proposition 2.9), we obtain for large t

$$0 < h_{p/(q+1-p)}(t) < \frac{N-p}{p-1}\,\upsilon_{p/(q+1-p)}(t). \tag{3.6}$$

Then $\lim_{t\to +\infty} h_{p/(q+1-p)}(t) = 0$. Hence, by equation (3.4), $\lim_{t\to +\infty} \omega'_{p/(q+1-p)}(t) = -l < 0$. But this contradicts the fact that $\lim_{t\to +\infty} \omega_{p/(q+1-p)}(t) = 0$.

Case 2. Let $\upsilon_{p/(q+1-p)}(t)$ oscillate for large t. Then there exists a sequence $\{\eta_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\upsilon_{p/(q+1-p)}$ has a local minimum in η_i. Hence, $\upsilon'_{p/(q+1-p)}(\eta_i) = 0$ and $\upsilon''_{p/(q+1-p)}(\eta_i) \geq 0$. Therefore,

$\lim_{i\to+\infty} \omega_{p/(q+1-p)}(\eta_i) = 0$ and $\omega'_{p/(q+1-p)}(\eta_i) \geq 0$. But according to equation (3.4), we have $\lim_{i\to+\infty} \omega'_{p/(q+1-p)}(\eta_i) = -l < 0$. This is a contradiction.

It follows from both cases that $\liminf_{r\to 0} r^{p/(q+1-p)} u(r) > 0$.

Step 2. $\limsup_{r\to 0} r^{p/(q+1-p)+1} u'(r) < 0$.

Since $u' < 0$ near 0, assume by contradiction that $\limsup_{r\to 0} r^{p/(q+1-p)+1} u'(r) = 0$. This means that $\liminf_{t\to+\infty} h_{p/(q+1-p)}(t) = 0$. In the same way as the Step 1, since $h_{p/(q+1-p)}(t)$ is bounded for large t (by Lemma 3.1), we distinguish two cases.

Case 1. Let $h_{p/(q+1-p)}(t)$ be monotone for large t. Then $h_{p/(q+1-p)}(t)$ converges to 0 when $t \to +\infty$. That is, $\lim_{r\to 0} r^{p/(q+1-p)+1} u'(r) = 0$. This implies using Hpital's rule that $\lim_{r\to 0} r^{p/(q+1-p)} u(r) = 0$. Which contradicts the fact that $\liminf_{r\to 0} r^{p/(q+1-p)} u(r) > 0$.

Case 2. Let $h_{p/(q+1-p)}(t)$ oscillate for large t. Then there exists a sequence $\{s_i\}$ going to $+\infty$ as $i \to +\infty$ such that $h_{p/(q+1-p)}$ has a local minimum in s_i. Hence, $\lim_{i\to+\infty} \omega_{p/(q+1-p)}(s_i) = 0$ and $\omega'_{p/(q+1-p)}(s_i) = 0$. But according to equation (3.4), we have $\lim_{i\to+\infty} \upsilon^q_{p/(q+1-p)}(s_i) = -l < 0$. This contradicts the fact that $\upsilon_{p/(q+1-p)}$ is positive. The proof of lemma is complete. □

We need also this classic result of [11] of which we recall the demonstration.

Lemma 3.6. *Let W be a positive differentiable function satisfying*

(i) $\displaystyle\int_{t_0}^{+\infty} W(t)\,dt < +\infty$ *for large t_0,*

(ii) $W'(t)$ *is bounded for large t.*

Then $\lim_{t\to+\infty} W(t) = 0$.

Proof. We claim that $W(t) \to 0$ as $t \to +\infty$. Suppose this is not the case. Then given $\varepsilon > 0$, there exists a sequence $\{t_j\}$ going to $+\infty$ as $j \to +\infty$ satisfying $W(t_j) \geq 2\varepsilon$. Since $W'(t)$ is bounded for large t, there exists a constant $K > 0$ such that $|W'(t)| \leq K$ for large t. By the mean value theorem for W, we have $W(t) \geq \varepsilon$ for $|t - t_j| < \dfrac{\varepsilon}{K}$. Choose a subsequence t'_j such that $t'_0 > t_0$ and $t'_j > t'_j{-}1 + \dfrac{2\varepsilon}{K} t'_0$ for $j > 1$. Therefore, we get

$$\sum_{j=1}^{m} \int_{t'_{j-1}}^{t'_j} W(t)\,dt > \sum_{j=1}^{m} \int_{t'_{j-1}}^{t'_{j-1}+\varepsilon/K} W(t)\,dt > \frac{\varepsilon^2}{K} m \to +\infty \ \text{ as } m \to +\infty.$$

This implies that

$$\int_{t_0}^{+\infty} W(t)\,dt = +\infty.$$

This contradiction completes the proof. □

The following theorem deals with the case $\lim_{r\to 0} r^{pq/(q+1-p)} f(r) = l > 0$ and $q \neq \frac{N(p-1)+p}{N-p}$.

Theorem 3.7. *Assume that* $N > p$. *Let* u *be a singular solution of equation* (2.1). *Suppose that* $\lim_{r\to 0} r^{pq/(q+1-p)} f(r) = l > 0$ *and* f *satisfies*

(H_1) *if* $q > \frac{N(p-1)+p}{N-p}$

or

(H_2) *if* $\frac{N(p-1)}{N-p} < q < \frac{N(p-1)+p}{N-p}$.

Then $l \le L$ *and* $r^{p/(q+1-p)} u(r)$ *converges when* $r \to 0$ *to one of the roots* z_1 *or* z_2 *of equation* (3.5) *such that* $0 < z_1 \le z_2$.

Proof. Define the following energy function associated with equation (3.4),

$$\begin{aligned} F(t) \ &= \frac{p-1}{p}\left|h_{p/(q+1-p)}(t)\right|^p - \frac{p}{q+1-p}\,\omega_{p/(q+1-p)}(t)\upsilon_{p/(q+1-p)}(t) \\ &\quad -\frac{A}{p}\left(\frac{p}{q+1-p}\right)^{p-1}\upsilon^p_{p/(q+1-p)}(t) \\ &\quad +\frac{1}{q+1}\upsilon^{q+1}_{p/(q+1-p)}(t) + l\upsilon_{p/(q+1-p)}(t), \end{aligned} \tag{3.7}$$

where

$$A = \frac{q(N-p) - (N(p-1)+p)}{q+1-p}. \tag{3.8}$$

Note that according to our hypothesis, we have $A \neq 0$. Moreover, by Lemma 3.1, $h_{p/(q+1-p)}(t)$ is bounded for large t, which in turn implies that $\omega_{p/(q+1-p)}(t)$ is bounded for large t. Therefore, $F(t)$ is bounded for large t.

For the rest of the proof we proceed in three steps.

Step 1. The function $F(t)$ converges when $t \to +\infty$.

A straightforward calculation gives

$$F'(t) = AX(t) - (g_{p/(q+1-p)}(t) - l)\upsilon'_{p/(q+1-p)}(t), \tag{3.9}$$

where

$$\begin{aligned} X(t) &= \left[\left|h_{p/(q+1-p)}(t)\right|^{p-1} - \left(\frac{p}{q+1-p} \right)^{p-1} \upsilon^{p-1}_{p/(q+1-p)}(t) \right] \\ &\quad \times \left[\left|h_{p/(q+1-p)}(t)\right| - \frac{p}{q+1-p} \upsilon_{p/(q+1-p)}(t) \right]. \end{aligned} \tag{3.10}$$

Integrating (3.9) on (T, t) for large T, we obtain

$$\begin{aligned} F(t) &= C(T) + AR(t) - (g_{p/(q+1-p)}(t) - l)\upsilon_{p/(q+1-p)} \\ &\quad + \int_T^t g'_{p/(q+1-p)}(s)\upsilon_{p/(q+1-p)}\, ds, \end{aligned} \tag{3.11}$$

where

$$R(t) = \int_T^t X(s)\, ds. \tag{3.12}$$

Since the function $s \to s^{p-1}$ is monotone, $X(t) \geq 0$. Therefore, the function $R(t)$ is positive and increasing. Moreover, by (3.11) and the fact that $A \neq 0$, $R(t)$ can be written as follows

$$\begin{aligned} R(t) &= \frac{F(t)}{A} + \frac{1}{A}(g_{p/(q+1-p)}(t) - l)\upsilon_{p/(q+1-p)} \\ &\quad - \frac{1}{A}\int_T^t g'_{p/(q+1-p)}\upsilon_{p/(q+1-p)} ds - \frac{C(T)}{A}. \end{aligned} \tag{3.13}$$

Since $\upsilon_{p/(q+1-p)}(t)$ and $F(t)$ are bounded for large t, $\lim_{t\to+\infty} g_{p/(q+1-p)}(t) = l$ and $-\left(g'_{p/(q+1-p)}(s)\right)^- \leq g'_{p/(q+1-p)}(s) \leq \left(g'_{p/(q+1-p)}(s)\right)^+$, according to the sign of A and the fact that $\int_T^{+\infty} \left(g'_{p/(q+1-p)}(s)\right)^- ds < +\infty$ from (H_1) or $\int_T^{+\infty} \left(g'_{p/(q+1-p)}(s)\right)^+ ds < +\infty$ from (H_2), we get $R(t)$ is bounded for large t. Hence, $R(t)$ converges when $t \to +\infty$, that is $\int_T^{+\infty} X(s)\, ds$ exists. Letting $t \to +\infty$ in (3.13), we deduce that $F(t)$ converges when $t \to +\infty$. Let $F = \lim_{t\to+\infty} F(t)$.

Step 2. $\lim_{t\to+\infty} \upsilon'_{p/(q+1-p)}(t) = 0$.

According to (3.10) and (2.9), and the fact that $h_{p/(q+1-p)}(t)$ is strictly positive for large t, it suffices to claim that $\lim_{t\to+\infty} X(t) = 0$. For this, since $\int_T^{+\infty} X(s)\, ds < +\infty$, by Lemma 3.6, it remains to show that $X'(t)$ is bounded for large t. Rewrite $X(t)$ as follows

$$\begin{aligned} X(t) &= \omega_{p/(q+1-p)}^{p/(p-1)}(t) - \frac{p}{q+1-p}\,\omega_{p/(q+1-p)}(t)\upsilon_{p/(q+1-p)}(t) \\ &\quad -\left(\frac{p}{q+1-p}\right)^{p-1}\upsilon_{p/(q+1-p)}^{p-1}(t)\upsilon'_{p/(q+1-p)}(t). \end{aligned} \tag{3.14}$$

Hence

$$\begin{aligned} X'(t) = {} & \frac{p}{p-1}\,h_{p/(q+1-p)}(t)\omega'_{p/(q+1-p)}(t) \\ & - \frac{p}{q+1-p}\,\omega_{p/(q+1-p)}(t)\upsilon'_{p/(q+1-p)}(t) \\ & - \frac{p}{q+1-p}\,\upsilon_{p/(q+1-p)}(t)\omega'_{p/(q+1-p)}(t) \\ & - (p-1)\left(\frac{p}{q+1-p}\right)^{p-1}\upsilon_{p/(q+1-p)}^{p-2}(t)\upsilon'^{2}_{p/(q+1-p)}(t) \\ & - \left(\frac{p}{q+1-p}\right)^{p-1}\upsilon_{p/(q+1-p)}^{p-1}(t)\upsilon''_{p/(q+1-p)}(t). \end{aligned} \tag{3.15}$$

Since $\upsilon_{p/(q+1-p)}(t)$, $h_{p/(q+1-p)}(t)$ and $g_{p/(q+1-p)}(t)$ are bounded for large t, according to (2.9) and (3.4), $\upsilon'_{p/(q+1-p)}(t)$ and $\omega'_{p/(q+1-p)}(t)$ are bounded for large t. Therefore, the first four terms of the second member of (3.15) are bounded for large t. It remains to prove that $\upsilon''_{p/(q+1-p)}(t)$ is bounded for large t. According to (2.9), we have

$$\upsilon''_{p/(q+1-p)}(t) = h'_{p/(q+1-p)}(t) - \frac{p}{q+1-p}\upsilon'_{p/(q+1-p)}(t). \tag{3.16}$$

Therefore, it suffices to prove that $h'_{p/(q+1-p)}(t)$ is bounded for large t.

Since $h_{p/(q+1-p)}(t) > 0$ for large t, we have by (2.8)

$$h'_{p/(q+1-p)}(t) = \frac{1}{p-1}\left(h_{p/(q+1-p)}(t)\right)^{2-p}\omega'_{p/(q+1-p)}(t). \tag{3.17}$$

Since $\lim\limits_{t\to+\infty} g_{p/(q+1-p)}(t) = l > 0$, by Lemma 3.5, $\liminf\limits_{t\to+\infty} h_{p/(q+1-p)}(t) > 0$. Therefore, there exists a constant $K > 0$ such that $h_{p/(q+1-p)}(t) \geq K$ for large t. Hence, $\left(h_{p/(q+1-p)}(t)\right)^{2-p}$ is bounded for large t. Consequently, $h'_{p/(q+1-p)}(t)$ is bounded for large t and according to (3.16) and (3.15), $X'(t)$ is bounded for large t. Hence, using Lemma 3.6, we get $\lim\limits_{t\to+\infty} X(t) = 0$.

Step 3. $\upsilon_{p/(q+1-p)}(t)$ converges when $t \to +\infty$.

Assume by contradiction that $\upsilon_{p/(q+1-p)}(t)$ oscillates for large t. Then there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{\eta_i\}$

and $\{\xi_i\}$ are local minimum and local maximum of $\upsilon_{p/(q+1-p)}$, respectively, satisfying $\eta_i < \xi_i < \eta_{i+1}$ and

$$\begin{aligned} 0 &\le \liminf_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = \lim_{i\to+\infty} \upsilon_{p/(q+1-p)}(\eta_i) = \alpha \\ &< \limsup_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = \lim_{i\to+\infty} \upsilon_{p/(q+1-p)}(\xi_i) = \beta < +\infty. \end{aligned} \tag{3.18}$$

Let

$$\psi(s) = -\frac{\Lambda^{q+1-p}}{p} s^p + \frac{s^{q+1}}{q+1} + ls = ls - \int_0^s \phi(r)\,dr, \quad s \ge 0, \tag{3.19}$$

where ϕ is given by (2.19). Since $\upsilon'_{p/(q+1-p)}(\eta_i) = \upsilon'_{p/(q+1-p)}(\xi_i) = 0$, using (3.7), (3.18), (2.8) and (2.9), we obtain

$$\lim_{i\to+\infty} F(\eta_i) = \psi(\alpha) \quad \text{and} \quad \lim_{i\to+\infty} F(\xi_i) = \psi(\beta). \tag{3.20}$$

Since $\lim_{t\to+\infty} F(t) = F$ (by Step 1), then

$$\psi(\alpha) = \psi(\beta) = F. \tag{3.21}$$

Therefore, there exist $\gamma \in (\alpha, \beta)$ and $t_i \in (\eta_i, \xi_i)$ such that $\upsilon_{p/(q+1-p)}(t_i) = \gamma$, $\psi'(\gamma) = 0$ and $\psi(\gamma) \ne F$.

On the other hand, according to the Step 2, $\lim_{i\to+\infty} \upsilon'_{p/(q+1-p)}(t_i) = 0$, which in turn implies by (2.9) that $\lim_{i\to+\infty} h_{p/(q+1-p)}(t_i) = \frac{p}{q+1-p}\gamma$. Hence,

$$\lim_{i\to+\infty} F(t_i) = \psi(\gamma) = F.$$

This is a contradiction. Therefore, $\upsilon_{p/(q+1-p)}(t)$ converges when $t \to +\infty$. Moreover, by Lemma 3.5 and Lemma 3.3, $\lim_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = d > 0$ and $l = \phi(d)$, hence $d = z_1$ or z_2 where z_1 and z_2 are two roots of equation (3.5) such that $0 < z_1 \le z_2$. Finally, by (2.20), it is clear that $l \le L$. The proof is complete. □

Now, if $\lim_{r\to0} r^{pq/(q+1-p)} f(r) = l \ge 0$ and $q = \frac{N(p-1)+p}{N-p}$, we have this result.

Theorem 3.8. *Assume that $N > p$ and $q = \dfrac{N(p-1)+p}{N-p}$. Let u be a singular solution of equation* (2.1)*. Suppose that $\lim_{r\to0} r^{pq/(q+1-p)} f(r) = l \ge 0$ and f satisfies (H_1) or (H_2). Then u satisfies one of the following cases:*

(i) $\lim_{r\to0} r^{p/(q+1-p)} u(r) = z_1$ *or* z_2.

(ii) $r^{p/(q+1-p)}u$ *oscillates and*

$$z_1 \le \alpha = \liminf_{r\to 0} r^{p/(q+1-p)}u(r) < z_2$$
$$< \beta = \limsup_{r\to 0} r^{p/(q+1-p)}u(r) \le Z, \tag{3.22}$$

where $Z \neq z_1$ *is the root of the equation*

$$\psi(Z) = \psi(z_1). \tag{3.23}$$

Moreover, α *and* β *satisfy the following estimates*

$$l = \frac{1}{p}\left(\frac{N-p}{p}\right)^p \frac{\beta^p - \alpha^p}{\beta - \alpha} - \frac{N-p}{Np}\,\frac{\beta^{Np/(N-p)} - \alpha^{Np/(N-p)}}{\beta - \alpha} \tag{3.24}$$

and

$$\frac{p^p}{N(N-p)^{p-1}} \le \frac{\beta^p - \alpha^p}{\beta^{Np/(N-p)} - \alpha^{Np/(N-p)}} \le \frac{1}{p-1}\left(\frac{p}{N-p}\right)^p \frac{N(p-1)+p}{N}. \tag{3.25}$$

In both cases, we have $l \le L$.

Remark 3.9. Since Z satisfies (3.23), $\phi(z_1) = l$ and $q = \frac{N(p-1)+p}{N-p}$, it is easy to see that

$$\psi(Z) = \frac{p-1}{p}\left(\frac{N-p}{p}\right)^p z_1^p - \frac{N(p-1)+p}{Np}\, z_1^{Np/(N-p)}. \tag{3.26}$$

Proof. We take the same notation as in the proof of Theorem 3.7 with $A = 0$, where A is given by (3.8).

Since $v_{p/(q+1-p)}(t)$ is bounded for large t, $\lim\limits_{t\to+\infty} g_{p/(q+1-p)}(t) = l$, using the fact that

$$g'_{p/(q+1-p)}(s) \ge -\left(g'_{p/(q+1-p)}(s)\right)^-$$

and

$$\int_T^{+\infty} \left(g'_{p/(q+1-p)}(s)\right)^- ds < +\infty,$$

from (H_1) or

$$g'_{p/(q+1-p)}(s) \le \left(g'_{p/(q+1-p)}(s)\right)^+$$

and

$$\int_T^{+\infty} \left(g'_{p/(q+1-p)}(s)\right)^+ ds < +\infty,$$

from (H_2), we get respectively according to (3.11) that $-F(t)$ converges or $F(t)$ converges, when $t \to +\infty$. Let $F = \lim\limits_{t\to+\infty} F(t)$.

Since $v_{p/(q+1-p)}(t)$ is bounded for large t, we have two possibilities, either $v_{p/(q+1-p)}(t)$ converges when $t \to +\infty$, therefore according to Lemma 3.3,

$\lim_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = z_1$ or z_2 where z_1 and z_2 are two roots of equation (3.5) such that $0 \le z_1 \le z_2$ and therefore by (2.20), we have $l \le L$. Either $\upsilon_{p/(q+1-p)}(t)$ oscillates, therefore $h'_{p/(q+1-p)}(\eta_i) = \upsilon''_{p/(q+1-p)}(\eta_i) \ge 0$ and $h'_{p/(q+1-p)}(\zeta_i) = \upsilon''_{p/(q+1-p)}(\zeta_i) \le 0$ where $\{\eta_i\}$ and $\{\zeta_i\}$ are respectively local minimum and local maximum of $\upsilon_{p/(q+1-p)}$. Therefore, by equation (3.4), we have

$$0 \le \omega'_{p/(q+1-p)}(\eta_i) = \phi\left(\upsilon_{p/(q+1-p)}(\eta_i)\right) - g_{p/(q+1-p)}(\eta_i) \tag{3.27}$$

and

$$0 \ge \omega'_{p/(q+1-p)}(\zeta_i) = \phi\left(\upsilon_{p/(q+1-p)}(\zeta_i)\right) - g_{p/(q+1-p)}(\zeta_i). \tag{3.28}$$

On the other hand, according to Theorem 3.7,

$$\lim_{t\to+\infty} F(t) = F = \psi(\alpha) = \psi(\beta),$$

which gives using expression of ψ and the fact that $q = \frac{N(p-1)+p}{N-p}$ and $\alpha < \beta$,

$$\begin{aligned} l =& \frac{\Lambda^{q+1-p}}{p}\frac{\beta^p - \alpha^p}{\beta - \alpha} - \frac{1}{q+1}\frac{\beta^{q+1} - \alpha^{q+1}}{\beta - \alpha} \\ =& \frac{1}{p}\left(\frac{N-p}{p}\right)^p \frac{\beta^p - \alpha^p}{\beta - \alpha} - \frac{N-p}{Np}\frac{\beta^{Np/(N-p)} - \alpha^{Np/(N-p)}}{\beta - \alpha}. \end{aligned}$$

This proves (3.24).

A simple study of the function ψ gives

$$\begin{array}{ll} \psi'(z_1) = \psi'(z_2) = 0, & \\ \psi'(s) > 0 & \text{for } 0 < s < z_1 \text{ (if } l > 0), \\ \psi'(s) < 0 & \text{for } z_1 < s < z_2, \\ \psi'(s) > 0 & \text{for } s > z_2, \\ \lim_{s\to+\infty} \psi(s) = +\infty. & \end{array} \tag{3.29}$$

Hence, there exists $Z > z_2$ such that $\psi(Z) = \psi(z_1)$.

Now to prove estimate (3.22), we let $i \to +\infty$ in (3.27) and (3.28) and we obtain

$$\phi(\beta) \le l \le \phi(\alpha), \tag{3.30}$$

that is, by (3.19)

$$\psi'(\alpha) \le 0 \le \psi'(\beta). \tag{3.31}$$

Combining this with (3.29) and the fact that $\psi(\alpha) = \psi(\beta)$, we deduce $z_1 \le \alpha < z_2 < \beta$. Moreover, we have $\beta \le Z$, otherwise $\psi(\beta) > \psi(Z) = \psi(z_1) \ge \psi(\alpha)$, which contradicts $\psi(\alpha) = \psi(\beta)$. Consequently estimate (3.22) is satisfied.

Concerning (3.25), we use the fact that $l \geq 0$ and (3.24) to obtain the left inequality. To prove the right inequality of (3.25), we begin with the case $l > 0$. Then $\beta > \alpha > 0$. Therefore, according to (3.30), we have

$$\beta\phi(\beta) \leq l\beta = \psi(\beta) + \frac{\Lambda^{q+1-p}}{p}\beta^p - \frac{\beta^{q+1}}{q+1}$$

and

$$\alpha\phi(\alpha) \geq l\alpha = \psi(\alpha) + \frac{\Lambda^{q+1-p}}{p}\alpha^p - \frac{\alpha^{q+1}}{q+1},$$

which in turn implies that

$$\beta\phi(\beta) - \frac{\Lambda^{q+1-p}}{p}\beta^p + \frac{\beta^{q+1}}{q+1} \leq \psi(\beta) = \psi(\alpha) \leq \alpha\phi(\alpha) - \frac{\Lambda^{q+1-p}}{p}\alpha^p + \frac{\alpha^{q+1}}{q+1}.$$

Using expression of ϕ, we get

$$\frac{p-1}{p}\Lambda^{q+1-p}\beta^p - \frac{q}{q+1}\beta^{q+1} \leq \frac{p-1}{p}\Lambda^{q+1-p}\alpha^p - \frac{q}{q+1}\alpha^{q+1}.$$

Using $q = \frac{N(p-1)+p}{N-p}$, we obtain the right inequality of (3.25).

If $l = 0$, it is easy to see that

$$z_1 = 0,\ z_2 = \Lambda,\ Z = \left(\frac{N(N-p)^{p-1}}{p^p}\right)^{(N-p)/p^2},$$

and

$$\frac{\beta^p - \alpha^p}{\beta^{Np/(N-p)} - \alpha^{Np/(N-p)}} = \frac{p^p}{N(N-p)^{p-1}} \leq \frac{1}{p-1}\left(\frac{p}{N-p}\right)^p \frac{N(p-1)+p}{N}.$$

Finally (3.30) and (2.20) give $l \leq L$. This completes the proof. □

Now we study the case $\lim_{r\to 0} r^{pq/(q+1-p)} f(r) = 0$ and $q \neq \frac{N(p-1)+p}{N-p}$. For this, we introduce the following condition:

(H_3) $r^{pq/(q+1-p)+1} f'(r)$ is bounded for small r and

$$\int_0^1 r^{pq/(q+1-p)-1} f(r)dr < \infty.$$

Theorem 3.10. *Assume that $N > p$. Let u be a singular solution of equation (2.1). Suppose that $\lim_{r\to 0} r^{pq/(q+1-p)} f(r) = 0$ and f satisfies one of the following conditions:*

(i) (H_1) *and* (H_3), *if* $q > \dfrac{N(p-1)+p}{N-p}$.

(ii) (H_2) *and* (H_3), *if* $\dfrac{N(p-1)}{N-p} < q < \dfrac{N(p-1)+p}{N-p}$.

Then $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = 0$ *or* $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = \Lambda$.

Proof. According to Lemma 3.3, if $\upsilon_{p/(q+1-p)}$ converges, $\lim_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = 0$ or Λ. Therefore, it suffices to show in the two cases that $\upsilon_{p/(q+1-p)}$ converges.

Note that the energy function $F(t)$ given by (3.7) does not allow as to prove that $\lim_{t\to+\infty} \upsilon'_{p/(q+1-p)}(t) = 0$. For this reason, we introduce another energy function E defined as follows:

$$\begin{aligned} E(t) &= \frac{p-1}{p}\left|h_{p/(q+1-p)}(t)\right|^p - \Gamma\omega_{p/(q+1-p)}(t)\upsilon_{p/(q+1-p)}(t) \\ &\quad + \frac{q}{q+1} A\,\Gamma^{1/q}\left|\omega_{p/(q+1-p)}(t)\right|^{(q+1)/q} + \frac{1}{q+1}\upsilon^{q+1}_{p/(q+1-p)}(t), \end{aligned} \tag{3.32}$$

where A is given by (3.8) and

$$\Gamma = \Gamma_{p/(q+1-p)} = N - \frac{pq}{q+1-p}. \tag{3.33}$$

Therefore

$$\begin{aligned} E'(t) &= AY(t) - g_{p/(q+1-p)}(t)\upsilon'_{p/(q+1-p)}(t) \\ &\quad - Ag_{p/(q+1-p)}(t)\left(\Gamma^{1/q}\left|\omega_{p/(q+1-p)}(t)\right|^{1/q} - \upsilon_{p/(q+1-p)}(t)\right), \end{aligned} \tag{3.34}$$

where

$$\begin{aligned} Y(t) &= \left(\upsilon_{p/(q+1-p)}(t) - \Gamma^{1/q}\left|\omega_{p/(q+1-p)}(t)\right|^{1/q}\right) \\ &\quad \times \left(\upsilon^q_{p/(q+1-p)}(t) - \Gamma\left|\omega_{p/(q+1-p)}(t)\right|\right). \end{aligned} \tag{3.35}$$

Similarly to the energy function F, it is easy to see that $E(t)$ is bounded for large t. The rest of the proof is done in two steps.

Step 1. $\lim_{t\to+\infty} \omega'_{p/(q+1-p)}(t) = 0$.

In the same way as the proof of Theorem 3.7, we integrate (3.34) on (T,t) for large T, we obtain

$$\begin{aligned} E(t) &= C(T) + AS(t) - g_{p/(q+1-p)}(t)\upsilon_{p/(q+1-p)}(t) \\ &\quad + \int_T^t g'_{p/(q+1-p)}(s)\upsilon_{p/(q+1-p)}(s)\,ds \\ &\quad - A\int_T^t g_{p/(q+1-p)}(s)\left(\Gamma^{1/q}\left|\omega_{p/(q+1-p)}(s)\right|^{1/q} - \upsilon_{p/(q+1-p)}(s)\right) ds, \end{aligned} \tag{3.36}$$

where

$$S(t) = \int_T^t Y(s)\,ds. \tag{3.37}$$

Since the function $s \to s^q$ is monotone, $Y(t) \geq 0$. Therefore, the function $S(t)$ is positive and increasing. Using the fact that $A \neq 0$, we have

$$\begin{aligned} S(t) \;\; &= \frac{E(t)}{A} + \frac{1}{A} g_{p/(q+1-p)}(t) \upsilon_{p/(q+1-p)}(t) \\ &\quad - \frac{1}{A} \int_T^t g'_{p/(q+1-p)}(s) \upsilon_{p/(q+1-p)}(s)\, ds \\ &\quad + \int_T^t g_{p/(q+1-p)}(s) \left(\Gamma^{1/q} \left| \omega_{p/(q+1-p)}(s) \right|^{1/q} - \upsilon_{p/(q+1-p)}(s) \right) ds \\ &\quad - \frac{C(T)}{A}. \end{aligned} \tag{3.38}$$

Since $\upsilon_{p/(q+1-p)}(t)$, $\omega_{p/(q+1-p)}(t)$ and $E(t)$ are bounded for large t, from (H_3)

$$\lim_{t \to +\infty} g_{p/(q+1-p)}(t) = 0, \;\; \int_T^{+\infty} g_{p/(q+1-p)}(s)\, ds < +\infty$$

and

$$-\left(g'_{p/(q+1-p)}(s) \right)^- \leq g'_{p/(q+1-p)}(s) \leq \left(g'_{p/(q+1-p)}(s) \right)^+,$$

then according to the sign of A and the fact that $\displaystyle\int_T^{+\infty} \left(g'_{p/(q+1-p)}(s) \right)^- ds < +\infty$ from (H_1) or $\displaystyle\int_T^{+\infty} \left(g'_{p/(q+1-p)}(s) \right)^+ ds < +\infty$ from (H_2), we get $S(t)$ is bounded for large t. Hence $S(t)$ converges when $t \to +\infty$. Letting $t \to +\infty$ in (3.38), we get $\lim_{t\to+\infty} E(t)$ exists and is finite.

Recall that for any $1 < \varrho \leq 2$, there is a c_ϱ such that

$$\left(|a|^{\varrho-2} a - |b|^{\varrho-2} b \right) (a - b) \geq c_\varrho (a-b)^2 \left(|a| + |b| \right)^{\varrho-2}, \tag{3.39}$$

for any a, $b \in \mathbb{R}$ such that $|a| + |b| > 0$. In particular, we have

$$\begin{aligned} &\left(\upsilon_{p/(q+1-p)}(t) - \Gamma^{1/q} \left| \omega_{p/(q+1-p)}(t) \right|^{1/q} \right) \\ &\qquad \times \left(\upsilon^q_{p/(q+1-p)}(t) - \Gamma \left| \omega_{p/(q+1-p)}(t) \right| \right) \\ &\geq c_q \left(\upsilon^q_{p/(q+1-p)}(t) - \Gamma \left| \omega_{p/(q+1-p)}(t) \right| \right)^2 \\ &\qquad \times \left(\upsilon^q_{p/(q+1-p)}(t) + \Gamma \left| \omega_{p/(q+1-p)}(t) \right| \right)^{-(1-1/q)}. \end{aligned} \tag{3.40}$$

Since $\omega_{p/(q+1-p)}(t) > 0$ for large t, according to (3.4) and (3.35), we have for large t

$$\begin{aligned} Y(t) \;\; &\geq \;\; c_q \left(\omega'_{p/(q+1-p)}(t) + g_{p/(q+1-p)}(t) \right)^2 \\ &\qquad \times \left(\upsilon^q_{p/(q+1-p)}(t) + \Gamma \left| \omega_{p/(q+1-p)}(t) \right| \right)^{-(1-1/q)}. \end{aligned} \tag{3.41}$$

But $\upsilon_{p/(q+1-p)}(t)$ and $\omega_{p/(q+1-p)}(t)$ are bounded for large t and $1-\dfrac{1}{q}>0$, there exists a constant $C>0$ such that for large t,

$$\left(\omega'_{p/(q+1-p)}(t)+g_{p/(q+1-p)}(t)\right)^2 \le CY(t).$$

Which implies that

$$\int_T^t \left(\omega'_{p/(q+1-p)}(s)+g_{p/(q+1-p)}(s)\right)^2 ds \le CS(t).$$

Hence, we have

$$\begin{aligned}\int_T^t \omega'^2_{p/(q+1-p)}(s)\,ds &\le CS(t)-2\int_T^t \omega'_{p/(q+1-p)}(s)g_{p/(q+1-p)}(s)\,ds\\ &\quad-\int_T^t g^2_{p/(q+1-p)}(s)\,ds\\ &\le CS(t)-2\int_T^t \omega'_{p/(q+1-p)}(s)g_{p/(q+1-p)}(s)\,ds.\end{aligned}$$

Since $S(t)$ and $\omega'_{p/(q+1-p)}(t)$ are bounded for large t and from (H_3) $\int_T^t g_{p/(q+1-p)}(s)\,ds<+\infty$, $\int_T^t \omega'^2_{p/(q+1-p)}(s)\,ds$ is also bounded. Moreover, $\int_T^t \omega'^2_{p/(q+1-p)}(s)\,ds$ is increasing, hence necessarily we have

$$\int_T^{+\infty} \omega'^2_{p/(q+1-p)}(s)\,ds<+\infty.$$

On the other hand, deriving equation (3.4), we obtain

$$\omega''_{p/(q+1-p)}(t)-\Gamma\omega'_{p/(q+1-p)}(t)+q\upsilon^{q-1}_{p/(q+1-p)}(t)\upsilon'_{p/(q+1-p)}(t)+g'_{p/(q+1-p)}(t)=0. \tag{3.42}$$

Since $\omega'_{p/(q+1-p)}(t)$, $\upsilon_{p/(q+1-p)}(t)$, $\upsilon'_{p/(q+1-p)}(t)$ and $g'_{p/(q+1-p)}(t)$ are bounded for large t, and ($g'_{p/(q+1-p)}(t)$ is bounded by (H_3)), $\omega''_{p/(q+1-p)}(t)$ is bounded for large t. Therefore, using Lemma 3.6, we have $\lim_{t\to+\infty}\omega'_{p/(q+1-p)}(t)=0$.

Step 2. $\upsilon_{p/(q+1-p)}(t)$ converges when $t\to+\infty$.

Letting $t\to+\infty$ in equation (3.4) and taking account of $\lim_{t\to+\infty} g_{p/(q+1-p)}(t) = 0$, we obtain

$$\lim_{t\to+\infty} -\Gamma\omega_{p/(q+1-p)}(t)+\upsilon^q_{p/(q+1-p)}(t)=0. \tag{3.43}$$

Assume by contradiction that $\upsilon_{p/(q+1-p)}(t)$ oscillates for large t. Then, inequality (3.18) is satisfied. But, $\upsilon'_{p/(q+1-p)}(\eta_i)=\upsilon'_{p/(q+1-p)}(\xi_i)=0$. Hence, by (3.43), we have $\phi(\alpha)=\phi(\beta)=0$, where ϕ is given by (2.19). Since $\alpha<\beta$, necessarily $\alpha = 0$ and $\beta = \Lambda$ (where Λ is given by (1.7)). But by (3.32), we have $\lim_{i\to+\infty} E(\eta_i) = 0$ and

$$\lim_{i\to+\infty} E(\xi_i) = \frac{-1}{q+1}\left(\frac{p}{q+1-p}\right)^{p-1}\Lambda^p < 0,$$

which contradicts the fact that $E(t)$ converges when $t \to +\infty$. Consequently, $\upsilon_{p/(q+1-p)}(t)$ converges when $t \to +\infty$. The proof of theorem is complete. $\square$

Next, we consider the case that $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = 0$. According to Lemma 3.3, this can only take place in the case $\lim_{r\to 0} r^{pq/(q+1-p)}f(r) = 0$.

Assume now that there exists $0 < k < \frac{pq}{q+1-p}$ such that $r^k f(r)$ is bounded for small r. We begin with the case $p \le k < \frac{pq}{q+1-p}$.

Theorem 3.11. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation (2.1) such that $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = 0$. Suppose that there exists a constant $p \le k < \frac{pq}{q+1-p}$ such that $r^k f(r)$ is bounded for small r. Then u has the following asymptotic behavior near 0*

(i) $r^{(k-p)/(p-1)}u(r)$ *is bounded, if* $p < k < \frac{pq}{q+1-p}$.

(ii) $\frac{u(r)}{|\ln r|}$ *is bounded,* $k = p$.

Proof. (i) Let $\sigma = \frac{k-p}{p-1}$. Using the change (2.5) (for $c = \sigma$), we claim that $\upsilon_\sigma(t)$ is bounded for large t. For contradiction, we distinguish two cases.

Case 1. $\lim_{t\to+\infty} \upsilon_\sigma(t) = +\infty$.

Since $0 < \sigma < \dfrac{p}{q+1-p}$ and $r^{\sigma(p-1)+p}f(r)$ is bounded for small r, according to Proposition 2.10, we have $E_{p/(q+1-p)}(r) > 0$ and $E_\sigma(r) < 0$, for small r. Hence using the fact that $u'(r) < 0$ for small r,

$$\sigma < \frac{r|u'|}{u} < \frac{p}{q+1-p}. \tag{3.44}$$

That is, using the change (2.5), we have for large t,

$$\sigma^{p-1} < \omega_\sigma(t)\upsilon_\sigma^{1-p}(t) < \left(\frac{p}{q+1-p}\right)^{p-1}. \tag{3.45}$$

Multiplying equation (2.6) (for $c = \sigma$) by $\upsilon_\sigma^{1-p}(t)$, we obtain

$$\left(\omega_\sigma(t)\upsilon_\sigma^{1-p}(t)\right)' + (p-1)\,|h_\sigma(t)|^p\,\upsilon_\sigma^{-p}(t) - (N-p)\,\omega_\sigma(t)\upsilon_\sigma^{1-p}(t) + G_\sigma(t) = 0, \tag{3.46}$$

where

$$G_\sigma(t) = e^{(\sigma(q+1-p)-p)t} \upsilon_\sigma^{q+1-p}(t) + g_\sigma(t)\upsilon_\sigma^{1-p}(t) = 0. \tag{3.47}$$

Since $\lim_{t\to+\infty} e^{(\sigma(q+1-p)-p)t}\upsilon_\sigma^{q+1-p}(t) = 0$ (because $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = 0$), $g_\sigma(t)$ is bounded for large t and $\lim_{t\to+\infty} \upsilon_\sigma^{p-1}(t) = +\infty$, we have $\lim_{t\to+\infty} G_\sigma(t) = 0$.

Let

$$\varphi_\sigma(t) = \omega_\sigma(t)\upsilon_\sigma^{1-p}(t). \tag{3.48}$$

Then estimation (3.45) implies that for large t

$$\sigma^{p-1} < \varphi_\sigma(t) < \left(\frac{p}{q+1-p}\right)^{p-1} \tag{3.49}$$

and moreover by (3.46), we have

$$-\varphi_\sigma'(t) = (p-1)\,|\varphi_\sigma(t)|^{p/(p-1)} - (N-p)\,\varphi_\sigma(t) + G_\sigma(t). \tag{3.50}$$

Let

$$\mu(s) = s^{p/(p-1)} - \frac{N-p}{p-1}s, \qquad s \ge 0. \tag{3.51}$$

Using the fact that $\varphi_\sigma(t) > 0$ for large t, we get for large t

$$-\varphi_\sigma'(t) = (p-1)\mu\left(\varphi_\sigma(t)\right) + G_\sigma(t). \tag{3.52}$$

A simple study of the function μ gives the existence of a constant $K > 0$ such that $\mu(s) < -K$ for $\sigma^{p-1} < s < \left(\frac{p}{q+1-p}\right)^{p-1} < \left(\frac{N-p}{p-1}\right)^{p-1}$. Since $\varphi_\sigma(t)$ satisfies (3.49) for large t and $\lim_{t\to+\infty} G_\sigma(t) = 0$, by (3.52), there exists a constant $K_1 > 0$ such that for large t

$$\varphi_\sigma'(t) > K_1.$$

Integrating this last inequality on (T,t) for large T, we obtain $\lim_{t\to+\infty} \varphi_\sigma(t) = +\infty$, which contradicts the fact $\varphi_\sigma(t)$ is bounded for large t by (3.49).

Case 2. Suppose that there exist two sequences $\{s_i\}$ and $\{t_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{s_i\}$ and $\{t_i\}$ are local minimum and local maximum of υ_σ, respectively, satisfying $s_i < t_i < s_{i+1}$ and $\lim_{t\to+\infty} \upsilon_\sigma(t_i) = +\infty$.

Taking $t = t_i$ in equation (2.6), we obtain

$$\omega_\sigma'(t_i) = \Gamma_\sigma\,\omega_\sigma(t_i) - e^{-(p-\sigma(q+1-p))t_i}\upsilon_\sigma^q(t_i) - g_\sigma(t_i) = 0. \tag{3.53}$$

Since $\upsilon_\sigma'(t_i) = 0$, by (2.9) and (2.8), we have

$$\frac{\upsilon_\sigma^{p-1}(t_i)}{\omega_\sigma(t_i)} = \sigma^{1-p}.$$

Hence, equation (3.53) can be written as

$$\omega'_\sigma(t_i) = \omega_\sigma(t_i)\left[\Gamma_\sigma - \sigma^{1-p}e^{-(p-\sigma(q+1-p))t_i}\upsilon_\sigma^{q+1-p}(t_i) - \frac{g_\sigma(t_i)}{\omega_\sigma(t_i)}\right]. \qquad (3.54)$$

According to our hypothesis, we have $\lim\limits_{i\to+\infty}\dfrac{\omega'_\sigma(t_i)}{\omega_\sigma(t_i)} = \Gamma_\sigma$, hence $\omega'_\sigma(t_i) > 0$ for large i. But $h'_\sigma(t_i) = \upsilon''_\sigma(t_i) \le 0$, which implies that $\omega'_\sigma(t_i) \le 0$. This is a contradiction. Consequently, $\upsilon_\sigma(t)$ is bounded for large t.

(ii) According to Proposition 2.7, we have $\lim\limits_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0$. Hence, integrating equation (2.1) on $(0,r)$ for small r and using the fact that $u'(r) < 0$ near 0, we obtain

$$r^{N-1}|u'|^{p-1} = \int_0^r s^{N-1}u^q(s)\,ds + \int_0^r s^{N-1}f(s)\,ds. \qquad (3.55)$$

According to (i), if there exists $0 < \varrho < \dfrac{p}{q+1-p}$ such that $r^{p+\varrho(p-1)}f(r)$ is bounded for small r, then $r^\varrho u(r)$ is bounded also for small r. In particular for $\varrho = \dfrac{k}{q}$, we have $r^{p+k(p-1)/q}f(r)$ is bounded for small r (because $p + \dfrac{k(p-1)}{q} > k$ and $r^k f(r)$ is bounded for small r) and therefore $r^{k/q}u(r)$ is bounded for small r. hence, by (3.55) and the fact that $N > \dfrac{pq}{q+1-p} > k$, there exists a constant $C > 0$ such that for small r, we have

$$|u'|^{p-1} \le Cr^{1-k}. \qquad (3.56)$$

Since $k = p$, the last inequality becomes for small r,

$$|u'| \le Cr^{-1}. \qquad (3.57)$$

Therefore, integrating (3.57) on (r, r_0) for $r_0 < 1$ and using the fact that $u' < 0$ near 0, we obtain

$$u(r) \le C(r_0) + C\,|\ln r|\,.$$

Which implies that $\dfrac{u(r)}{|\ln r|}$ is bounded for small r. □

Lemma 3.12. *Under hypotheses of Theorem* 3.11 *for* $p < k < \frac{pq}{q+1-p}$, *we have*

$$\liminf_{r\to 0} r^k f(r) \le (N-k)\left(\frac{k-p}{p-1}\right)^{p-1} \limsup_{r\to 0} r^{k-p}u^{p-1}(r) \qquad (3.58)$$

and

$$\limsup_{r\to 0} r^k f(r) \ge (N-k)\left(\frac{k-p}{p-1}\right)^{p-1} \liminf_{r\to 0} r^{k-p}u^{p-1}(r). \qquad (3.59)$$

Proof. To show estimate (3.58), we assume by contradiction that

$$\liminf_{r\to 0} r^k f(r) > (N-k)\left(\frac{k-p}{p-1}\right)^{p-1} \limsup_{r\to 0} r^{k-p}u^{p-1}(r).$$

Hence, using the change (2.5) and taking $c=\sigma=\dfrac{k-p}{p-1}$ in (2.5), there exists $b_1>0$ such that for large t,

$$g_\sigma(t)=e^{-kt}f(e^{-t})\geq (N-k)\,\sigma^{p-1}\upsilon_\sigma^{p-1}(t)+b_1. \tag{3.60}$$

We claim that $\upsilon_\sigma(t)$ is strictly monotone for large t. For this, using (2.14), it suffices to show that $E_\sigma(r)\neq 0$ for small r. Suppose by contradiction that there exists a small r such that $E_\sigma(r)=0$. Then, according to (2.16), we have

$$\begin{aligned}(p-1)r^{k-1}|u'|^{p-2}E'_\sigma(r) &= (N-k)\,\sigma^{p-1}\upsilon_\sigma^{p-1}(t) \\ &\quad -e^{-((pq-k(q+1-p))/(p-1))t}\upsilon_\sigma^{q}(t)-g_\sigma(t).\end{aligned} \tag{3.61}$$

Using (3.60), we get for small r,

$$(p-1)r^{k-1}|u'|^{p-2}E'_\sigma(r)<(N-k)\,\sigma^{p-1}\upsilon_\sigma^{p-1}(t)-g_\sigma(t)\leq -b_1<0. \tag{3.62}$$

Hence, $E_\sigma(r)\neq 0$ for small r. Consequently $\upsilon_\sigma(t)$ is strictly monotone for large t.

Moreover, since $\upsilon_\sigma(t)$ is bounded for large t by Theorem 3.11, $\upsilon_\sigma(t)$ converges when $t\to+\infty$. Let $\lim\limits_{t\to+\infty}\upsilon_\sigma(t)=d_1\geq 0$. We distinguish two cases.

Case 1. $\upsilon'_\sigma(t)$ is monotone for large t.

Since $\upsilon'_\sigma(t)\neq 0$ and $\upsilon_\sigma(t)$ is bounded, for large t, necessarily $\lim\limits_{t\to+\infty}\upsilon'_\sigma(t)=0$, which implies by (2.9) that $\lim\limits_{t\to+\infty}h_\sigma(t)=\sigma d_1$ and therefore $\lim\limits_{t\to+\infty}\omega_\sigma(t)=\sigma^{p-1}d_1^{p-1}$.

On the other hand, according to (2.6), we have

$$\omega'_\sigma(t)=(N-k)\omega_\sigma(t)-e^{-((pq-k(q+1-p))/(p-1))t}\upsilon_\sigma^{q}(t)-g_\sigma(t). \tag{3.63}$$

Therefore, according to (3.60), we have for large t,

$$\omega'_\sigma(t)\leq\chi(t)-b_1, \tag{3.64}$$

where

$$\begin{aligned}\chi(t) &= (N-k)\omega_\sigma(t)-e^{-((pq-k(q+1-p))/(p-1))t}\upsilon_\sigma^{q}(t) \\ &\quad -(N-k)\,\sigma^{p-1}\upsilon_\sigma^{p-1}(t).\end{aligned} \tag{3.65}$$

From $k<\frac{pq}{q+1-p}$, $\lim\limits_{t\to+\infty}\upsilon_\sigma(t)=d_1$ and $\lim\limits_{t\to+\infty}\omega_\sigma(t)=\sigma^{p-1}d_1^{p-1}$, we have $\lim\limits_{t\to+\infty}\chi(t)=0$. Hence, there exists a constant $C>0$ such that $\omega'_\sigma(t)\leq -C$

for large t. Integrating this last inequality on (T,t) for large T, we obtain $\lim_{t\to+\infty}\omega_\sigma(t)=-\infty$. This is a contradiction.

Case 2. $\upsilon'_\sigma(t)$ is not monotone for large t.

Since $\upsilon'_\sigma(t)\neq 0$ and $\upsilon_\sigma(t)$ is bounded, for large t, we have two possibilities, $\liminf_{t\to+\infty}\upsilon'_\sigma(t)=0$ if $\upsilon'_\sigma(t)>0$ for large t or $\limsup_{t\to+\infty}\upsilon'_\sigma(t)=0$ if $\upsilon'_\sigma(t)<0$ for large t. Therefore, there exists a sequence $\{\gamma_i\}$ going to $+\infty$ as $i\to+\infty$ such that $\{\gamma_i\}$ is a local extrema of υ'_σ satisfying $\lim_{i\to+\infty}\upsilon'_\sigma(\gamma_i)=0$. Therefore, $\lim_{i\to+\infty}\omega_\sigma(\gamma_i)=\sigma^{p-1}d_1^{p-1}$ and $\lim_{i\to+\infty}\omega'_\sigma(\gamma_i)=0$ (because $\upsilon''_\sigma(\gamma_i)=0$ and therefore $\lim_{i\to+\infty}h'_\sigma(\gamma_i)=0$). Taking $t=\gamma_i$ in equation (3.63) and letting $i\to+\infty$, we obtain

$$\lim_{i\to+\infty}g_\sigma(\gamma_i)=(N-k)\sigma^{p-1}d_1^{p-1}.$$

This is impossible according to (3.60) because $b_1>0$. Consequently, estimate (3.58) is satisfied.

Now, we show estimate (3.59). Suppose by contradiction that

$$\limsup_{r\to 0} r^k f(r)<(N-k)\left(\frac{k-p}{p-1}\right)^{p-1}\liminf_{r\to 0} r^{k-p}u^{p-1}(r).$$

Taking $\sigma=\frac{k-p}{p-1}$, there exists $b_2>0$ such that for large t

$$g_\sigma(t)\le (N-k)\,\sigma^{p-1}\upsilon_\sigma^{p-1}(t)-b_2. \tag{3.66}$$

In the same way, we show that the last inequality implies that $\upsilon_\sigma(t)$ is strictly monotone for large t. Suppose by contradiction that there exists a small r such that $E_\sigma(r)=0$. Then, according to (3.61) and (3.66), we have for small r,

$$(p-1)r^{k-1}|u'|^{p-2}E'_\sigma(r)\ge b_2-e^{-((pq-k(q+1-p))/(p-1))t}\upsilon_\sigma^q(t). \tag{3.67}$$

Since $\lim_{t\to+\infty}e^{-((pq-k(q+1-p))/(p-1))t}\upsilon_\sigma^q(t)=0$ because $k<\frac{pq}{q+1-p}$ and $\upsilon_\sigma(t)$ is bounded for large t by Theorem 3.11, for small r, we have

$$(p-1)r^{k-1}|u'|^{p-2}E'_\sigma(r)\ge\frac{b}{2}>0.$$

Hence, $E_\sigma(r)\neq 0$ for small r and therefore $\upsilon_\sigma(t)$ is strictly monotone for large t.

Now, it is enough to repeat the same reasoning as the first case using equation (3.63) and estimate (3.66) to obtain the contradiction. Consequently, estimate (3.59) is satisfied. This completes the proof. □

Now, the following result proves that if $f(r)\underset{0}{\sim} l\,r^{-k}$ for $p\le k<\frac{pq}{q+1-p}$, then the singular solution u has also an equivalent near 0.

Theorem 3.13. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation (2.1) such that $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = 0$. Suppose that $\lim_{r\to 0} r^k f(r) = l > 0$ for $p \le k < \frac{pq}{q+1-p}$. Then*

$$\text{(i)}\ \lim_{r\to 0} r^{(k-p)/(p-1)}u(r) = \frac{p-1}{k-p}\left(\frac{l}{N-k}\right)^{1/(p-1)}, \quad \textit{if}\ \ p < k < \frac{pq}{q+1-p}.$$

$$\text{(ii)}\ \lim_{r\to 0} \frac{u(r)}{|\ln r|} = \left(\frac{l}{N-p}\right)^{1/(p-1)}, \quad \textit{if}\ \ k = p.$$

Proof. (i) Let $\sigma = \frac{k-p}{p-1}$. We know by Theorem 3.11 that $\upsilon_\sigma(t)$ is bounded for large t. Assume that $\upsilon_\sigma(t)$ oscillates for large t. Then there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{\eta_i\}$ and $\{\xi_i\}$ are local minimum and local maximum of υ_σ, respectively, satisfying $\eta_i < \xi_i < \eta_{i+1}$ and

$$\begin{aligned} 0 \ &\le\ \liminf_{t\to+\infty} \upsilon_\sigma(t) = \lim_{i\to+\infty} \upsilon_\sigma(\eta_i) = \alpha \\ &<\ \limsup_{t\to+\infty} \upsilon_\sigma(t) = \lim_{i\to+\infty} \upsilon_\sigma(\xi_i) = \beta < +\infty. \end{aligned} \tag{3.68}$$

Since $\upsilon'_\sigma(\eta_i) = \upsilon'_\sigma(\xi_i) = 0$, $\upsilon''_\sigma(\eta_i) \ge 0$ and $\upsilon''_\sigma(\xi_i) \le 0$, using (2.8) and (2.9), we have

$$\begin{gathered} \lim_{i\to+\infty} \omega_\sigma(\eta_i) = \sigma^{p-1}\alpha^{p-1}, \\ \lim_{i\to+\infty} \omega_\sigma(\xi_i) = \sigma^{p-1}\beta^{p-1}, \\ \omega'_\sigma(\eta_i) \ge 0 \ \text{ and } \ \omega'_\sigma(\xi_i) \le 0. \end{gathered}$$

Therefore according to (3.63), we have

$$(N-k)\omega_\sigma(\eta_i) - e^{-((pq-k(q+1-p))/(p-1))\eta_i}\upsilon_\sigma^q(\eta_i) - g_\sigma(\eta_i) \ge 0 \tag{3.69}$$

and

$$(N-k)\omega_\sigma(\xi_i) - e^{-((pq-k(q+1-p))/(p-1))\xi_i}\upsilon_\sigma^q(\xi_i) - g_\sigma(\xi_i) \le 0. \tag{3.70}$$

Letting $i \to +\infty$ in the two previous inequalities and using the fact that $\lim_{t\to+\infty} g_\sigma(t) = l$, we obtain

$$\beta^{p-1} \le \frac{l}{(N-k)\sigma^{p-1}} \le \alpha^{p-1}.$$

But this contradicts (3.68). Therefore, $\upsilon_\sigma(t)$ converges when $t \to +\infty$.

On the other hand, we have by Lemma 3.12,

$$\liminf_{t\to+\infty} \upsilon_\sigma^{p-1}(t) \le \frac{l}{(N-k)\sigma^{p-1}} \le \limsup_{t\to+\infty} \upsilon_\sigma^{p-1}(t).$$

Hence, $\lim_{t\to+\infty} \upsilon_\sigma(t) = \frac{1}{\sigma}\left(\frac{l}{N-k}\right)^{1/(p-1)} = \frac{p-1}{k-p}\left(\frac{l}{N-k}\right)^{1/(p-1)}$.

(ii) Multiplying equation (2.2) by r^{p+1-N}, we obtain

$$r^{p+1-N}\left(r^{N-1}\left|u'\right|^{p-2}u'\right)' + r^p u^q(r) + r^p f(r) = 0. \tag{3.71}$$

Since $\frac{u(r)}{|\ln r|}$ is bounded for small r, $\lim_{r\to 0} r^{p/q}u(r) = 0$. therefore, by (3.71),

$$\lim_{r\to 0} r^{p+1-N}\left(r^{N-1}\left|u'\right|^{p-2}u'\right)' = -l.$$

Hence, since $\lim_{r\to 0} r^{N-1}\left|u'\right|^{p-2}u'(r) = 0$ (by Proposition 2.7) and $N > p$, we obtain using Hopital's rule,

$$\lim_{r\to 0} r^{p-1}\left|u'\right|^{p-2}u'(r) = \frac{-l}{N-p}.$$

Taking into account the fact that $u' < 0$ near 0, we have

$$\lim_{r\to 0} r\left|u'(r)\right| = \left(\frac{l}{N-p}\right)^{1(p-1)}.$$

Since $\lim_{r\to 0} u(r) = +\infty$ and $\lim_{r\to 0} \ln r = -\infty$, using again Hopital's rule, we obtain

$$\lim_{r\to 0}\frac{u(r)}{|\ln r|} = -\lim_{r\to 0}\frac{u(r)}{\ln r} = -\lim_{r\to 0} ru'(r) = \left(\frac{l}{N-p}\right)^{1(p-1)}.$$

The proof is complete. □

The following result gives an equivalent of u' near the origin.

Theorem 3.14. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a singular solution of equation* (2.1) *such that $\lim_{r\to 0} r^{p/(q+1-p)}u(r) = 0$. Suppose that $\lim_{r\to 0} r^k f(r) = l > 0$ for $p \le k < \frac{pq}{q+1-p}$. Then we have*

(i) $\lim_{r\to 0} r^{(k-1)/(p-1)}u'(r) = -\left(\frac{l}{N-k}\right)^{1/(p-1)}$, *if* $p < k < \frac{pq}{q+1-p}$.

(ii) $\lim_{r\to 0} ru'(r) = -\left(\frac{l}{N-p}\right)^{1/(p-1)}$, *if* $k = p$.

Proof. (i) Let $\sigma = \frac{k-p}{p-1}$. Then, by the change (2.5) and Theorem 3.13, we have $\lim_{t\to+\infty} \upsilon_\sigma(t)= \frac{p-1}{k-p}\left(\frac{l}{N-k}\right)^{1/(p-1)}$.

Now, we show that $\lim_{t\to+\infty} h_\sigma(t) = \left(\frac{l}{N-k}\right)^{1/(p-1)}$. Since $E_{(N-p)/(p-1)}(r) > 0$ for small r by Proposition 2.9, $h_\sigma(t)$ is bounded for large t. Suppose by contradiction that $h_\sigma(t)$ oscillates for large t. Then there exist two sequences $\{s_i\}$ and $\{k_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{s_i\}$ and $\{k_i\}$ are local minimum and local maximum of h_σ, respectively, satisfying $s_i < k_i < s_{i+1}$ and

$$\liminf_{t\to+\infty} h_\sigma(t) = \lim_{i\to+\infty} h_\sigma(s_i) = m_1 < \limsup_{t\to+\infty} h_\sigma(t) = \lim_{i\to+\infty} h_\sigma(k_i) = M_1. \tag{3.72}$$

Therefore $\omega'_\sigma(s_i) = \omega'_\sigma(k_i) = 0$ (because $h'_\sigma(s_i) = h'_\sigma(k_i) = 0$), $\lim_{i\to+\infty} \omega_\sigma(s_i) = |m_1|^{p-2} m_1$ and $\lim_{i\to+\infty} \omega_\sigma(k_i) = |M_1|^{p-2} M_1$. Since $k < \frac{pq}{q+1-p}$, υ_σ converges and $\lim_{t\to+\infty} e^{-kt} f(e^{-t}) = l$, we deduce, by taking respectively $t = s_i$ and $t = k_i$ in equation (2.6) and letting $i \to +\infty$, that

$$-(N-k)\,|m_1|^{p-2}\, m_1 = -l = -(N-k)\,|M_1|^{p-2}\, M_1.$$

Since $N > \frac{pq}{q+1-p} > k$ and $l > 0$, we have

$$|m_1|^{p-2}\, m_1 = |M_1|^{p-2}\, M_1 > 0.$$

That is, $m_1 = M_1$. Which contradicts (3.72). Therefore, $h_\sigma(t)$ converges when $t \to +\infty$, hence by (2.9), we have $\lim_{t\to+\infty} \upsilon'_\sigma(t) = 0$ (because υ_σ converges). Consequently, $\lim_{t\to+\infty} h_\sigma(t) = \left(\frac{l}{N-k}\right)^{1/(p-1)}$, which is equivalent by (2.11) to $\lim_{r\to 0} r^{(k-1)/(p-1)} u'(r) = -\left(\frac{l}{N-k}\right)^{1/(p-1)}$.

(ii) It can be easily deduced from (ii) of Theorem 3.13 by using Hopital's rule. The proof is complete. □

Finally concerning the case $0 < k < p$, we will show under some assumptions that the solution u of equation (2.1) can be extended by continuity at 0.

Theorem 3.15. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Let u be a solution of equation (2.1) such that $\lim_{r\to 0} r^{p/(q+1-p)} u(r) = 0$. Suppose that there exists a constant $0 < k < p$ such that $r^k f(r)$ is bounded for small r. Then $\lim_{r\to 0} u(r)$ is finite and strictly positive.*

Proof. According to Proposition 2.1, we have u is strictly decreasing near 0 and $\lim_{r\to 0} u(r) \in]0, +\infty]$. Assume that u is singular, that is $\lim_{r\to 0} u(r) = +\infty$. Then, since $0 < k < p < \frac{pq}{q+1-p}$, according to Theorem 3.11, $r^{k/q} u(r)$ is bounded for small r. Therefore, inequality (3.56) is satisfied for small r. Integrating this last inequality on (r, r_0) for small r_0 and using the fact that $p > k$, we obtain

$$u(r) \leq C(r_0) - Cr^{(p-k)/(p-1)}.$$

That is, $u(r)$ is bounded for small r. But this contradicts the fact that u is singular. Consequently, $\lim_{r\to 0} u(r) \in]0, +\infty[$. The proof is complete. □

4. Existence of a singular solution

In this section, we establish the existence of singular solutions of equation (2.1) under some assumptions on f. We use the technical results introduced by [15].

In view of the first section, if u is a singular solution of (2.1), then

$$\lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0$$

when $N > p$ and $q > \frac{N(p-1)}{N-p}$. Hence a natural problem arises:

Find a function u defined on $]0, r_{\max}[$ such that $u \in C^0(]0, r_{\max}[) \cap C^1(]0, r_{\max}[)$ and $|u'|^{p-2}u' \in C^1(]0, r_{\max}[)$, where $0 < r_{\max} \leq +\infty$ and satisfying

$$(\mathbf{P}) \begin{cases} (|u'|^{p-2}u')'(r) + \dfrac{N-1}{r}|u'|^{p-2}u'(r) + u^q(r) + f(r), \quad r > 0, \\ \lim_{r\to 0} u(r) = +\infty, \quad \lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0, \end{cases}$$

where $p > 2$, $q > 1$, $N \geq 1$ and f is a continuous radial function and strictly positive on $]0, +\infty[$.

Theorem 4.1. *Assume that $N > p$ and $q > \frac{N(p-1)}{N-p}$. Suppose that there exists a constant $0 < k < p$ such that $r^k f(r)$ is bounded for small r. Then problem (P) has a unique solution defined on a maximal interval $]0, r_{\max}[$, where $0 < r_{\max} \leq +\infty$.*

Proof. Recall by Remark 2.2 that if u is a solution of (2.1) such that $\lim_{r\to 0} u(r) = \gamma > 0$, then necessarily $\lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0$. Hence, for any $\gamma > 0$, we consider the problem

$$(\mathbf{Q}) \begin{cases} (|u'|^{p-2}u')'(r) + \dfrac{N-1}{r}|u'|^{p-2}u'(r) + u^q(r) + f(r), \quad r > 0, \\ u(0) = \gamma, \quad \lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0. \end{cases}$$

First, we show that for any $\gamma > 0$, problem (Q) has a unique solution u defined on a maximal interval $[0, r_{max}[$. To establish local existence and uniqueness, we use a method introduced in [5] and we will try to convert the problem (Q) into a fixed point problem of some operator.

Note that the difficulty lies in the fact that there was no initial data, but has only a limited condition.

Let u be a solution of problem (Q) on $[0, r_{max}[$. Then, integrating equation (2.2) on $(0, r)$ for any $r \in [0, r_{max}[$ and using the fact that $\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0$, we obtain

$$u(r) = \gamma - \int_0^r G(F[u](s))ds, \tag{4.1}$$

where

$$G(s) = |s|^{(2-p)/(p-1)}s, \;\; s \in \mathbb{R} \tag{4.2}$$

and the nonlinear mapping F is given by

$$F[\varphi](s) = s^{1-N} \int_0^s \sigma^{N-1}\big[\varphi^q(\sigma) + f(\sigma)\big]d\sigma. \tag{4.3}$$

Let $R > 0$, $\gamma > M > 0$ and consider the following complete metric space:

$$E_{\gamma,M,R} = \{\varphi \in C([0,\, R]) : \|\varphi - \gamma\|_0 \le M\}, \tag{4.4}$$

where $C([0, R])$ is the Banach space of real continuous functions on $[0, R]$ with the uniform norm, denoted by $\|\cdot\|_0$.

Next we define the mapping $\mathcal{T}$ on $E_{\gamma,M,R}$ by

$$\mathcal{T}[\varphi](r) = \gamma - \int_0^r G(F[\varphi](s))ds. \tag{4.5}$$

The idea is to show that $\mathcal{T}$ is a contraction from $E_{\gamma,M,R}$ into itself for small R. We will do it in two steps.

Step 1. $\mathcal{T}$ maps $E_{\gamma,M,R}$ into itself for small M and R.

Since $\varphi(r) \in [\gamma - M, \gamma + M]$ and $0 \le f(r) \le Cr^{-k}$ near the origin with $k < p < N$, for small R

$$\frac{(\gamma - M)^q}{N} s \le F[\varphi](s) \le \frac{(\gamma + M)^q}{N} s + \frac{C}{N-k} s^{1-k} \;\text{ for } s \in]0, R]. \tag{4.6}$$

Therefore for sufficiently small R, we have

$$C_1 s \le F[\varphi](s) \le C_2 s^{1-k}. \tag{4.7}$$

On the other hand, we have by (4.5)

$$|\mathcal{T}[\varphi](r) - \gamma| \le \int_0^r |F[\varphi](s)|^{1/(p-1)}\, ds \;\text{ for any } r \in [0, R]. \tag{4.8}$$

Hence, owing to (4.5) and (4.2), we obtain for any $r \in [0, R]$

$$|\mathcal{T}[\varphi](r) - \gamma| \le \frac{C_2^{1/(p-1)}(p-1)}{p-k} R^{(p-k)/(p-1)}. \tag{4.9}$$

So we can choose R sufficiently small such that

$$|\mathcal{T}[\varphi](r) - \gamma| \leq M, \quad \text{for } \varphi \in E_{\gamma,M,R}. \tag{4.10}$$

That is, $\mathcal{T}[\varphi] \in E_{\gamma,M,R}$.

Step 2. $\mathcal{T}$ is a contraction from $E_{\gamma,M,R}$ into itself for small R.

For any $r \in [0,R]$ and any $\varphi, \psi \in E_{\gamma,M,R}$, we have

$$|\mathcal{T}[\varphi](r) - |\mathcal{T}\psi](r)| \leq \int_0^r |G(F[\varphi](s)) - G(F[\psi](s))| ds, \tag{4.11}$$

where $F[\varphi]$ is given by (4.3). Next, let

$$\Phi(s) = \min(|F[\varphi](s)|, |F[\psi](s)|).$$

Then

$$|\mathcal{T}[\varphi](r) - \mathcal{T}[\psi](r)| \leq \int_0^r (\Phi(s))^{(2-p)/(p-1)} |F[\varphi](s) - F[\psi](s)| ds. \tag{4.12}$$

Using (4.7), we have

$$\Phi(s) \geq C_1 s. \tag{4.13}$$

And according to (4.3) and (4.4), we have

$$|F[\varphi](s) - F[\psi](s)| \leq \frac{q(M+\gamma)^{q-1}}{N}\, s \, \|\varphi - \psi\|_0 . \tag{4.14}$$

Therefore for any $r \in [0,R]$

$$|\mathcal{T}[\varphi](r) - \mathcal{T}[\psi](r)| \leq \frac{q(p-1)C_1^{(2-p)/(p-1)}(M+\gamma)^{q-1}}{Np}\, R^{p/(p-1)} \, \|\varphi - \psi\|_0 . \tag{4.15}$$

So we can choose R small enough such that $\mathcal{T}$ is a contraction. Consequently, the Banach Fixed Point Theorem implies the existence of unique fixed point $u = u_\gamma$ of $\mathcal{T}$ which is a solution of (4.1), that is solution of problem (Q). This solution can be extended to a maximal interval $[0, r_{\max}[$, $0 < r_{\max} \leq +\infty$.

Now, we have by the maximum principle $\gamma \mapsto u_\gamma$ is increasing and by Proposition 2.4, $u_\gamma(r) \leq C(N,p,q)\, r^{-p/(q+1-p)}$, where $C(N,p,q)$ is explicitly given by (2.4). Therefore u_γ converges when $\gamma \to +\infty$ to u which is a solution of problem (P) on a maximal interval $]0, r_{\max}[$, $0 < r_{\max} \leq +\infty$. The proof is complete. □

5. Nonexistence result

In this section, we present a nonexistence result concerning singular solutions of equation (2.1). Note that the comparison near the origin between the functions f and $L\, r^{-pq/(q+1-p)}$ where L is given by (2.20), will play an important role in proving this result.

Theorem 5.1. *Let* $N > p$ *and* $q > \frac{N(p-1)}{N-p}$. *If* $\liminf_{r\to 0} r^{pq/(q+1-p)} f(r) > L$, *then equation* (2.1) *does not possess any singular solutions.*

Proof. Suppose that $\liminf_{r\to 0} r^{pq/(q+1-p)} f(r) > L$. Let u be a singular solution of equation (2.1). According to Proposition 2.9, we have $E_{p/(q+1-p)}(r) \neq 0$ for small r, that is, by (2.14), $\upsilon_{p/(q+1-p)}(t)$ is strictly monotone for large t.

Moreover $\upsilon_{p/(q+1-p)}$ is bounded by Proposition 2.4 and Remark 2.5, hence it converges. Let $\lim_{t\to+\infty} \upsilon_{p/(q+1-p)}(t) = d \geq 0$. This being said, we have then two possibilities, $\liminf_{t\to+\infty} \upsilon'_{p/(q+1-p)}(t) = 0$ if $\upsilon'_{p/(q+1-p)}(t) > 0$ for large t or $\limsup_{t\to+\infty} \upsilon'_{p/(q+1-p)}(t) = 0$ if $\upsilon'_{p/(q+1-p)}(t) < 0$ for large t. Therefore, there exists a sequence $\{\gamma_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{\gamma_i\}$ is a local extrema of $\upsilon'_{p/(q+1-p)}$ satisfying $\lim_{i\to+\infty} \upsilon'_{p/(q+1-p)}(\gamma_i) = 0$. Therefore, we have

$$\lim_{i\to+\infty} \omega_{p/(q+1-p)}(\gamma_i) = \left(\frac{p}{q+1-p}\right)^{p-1} d^{p-1}.$$

Moreover, by deriving equation (2.9) and using the fact that $\upsilon''_{p/(q+1-p)}(\gamma_i) = 0$, we obtain $\lim_{i\to+\infty} h'_{p/(q+1-p)}(\gamma_i) = 0$. Hence, $\lim_{i\to+\infty} \omega'_{p/(q+1-p)}(\gamma_i) = 0$. Taking $t = \gamma_i$ in equation (3.4) and letting $i \to +\infty$, we obtain

$$\lim_{i\to+\infty} g_{p/(q+1-p)}(\gamma_i) = \phi(d) \leq L,$$

where ϕ and L are given respectively by (2.19) and (2.20). But this contradicts the fact that $\liminf_{t\to+\infty} g_{p/(q+1-p)}(t) > L$. The proof is complete. □

Acknowledgments: The authors thank the reviewer for several remarks on a preliminary version of the paper.

References

[1] P. Aviles, *Local behavior of positive solutions of some elliptic equation*, Comm. Math. Phys., **108** (1987), 177–192.

[2] S. Bae, *Asymptotic behavior of positive solutions of inhomogeneous semilinear elliptic equations*, Nonlinear Analysis, **51** (2002), 1373–1403.

[3] M.F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type*, Archive for Rational Mech. and Anal., **107**(4) (1989), 293–324.

[4] M.F. Bidaut-Véron and S. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math., **84** (2001), 1–49.

[5] A. Bouzelmate, A. Gmira and G. Reyes, *Radial selfsimilar solutions of a nonlinear Ornstein-Uhlenbeck equation*, Elect. J. Diff. Equ., **67** (2007), 1–21.

[6] A. Bouzelmate and A. Gmira, *On the radial solutions of a nonlinear singular elliptic equation*, Inter. J. Math. Anal., **9**(26) (2015), 1279–1297.

[7] L.A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure and Appl. Math., **42** (1989), 271–297.

[8] R.H. Fowler, *The form near infinity of real continuous solutions of a certain differential equation of second order*, Quart. J. Math., **45** (1914), 289–350.

[9] R.H. Fowler, *The solutions of Emden's and similar differential equation*, Monthly notices Roy. Astr. Soc., **91** (1920), 63–91.

[10] R.H. Fowler, *Further studies on Emden's and similar differential equation*, Quart. J. Math., **2** (1931), 259–288.

[11] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure and Appl. Math., **34** (1980), 525–598.

[12] M. Guedda and L. Véron, *Local and global properties of solutions of quasilinear equations*, J. Diff. Equ., **76** (1988), 159–189.

[13] P.L. Lions, *Isolated singularities in semilinear problems*, J. Diff. Equ., **38** (1980), 441–450.

[14] W.M. Ni and J. Serrin, *Nonexistence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure and Appl. Math., **38** (1986), 379–399.

[15] L. Véron, *Singularities of solutions of second order quasilinear equations*, Université François Rabelais-Tours, 1996.