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$\psi-$COUPLED FIXED POINT THEOREM VIA SIMULATION FUNCTIONS IN COMPLETE PARTIALLY ORDERED METRIC SPACE AND ITS APPLICATIONS

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Abstract. We proposed to give some new $\psi-$coupled fixed point theorems using simulation function coupled with other control functions in a complete partially ordered metric space which includes many related results. Further we prove the existence of solution of a fractional integral equation by using this fixed point theorem and explain it with the help of an example.

1. Introduction and preliminaries

The importance and usefulness of the metric fixed point can be seen in different areas e.g. approximation theory, optimization etc. The idea of a

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coupled fixed point was introduced and studied by Opoitsev [19, 20]. Later, it was studied by Guo and Lakhsmikanthan [10] and other researchers. Application of contraction type conditions on coupled fixed points was first initiated by Bhaskar and Lakhsmikanthan [6] which is further studied and extended to tripled fixed point theorems in ordered and cone metric spaces, for instance, see [1, 2, 3, 5, 7, 9, 8, 12, 13, 18, 22, 23, 24, 26] and references therein.

Ran and Reurings [25] studied the fixed point theorem in ordered metric spaces and applied it to linear and nonlinear matrix equations. In [6] Bhaskar and Lakhsmikanthan studied a coupled fixed point in ordered metric spaces and applied to the solvability for a periodic boundary value problem. Fan et al. [9] modified the notion of F-control function [15] and established some new coupled fixed point in metric spaces, and applied it to prove the existence of solution of integral equations.

We recall the following notations introduced by Bhaskar and Lakshminathan [6].

Definition 1.1. ([6]) Let $\left(\hat{X}, \preccurlyeq\right)$ be a partially ordered set and endow the product space $\hat{X} \times \hat{X}$ with the following partial order:

$$\text{for } (x,y), (u,v) \in \hat{X} \times \hat{X},\ (u,v) \preccurlyeq (x,y) \Longleftrightarrow x \succcurlyeq u,\ y \preccurlyeq v.$$

Definition 1.2. ([6]) Let $\left(\hat{X}, \preccurlyeq\right)$ be a partially ordered set and $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$. We say that $\hat{F}$ has the mixed monotone property if $\hat{F}(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in \hat{X}$,

$$x_1, x_2 \in \hat{X},\ x_1 \preccurlyeq x_2 \Rightarrow \hat{F}(x_1, y) \preccurlyeq \hat{F}(x_2, y)$$

and

$$y_1, y_2 \in \hat{X},\ y_1 \preccurlyeq y_2 \Rightarrow \hat{F}(x, y_1) \succcurlyeq \hat{F}(x, y_2).$$

Definition 1.3. ([6]) An element $(x,y) \in \hat{X} \times \hat{X}$ is called a coupled fixed point of the mapping $\hat{F}$ if $\hat{F}(x,y) = x$ and $\hat{F}(x,y) = y$, for all $x, y \in \hat{X}$.

In [17], Matthews introduced the notations of partial metric, quasi-metric spaces and presented the relation among the metric, partial metric and quasi-metric as follows:

If $\bar{d}$ is a quasi-metric on $\hat{X}$, then $d : \hat{X} \times \hat{X} \to [0, \infty)$ given by

$$d(x,y) = \bar{d}(x,y) + \bar{d}(y,x),\ (x,y) \in \hat{X} \times \hat{X}$$

is a metric on $\hat{X}$ and if d_p is a partial metric on X, then $d_1 : \hat{X} \times \hat{X} \to [0, \infty)$ given by

$$d_1(x,y) = 2d_p(x,y) - d_p(x,x) - d_p(y,y),\ (x,y) \in \hat{X} \times \hat{X}$$

is a metric on $\hat{X}$.

Lemma 1.4. ([17]) *Let $(\hat{X}, d_p)$ be a partial metric space. Then*

(1) *$\{x_n\}$ is a Cauchy sequence in $(\hat{X}, d_p)$ if and only if $\{x_n\}$ is a Cauchy sequence in the metric space $(\hat{X}, d_1)$.*
(2) *the partial metric space $(\hat{X}, d_p)$ is complete if and only if the metric space $(\hat{X}, d_1)$ is complete.*

Motivated from [15], Fan [9] introduced the following notion:

Definition 1.5. ([9]) Let $\hat{X}$ be a nonempty set, $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ be a mapping, $\psi : \hat{X} \times \hat{X} \to [0, \infty)$ be a given function, we say that a pair $(x, y) \in \hat{X} \times \hat{X}$ is a $\psi-$coupled fixed point of the mapping $\hat{F}$ if (x, y) is a coupled fixed point of the mapping $\hat{F}$ and satisfies $\psi(x, y) = 0$.

We denote by $C_{\hat{F}}$ the set of all coupled fixed points of the mapping $\hat{F}$, that is

$$C_{\hat{F}} = \left\{(x, y) \in \hat{X} \times \hat{X} : \hat{F}(x, y) = x,\ \hat{F}(y, x) = y\right\}$$

and

$$Z_\psi = \left\{(x, y) \in \hat{X} \times \hat{X} : \psi(x, y) = 0\right\}.$$

Hence, a $\psi-$coupled fixed point (x, y) is also represented as $(x, y) \in C_{\hat{F}} \cap Z_\psi$.

Definition 1.6. ([9]) Let τ be the set of functions $T : [0, \infty)^3 \to [0, \infty)$. For any $x, y, z, u, v \in [0, \infty)$, the function T satisfies the following conditions:

(i) $x \le T(x, y, z)$;
(ii) $T(x, y, z) = 0 \Longleftrightarrow x = y = z = 0$;
(iii) $T(., x, y) = T(., y, x)$ and $T(u + v, x, y) \le T(u, x, z) + T(v, z, y)$;
(iv) $T(x, ., .)$ is nondecreasing on $[0, \infty)$.

The function T is called a modified $F-$control function.

For example

(1) $T(x, y, z) = x + y + z$,
(2) $T(x, y, z) = x + \max\{y, z\}$,
(3) $T(x, y, z) = \max\{x, y, z\} + y + z$.

We shall use the following notations:

$$T^0(x, y) := 1_X, T^1(x, y) := T(x, y), ..., T^{n+1}(x, y) := T^n o\left(T(x, y), T(x, y)\right).$$

In order to establish our results, we need following related notions.

Definition 1.7. ([14]) A function $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a Jachymski function if

(1) $\hat{\phi}(0)=0$,
(2) for each $\epsilon>0$ there exists $\delta>0$ such that for $t>0$ with $\epsilon<t<\epsilon+\delta$, we have $\hat{\phi}(t)\leq\epsilon$.

Definition 1.8. ([11]) A function $\hat{\theta}:\mathbb{R}_+\to\mathbb{R}_+\to\mathbb{R}$ satisfying the following conditions:

(1) $\hat{\theta}(t,s)<s-t$ for all $t,s>0$,
(2) if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n=l>0$, $\lim_{n\to\infty} s_n=s>0$, then $\limsup_{n\to\infty}\hat{\theta}(t_n,s_n)<s-l$.

For example, let θ_1 and θ_2 be two altering distance functions such that $\theta_2(t)<t\leq\theta_1(t)$ for all $t>0$. Then $\hat{\theta}(t,s)=\theta_2(s)-\theta_1(t)$ for all $t,s\in[0,\infty)$ is a simulation function [4, 16, 21].

If we take $\theta_2(t)=\lambda t$ for all $t\geq 0,\ \lambda\in[0,1)$ and $\theta_1(t)=t$ then we obtain the simulation function $\hat{\theta}(t,s)=\lambda s-t$ for all $t,s\in[0,\infty)$.

In the literature, there are several contractions which produce unique fixed points and coupled fixed point and that are more general than Bhaskar and Lakshminathan [6]. Thus, it is interesting to see whether the underlying fractional integral equation can be handled with more general contraction which includes many other related contractions. Inspired by these observations, we wish to continue this study of solvability of a fractional integral equation by considering new contraction. We verify the application part by an example.

2. Main results

In this section, we prove new $\psi-$coupled fixed point theorems using simulation function coupled with other control functions in a complete partially ordered metric space.

Theorem 2.1. *Let $\left(\hat{X},d,\preccurlyeq\right)$ be a complete partially ordered metric space. Let $\hat{F}:\hat{X}\times\hat{X}\to\hat{X}$ be a mixed monotone mapping, $\hat{\phi}:\mathbb{R}_+\to\mathbb{R}_+$ be a Jachymski function and for any $s>0,\ \hat{\phi}(s)<s$ continuous and nondecreasing functions. Assume that*

(a) *$\hat{F}^2$ is continuous on $\hat{X}\times\hat{X}$,*
(b) *there exists $\hat{T}\in\tau$ such that for any $x,y,u,v\in\hat{X}$ with $u\preccurlyeq\hat{F}(x,y),\ v\succcurlyeq\hat{F}(y,x)$,*

$$\hat{\theta}\begin{pmatrix} \hat{T}\left(d\left(\hat{F}(x,y),\hat{F}(u,v)\right),\psi\left(\hat{F}(x,y),\hat{F}(y,x)\right),\psi\left(\hat{F}(u,v),\hat{F}(v,u)\right)\right), \\ \hat{\phi}\left(\hat{T}\left(d(x,u),\psi(x,y),\psi(u,v)\right)\right) \end{pmatrix}$$
$$\geq 0, \tag{2.1}$$

where $\psi:\hat{X}\times\hat{X}\to\mathbb{R}_+$ *is any given function and* $\hat{\theta}$ *is a simulation function.*

Then there exists a $\psi-$*coupled fixed point of* $\hat{F}$.

Proof. For any $(l,m)\in C_{\hat{F}}$, if we choose $x=u=l,\ y=v=m$ in (2.1) we get

$$\hat{\theta}\left(\hat{T}\left(0,\psi\left(l,m\right),\psi\left(l,m\right)\right),\hat{\phi}\left(\hat{T}\left(0,\psi(l,m),\psi(l,m)\right)\right)\right)\geq 0,$$

that is,

$$\hat{T}\left(0,\psi\left(l,m\right),\psi\left(l,m\right)\right)\leq\hat{\phi}\left(\hat{T}\left(0,\psi(l,m),\psi(l,m)\right)\right).$$

If $\psi(l,m)\neq 0$ then $\hat{T}\left(0,\psi\left(l,m\right),\psi\left(l,m\right)\right)>0$.

Now by using the property of $\hat{\phi}$ we get,

$$\hat{T}\left(0,\psi\left(l,m\right),\psi\left(l,m\right)\right)\leq\hat{\phi}\left(\hat{T}\left(0,\psi(l,m),\psi(l,m)\right)\right)<\hat{T}\left(0,\psi(l,m),\psi(l,m)\right),$$

a contradiction and hence $\psi(l,m)=0$. Therefore, $C_{\hat{F}}\subseteq Z_{\psi}$.

Now, we prove that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are Cauchy sequences, where

$$\alpha_n=\hat{F}\left(\alpha_{n-1},\beta_{n-1}\right)\preccurlyeq\hat{F}\left(\alpha_n,\beta_n\right)=\alpha_{n+1}$$

and

$$\beta_n=\hat{F}\left(\beta_{n-1},\alpha_{n-1}\right)\succcurlyeq\hat{F}\left(\beta_n,\alpha_n\right)=\beta_{n+1}$$

for $n=1,2,....$

We also have,

$$\alpha_n\preccurlyeq\hat{F}\left(\alpha_{n+1},\beta_{n+1}\right) \tag{2.2}$$

and

$$\beta_n\succcurlyeq\hat{F}\left(\beta_{n+1},\alpha_{n+1}\right) \tag{2.3}$$

for $n=1,2,...$, since $\hat{F}\left(\alpha_n,\beta_n\right)\preccurlyeq\hat{F}(\alpha_{n+1},\beta_{n+1}),\ \hat{F}\left(\beta_n,\alpha_n\right)\succcurlyeq\hat{F}(\beta_{n+1},\alpha_{n+1})$.

Let $b_{n+1}=\hat{T}\left(d(\alpha_{n+1},\alpha_n),\psi\left(\alpha_{n+1},\beta_{n+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)$. From (2.1)-(2.2) we get,

$$b_{n+1}=\hat{T}\begin{pmatrix} d(\hat{F}\left(\alpha_n,\beta_n\right),\hat{F}\left(\alpha_{n-1},\beta_{n-1}\right)), \\ \psi\left(\hat{F}\left(\alpha_n,\beta_n\right),\hat{F}\left(\beta_n,\alpha_n\right)\right),\psi\left(\hat{F}\left(\alpha_{n-1},\beta_{n-1}\right),\hat{F}\left(\beta_{n-1},\alpha_{n-1}\right)\right) \end{pmatrix}.$$

Again,

$$\hat{\theta}\left(\begin{array}{c}\hat{T}\left(\begin{array}{c} d\left(\hat{F}(\alpha_n,\beta_n),\hat{F}(\alpha_{n-1},\beta_{n-1})\right),\psi\left(\hat{F}(\alpha_n,\beta_n),\hat{F}(\beta_n,\alpha_n)\right),\\ \psi\left(\hat{F}(\alpha_{n-1},\beta_{n-1}),\hat{F}(\beta_{n-1},\alpha_{n-1})\right)\end{array}\right),\\ \hat{\phi}\left(\hat{T}\left(d(\alpha_n,\alpha_{n-1}),\psi\left(\alpha_n,\beta_n\right),\psi\left(\alpha_{n-1},\beta_{n-1}\right)\right)\right)\end{array}\right)$$
$$\geq 0$$

gives

$$\begin{aligned} & b_{n+1} \\ & = \hat{T}\left(\begin{array}{c} d\left(\hat{F}(\alpha_n,\beta_n),\hat{F}(\alpha_{n-1},\beta_{n-1})\right),\psi\left(\hat{F}(\alpha_n,\beta_n),\hat{F}(\beta_n,\alpha_n)\right),\\ \psi\left(\hat{F}(\alpha_{n-1},\beta_{n-1}),\hat{F}(\beta_{n-1},\alpha_{n-1})\right)\end{array}\right) \\ & \leq \hat{\phi}\left(\hat{T}\left(d(\alpha_n,\alpha_{n-1}),\psi\left(\alpha_n,\beta_n\right),\psi\left(\alpha_{n-1},\beta_{n-1}\right)\right)\right) \\ & = \hat{\phi}\left(b_n\right) \\ & < b_n \end{aligned}$$

for $n = 1, 2, ...$ Thus, $\{b_n\}$ is a strictly decreasing sequence in $\hat{X}$, therefore $\{b_n\}$ is a convergent sequence and converges to $b \in \mathbb{R}_+$ (say).

Now, we discuss the following two cases:

Case 1: If $b_n = 0$ for all $n \geq N_1$, where N_1 is a natural number then we have $b = 0$.

Case 2: If $\{b_n\}$ is a positive sequence in $\hat{X}$ then $\hat{\phi}(b_n) < b_n$ for all $n \in \mathbb{N}$.

Again, we have $b_{n+1} < b_n$ for all $n \in \mathbb{N}$ i.e., $\{b_n\}$ is strictly decreasing and $b < b_{n+1} < b_n$.

If $a > 0$, then from Definition 1.7 we can find $N_2 \in \mathbb{N}$, $\delta > 0$ such that $n \geq N_2$ we get $b < b_n < b + \delta$ gives $\hat{\phi}(b_n) \leq b$, a contradiction. Therefore we have $b = 0$, i.e., $\lim\limits_{n\to\infty} b_n = 0$.

Using $\epsilon - \delta$ definition of limit and Definition 1.7 for $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $\delta = \frac{\epsilon}{2}$ and $N_3 \in \mathbb{N}$ such that for $n \geq \max\{N_2, N_3\}$ we get $b_{n+k} < b_n < \frac{\epsilon}{2} < \frac{\epsilon}{2} + \delta$, that is,

$$\begin{aligned} & \hat{T}\left(d(\alpha_{n+k},\alpha_{n+k-1}),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_{n+k-1},\beta_{n+k-1}\right)\right) \\ & < \hat{T}\left(d(\alpha_n,\alpha_{n-1}),\psi\left(\alpha_n,\beta_n\right),\psi\left(\alpha_{n-1},\beta_{n-1}\right)\right) \\ & < \frac{\epsilon}{2} \\ & < \frac{\epsilon}{2} + \delta. \end{aligned}$$

Now we show that the following holds:

$$\hat{T}\left(d(\alpha_{n+k},\alpha_n),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)<\frac{\epsilon}{2}+\delta \tag{2.4}$$

for $k\in\mathbb{N}$.

It is obvious that (2.4) holds for $k=1$. Suppose that (2.4) holds for k and we shall try to prove (2.4) holds for $k+1$. We consider the following cases:

Case 3: Let $\hat{T}\left(d(\alpha_{n+k},\alpha_n),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)>\frac{\epsilon}{2}$.
We have from Definition 1.7,

$$\begin{aligned}
&\hat{T}\left(d(\alpha_{n+k+1},\alpha_{n+1}),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right)\right)\\
&\leq\hat{\phi}\Big(\hat{T}\left(d(\alpha_{n+k},\alpha_n),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)\Big)\\
&\leq\frac{\epsilon}{2}.
\end{aligned}$$

In addition we have

$$\begin{aligned}
&\hat{T}\left(d(\alpha_{n+k+1},\alpha_n),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\leq\hat{T}\left(d(\alpha_{n+k+1},\alpha_{n+1})+d(\alpha_{n+1},\alpha_n),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\leq\hat{T}\left(d(\alpha_{n+k+1},\alpha_{n+1}),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right)\right)\\
&\quad+\hat{T}\left(d(\alpha_{n+1},\alpha_n),\psi\left(\alpha_{n+1},\beta_{n+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\frac{\epsilon}{2}+\delta.
\end{aligned}$$

Case 4: Let $\hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)\leq\frac{\epsilon}{2}$.
If $\hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)=0$, then applying Definition 1.7 we get,

$$\hat{T}\left(d\left(\alpha_{n+k+1},\alpha_{n+1}\right),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right)\right)=0.$$

Again,

$$\begin{aligned}
&\hat{T}\left(d\left(\alpha_{n+k+1},\alpha_n\right),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\leq\hat{T}\left(d\left(\alpha_{n+k+1},\alpha_{n+1}\right),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right)\right)\\
&\quad+\hat{T}\left(d\left(\alpha_{n+1},\alpha_n\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\leq\frac{\epsilon}{2}\\
&<\frac{\epsilon}{2}+\delta.
\end{aligned}$$

If $\hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)>0$ then we get,

$$\begin{aligned}
&\hat{\phi}\left(\hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)\right)\\
&<\hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\le \frac{\epsilon}{2}.
\end{aligned}$$

Again,

$$\begin{aligned}
&\hat{T}\left(d\left(\alpha_{n+k+1},\alpha_n\right),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\le \hat{T}\left(d\left(\alpha_{n+k+1},\alpha_{n+1}\right)+d\left(\alpha_{n+1},\alpha_n\right),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&\le \hat{T}\left(d\left(\alpha_{n+k+1},\alpha_{n+1}\right),\psi\left(\alpha_{n+k+1},\beta_{n+k+1}\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right)\right)\\
&+\hat{T}\left(d\left(\alpha_{n+1},\alpha_n\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&<\hat{\phi}\left(\hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)\right)\\
&+\hat{T}\left(d\left(\alpha_{n+1},\alpha_n\right),\psi\left(\alpha_{n+1},\beta_{n+1}\right),\psi\left(\alpha_n,\beta_n\right)\right)\\
&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\frac{\epsilon}{2}+\delta.
\end{aligned}$$

Thus (2.4) holds for $k+1$. We also have

$$d\left(\alpha_{n+k},\alpha_n\right)\le \hat{T}\left(d\left(\alpha_{n+k},\alpha_n\right),\psi\left(\alpha_{n+k},\beta_{n+k}\right),\psi\left(\alpha_n,\beta_n\right)\right)<\frac{\epsilon}{2}+\delta=\epsilon.$$

This proves that $\{\alpha_n\}$ is a Cauchy sequence. Similarly, we can show that $\{\beta_n\}$ is a Cauchy sequence.

Now we prove that $\hat{F}$ has a ψ-coupled fixed point. Since $\left(\hat{X},d\right)$ is a complete metric space, therefore there exists $\bar{\alpha}$, $\bar{\beta}$ in $\hat{X}$ such that $\alpha_n\to\bar{\alpha}$ and $\beta_n\to\bar{\beta}$ as $n\to\infty$. We have

$$\begin{aligned}
&\alpha_{n+2}\\
&=\hat{F}\left(\alpha_{n+1},\beta_{n+1}\right)\\
&=\hat{F}\circ\left(\hat{F}\left(\alpha_n,\beta_n\right),\hat{F}\left(\beta_n,\alpha_n\right)\right)\\
&=\hat{F}^2\left(\alpha_n,\beta_n\right)\\
&\to\bar{\alpha}
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{n+2} \\
&= \hat{F}\left(\beta_{n+1}, \alpha_{n+1}\right) \\
&= \hat{F} \circ \left(\hat{F}\left(\beta_n, \alpha_n\right), \hat{F}\left(\alpha_n, \beta_n\right)\right) \\
&= \hat{F}^2\left(\beta_n, \alpha_n\right) \\
&\to \bar{\beta}
\end{aligned}$$

as $n \to \infty$. As $\hat{F}^2$ is continuous, we also have

$$\hat{F}^2\left(\alpha_n, \beta_n\right) \to \hat{F}^2\left(\bar{\alpha}, \bar{\beta}\right)$$

and

$$\hat{F}^2\left(\beta_n, \alpha_n\right) \to \hat{F}^2\left(\bar{\beta}, \bar{\alpha}\right).$$

By the uniqueness property of the limit, we get

$$\hat{F}^2\left(\bar{\alpha}, \bar{\beta}\right) = \bar{\alpha}$$

and

$$\hat{F}^2\left(\bar{\beta}, \bar{\alpha}\right) = \bar{\beta}.$$

If possible assume $\bar{\alpha} \neq \hat{F}\left(\bar{\alpha}, \bar{\beta}\right)$ and $\bar{\beta} \neq \hat{F}\left(\bar{\beta}, \bar{\alpha}\right)$ then we get,

$$\begin{aligned}
&\hat{T}\left(d\left(\bar{\alpha}, \hat{F}\left(\bar{\alpha}, \bar{\beta}\right)\right), \psi\left(\bar{\alpha}, \bar{\beta}\right), \psi\left(\hat{F}\left(\bar{\alpha}, \bar{\beta}\right), \hat{F}\left(\bar{\beta}, \bar{\alpha}\right)\right)\right) \\
&= \hat{T}\left(d\left(\hat{F}^2\left(\bar{\alpha}, \bar{\beta}\right), \hat{F}\left(\bar{\alpha}, \bar{\beta}\right)\right), \psi\left(\hat{F}^2\left(\bar{\alpha}, \bar{\beta}\right), \hat{F}^2\left(\bar{\beta}, \bar{\alpha}\right)\right), \psi\left(\hat{F}\left(\bar{\alpha}, \bar{\beta}\right), \hat{F}\left(\bar{\beta}, \bar{\alpha}\right)\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\theta}\Big(\hat{T}\left(d\left(\hat{F}^2(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\alpha}, \bar{\beta})\right), \psi\left(\hat{F}^2(\bar{\alpha}, \bar{\beta}), \hat{F}^2(\bar{\beta}, \bar{\alpha})\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right)\right), \\
&\hat{\phi}\left(\hat{T}\left(d\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \bar{\alpha}\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right), \psi\left(\bar{\alpha}, \bar{\beta}\right)\right)\right)\Big) \geq 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&\hat{T}\left(d\left(\hat{F}^2(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\alpha}, \bar{\beta})\right), \psi\left(\hat{F}^2(\bar{\alpha}, \bar{\beta}), \hat{F}^2(\bar{\beta}, \bar{\alpha})\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right)\right) \\
&\leq \hat{\phi}\left(\hat{T}\left(d\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \bar{\alpha}\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right), \psi\left(\bar{\alpha}, \bar{\beta}\right)\right)\right) \\
&< \hat{T}\left(d\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \bar{\alpha}\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right), \psi\left(\bar{\alpha}, \bar{\beta}\right)\right)
\end{aligned}$$

which gives

$$\begin{aligned}
&\hat{T}\left(d\left(\bar{\alpha}, \hat{F}(\bar{\alpha}, \bar{\beta})\right), \psi\left(\bar{\alpha}, \bar{\beta}\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right)\right) \\
&< \hat{T}\left(d\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \bar{\alpha}\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right), \psi\left(\bar{\alpha}, \bar{\beta}\right)\right)
\end{aligned}$$

a contradiction to Definition 1.6 (iii) as $d\left(\bar{\alpha}, \hat{F}\left(\bar{\alpha}, \bar{\beta}\right)\right) = d\left(\hat{F}\left(\bar{\alpha}, \bar{\beta}\right), \bar{\alpha}\right)$.

This gives

$$\hat{T}\left(d\left(\bar{\alpha}, \hat{F}(\bar{\alpha}, \bar{\beta})\right), \psi\left(\bar{\alpha}, \bar{\beta}\right), \psi\left(\hat{F}(\bar{\alpha}, \bar{\beta}), \hat{F}(\bar{\beta}, \bar{\alpha})\right)\right) = 0.$$

By Definition 1.6(ii), we get

$$d\left(\bar{\alpha}, \hat{F}(\bar{\alpha}, \bar{\beta})\right) = \psi\left(\bar{\alpha}, \bar{\beta}\right) = 0$$

implies $\bar{\alpha} = \hat{F}(\bar{\alpha}, \bar{\beta})$.

In a similar manner it can be shown that $\bar{\beta} = \hat{F}(\bar{\beta}, \bar{\alpha})$. Thus we conclude that $(\bar{\alpha}, \bar{\beta})$ is $\psi-$coupled fixed point of $\hat{F}$. □

Taking various concrete functions, we can get several classes of contractive conditions in a metric space such as:

Corollary 2.2. *Let $\left(\hat{X}, d, \preccurlyeq\right)$ be a complete partially ordered metric space. Let $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ be a mixed monotone mapping, $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a Jachymski function and for any $s > 0$, $\hat{\phi}(s) < s$ continuous and nondecreasing functions. Assume that*

(a) *$\hat{F}^2$ is continuous on $\hat{X} \times \hat{X}$,*
(b) *there exists $\hat{T} \in \tau$ such that for any $x, y, u, v \in \hat{X}$ with $u \preccurlyeq \hat{F}(x, y)$, $v \succcurlyeq \hat{F}(y, x)$ satisfying*

$$\begin{aligned}&\theta_1\Big(\hat{T}\left(d\left(\hat{F}(x, y), \hat{F}(u, v)\right), \psi\left(\hat{F}(x, y), \hat{F}(y, x)\right), \psi\left(\hat{F}(u, v), \hat{F}(v, u)\right)\right)\Big)\\ &\leq \theta_2\left(\hat{\phi}\left(\hat{T}\left(d(x, u), \psi(x, y), \psi(u, v)\right)\right)\right),\end{aligned}$$

where $\psi : \hat{X} \times \hat{X} \to \mathbb{R}_+$ is any given function and θ_1, θ_2 are two altering distance functions such that $\theta_2(t) < t \leq \theta_1(t)$ for all $t > 0$.

Then there exists a ψ-coupled fixed point of $\hat{F}$.

Proof. The result follows by taking $\hat{\theta}(t, s) = \theta_2(s) - \theta_1(t)$ in Theorem 2.1. □

Corollary 2.3. *Let $\left(\hat{X}, d, \preccurlyeq\right)$ be a complete partially ordered metric space. Let $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ be a mixed monotone mapping, $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a Jachymski function and for any $s > 0$, $\hat{\phi}(s) < s$ continuous and nondecreasing functions. Assume that*

(a) *$\hat{F}^2$ is continuous on $\hat{X} \times \hat{X}$,*

(b) *there exists $\hat{T} \in \tau$ such that for any $x, y, u, v \in \hat{X}$ with $u \preccurlyeq \hat{F}(x, y),\ v \succcurlyeq \hat{F}(y, x)$, satisfying*

$$\hat{T}\left(d\left(\hat{F}(x,y),\hat{F}(u,v)\right),\psi\left(\hat{F}(x,y),\hat{F}(y,x)\right),\psi\left(\hat{F}(u,v),\hat{F}(v,u)\right)\right)$$
$$\leq \lambda\hat{\phi}\left(\hat{T}\left(d(x,u),\psi(x,y),\psi(u,v)\right)\right),$$

where $\psi : \hat{X} \times \hat{X} \to \mathbb{R}_+$ is any given function and $\lambda \in [0,1)$.

Then there exists a ψ-coupled fixed point of $\hat{F}$.

Proof. The result follows by taking $\hat{\theta}(t,s) = \lambda s - t$ in Theorem 2.1. □

Corollary 2.4. *Let $\left(\hat{X}, d, \preccurlyeq\right)$ be a complete partially ordered metric space. Let $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ be a mixed monotone mapping, $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a Jachymski function and for any $s > 0,\ \hat{\phi}(s) < s$ continuous and nondecreasing functions. Assume that*

(a) *$\hat{F}^2$ is continuous on $\hat{X} \times \hat{X}$,*

(b) *for any $x, y, u, v \in \hat{X}$ with $u \preccurlyeq \hat{F}(x,y),\ v \succcurlyeq \hat{F}(y,x)$, satisfying*

$$d\left(\hat{F}(x,y),\hat{F}(u,v)\right)+\psi\left(\hat{F}(x,y),\hat{F}(y,x)\right)+\psi\left(\hat{F}(u,v),\hat{F}(v,u)\right)$$
$$\leq \lambda\hat{\phi}\left(d(x,u)+\psi(x,y)+\psi(u,v)\right),$$

where $\psi : \hat{X} \times \hat{X} \to \mathbb{R}_+$ is any given function and $\lambda \in [0,1)$.

Then there exists a ψ-coupled fixed point of $\hat{F}$.

Proof. The result follows by taking $\hat{T}(x,y,z) = x+y+z$ in Corollary 2.3. □

Corollary 2.5. *Let $\left(\hat{X}, d, \preccurlyeq\right)$ be a complete partially ordered metric space. Let $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ be a mixed monotone mapping, $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ be a Jachymski function and for any $s > 0,\ \hat{\phi}(s) < s$ continuous and nondecreasing functions. Assume that*

(a) *$\hat{F}^2$ is continuous on $\hat{X} \times \hat{X}$,*

(b) *for any $x, y, u, v \in \hat{X}$ with $u \preccurlyeq \hat{F}(x,y),\ v \succcurlyeq \hat{F}(y,x)$ satisfying*

$$d\left(\hat{F}(x,y),\hat{F}(u,v)\right) \leq \lambda\hat{\phi}\left(d(x,u)\right),$$

where $\psi : \hat{X} \times \hat{X} \to \mathbb{R}_+$ is any given function and $\lambda \in [0,1)$.

Then there exists a $\psi-$coupled fixed point of $\hat{F}$.

Proof. The result follows by taking $\psi(x,y) = \psi(u,v) = 0$ in Corollary 2.4. □

Corollary 2.6. *Let* $\left(\hat{X}, d, \preccurlyeq\right)$ *be a complete partially ordered metric space with*

$$d(x,y) = 2p(x,y) - p(x,x) - p(y,y), \ x, y \in \hat{X},$$

where p *is a partial metric on* $\hat{X}$. *Let* $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ *be a mixed monotone mapping,* $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ *be a Jachymski function and for any* $s > 0$, $\hat{\phi}(s) < s$, *continuous and nondecreasing functions. Assume that*

(a) $\hat{F}^2$ *is continuous on* $\hat{X} \times \hat{X}$,

(b) *for any* $x, y, u, v \in \hat{X}$ *with* $u \preccurlyeq \hat{F}(x,y)$, $v \succcurlyeq \hat{F}(y,x)$ *satisfying*

$$\begin{aligned} &2p\left(\hat{F}(x,y), \hat{F}(u,v)\right) - p\left(\hat{F}(x,y), \hat{F}(x,y)\right) - p\left(\hat{F}(u,v), \hat{F}(u,v)\right) \\ &\leq \lambda\hat{\phi}\left(2p(x,u) - p(x,x) - p(u,u)\right), \end{aligned}$$

where $\psi : \hat{X} \times \hat{X} \to \mathbb{R}_+$ *is any given function and* $\lambda \in [0,1)$.

Then there exists a ψ*-coupled fixed point of* $\hat{F}$.

Proof. The result follows by taking $d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ in Corollary 2.5. □

Corollary 2.7. *Let* $\left(\hat{X}, d, \preccurlyeq\right)$ *be a complete partially ordered metric space. Let* $\hat{F} : \hat{X} \times \hat{X} \to \hat{X}$ *be a mixed monotone mapping,* $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ *be a Jachymski function and for any* $s > 0$, $\hat{\phi}(s) < s$, *continuous and nondecreasing functions. Assume that*

(a) $\hat{F}^2$ *is continuous on* $\hat{X} \times \hat{X}$,

(b) *for any* $x, y, u, v \in \hat{X}$ *with* $u \preccurlyeq \hat{F}(x,y)$, $v \succcurlyeq \hat{F}(y,x)$, *satisfying*

$$\begin{aligned} &d\left(\hat{F}(x,y), \hat{F}(u,v)\right) + \max\left\{\psi\left(\hat{F}(x,y), \hat{F}(y,x)\right), \psi\left(\hat{F}(u,v), \hat{F}(v,u)\right)\right\} \\ &\leq \lambda\hat{\phi}\left(d(x,u) + \max\left\{\psi(x,y), \psi(u,v)\right\}\right), \end{aligned}$$

where $\psi : \hat{X} \times \hat{X} \to \mathbb{R}_+$ *is any given function and* $\lambda \in [0,1)$.

Then there exists a ψ*-coupled fixed point of* $\hat{F}$.

Proof. The result follows by taking $\hat{T}(x,y,z) = x + \max\{y, z\}$ in Corollary 2.3. □

3. Application on fractional integral equation

Consider the following fractional integral equation:

$$p(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-w)^{1-\alpha}} \left[\left(A\left(t, w, p(w)\right) - B\left(t, w, p(w)\right)\right)\right] dw \tag{3.1}$$

for $0 \leq t \leq d$. The functions $A : [0,d] \times [0,d] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $B : [0,d] \times [0,d] \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions.

Assume

(1) $A(.,.,p(w))$ and $B(.,.,p(w))$ are both nondecreasing on $\mathbb{R}_+$.
(2) for any $p, \hat{p}, v, \hat{v} \in \mathbb{R}_+$ with

$$\begin{aligned} v(t) &\preccurlyeq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-w)^{1-\alpha}} \left[\left(A\left(t,w,p(w)\right) - B\left(t,w,\hat{p}(w)\right) \right) \right] dw, \\ \hat{v}(t) &\succcurlyeq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-w)^{1-\alpha}} \left[\left(A\left(t,w,\hat{p}(w)\right) - B\left(t,w,p(w)\right) \right) \right] dw \end{aligned}$$

satisfying

$$\begin{aligned} &\left| A\left(t,w,p(w)\right) - B\left(t,w,\hat{p}(w)\right) - A\left(t,w,v(w)\right) + B\left(t,w,\hat{v}(w)\right) \right| \\ &\quad \leq \frac{\lambda \hat{\phi}\left(|p(w)-v(w)|\right)}{d^{\alpha}} \end{aligned}$$

with $\frac{\lambda}{\Gamma(\alpha+1)} \in [0,1)$, where $\hat{\phi} : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing Jachymski function satisfying that for $s > 0$, $\hat{\phi}(s) < s$.

Theorem 3.1. *Under the assumption* (1)-(2)*, equation* (3.1) *has a unique solution.*

Proof. Let $X := C[0,d]$ be a real valued continuous functions on $[0,d]$. Let $\delta : X \times X \to \mathbb{R}_+$ be a metric on X defined by $\delta(p_1,p_2) = \max|p_1(t) - p_2(t)|$ for $p_1, p_2 \in$ and $z \in [0,d]$. Then it is trivial that (X,δ) is a metric space. We denote (X,δ) with the partial order $\preccurlyeq$ as

$$p(t), \hat{p}(t) \in X,\ p(t) \preccurlyeq \hat{p}(t) \Leftrightarrow \left(p(t), \hat{p}(t)\right) \preccurlyeq \left(\hat{p}(t), p(t)\right).$$

Define $L : X \times X \to X$ by

$$L(p,\hat{p})(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-w)^{1-\alpha}} \left[\left(A\left(t,w,p(w)\right) - B\left(t,w,\hat{p}(w)\right) \right) \right] dw. \tag{3.2}$$

By condition (1), we obtain that D is mixed monotone and continuous mapping. By (3.1) and (3.2) we obtain

$$\left| L\left(p,\hat{p}\right)(t) - L\left(v,\hat{v}\right)(t) \right| = \left| \begin{array}{c} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-w)^{1-\alpha}} [\left(A\left(t,w,p(w)\right) - B\left(t,w,\hat{p}(w)\right)\right) \\ - \left(A\left(t,w,v(w)\right) - B\left(t,w,\hat{v}(w)\right)\right)] dw \end{array} \right|$$

$$\begin{aligned}
&\leq \lambda \int_0^t \frac{\hat{\phi}\left(|p(w)-v(w|\right)}{(t-w)^{1-\alpha} d^\alpha \Gamma(\alpha)} dw \\
&= \frac{\lambda \hat{\phi}\left(|p(w)-v(w|\right)}{d^\alpha \Gamma(\alpha)} \int_0^t \frac{dw}{(t-w)^{1-\alpha}} \\
&\leq \frac{\lambda \hat{\phi}\left(|p(w)-v(w|\right)}{d^\alpha \Gamma(\alpha)} . \frac{d^\alpha}{\alpha} \\
&= \frac{\lambda}{\Gamma(\alpha+1)} \hat{\phi}\left(|p(w)-v(w|\right) \\
&\leq \frac{\lambda}{\Gamma(\alpha+1)} \hat{\phi}\left(\delta(p,v)\right).
\end{aligned}$$

All assumptions of Corollary 2.5 are satisfied, so operator L has a $\psi-$coupled fixed point on $X \times X$ hence we conclude that the equation 3.1 has a unique solution. □

Example 3.2. Consider the following example

$$p(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{t^2 w^{\frac{1}{2}} p(w) - tw^2}{8(t-w)^{\frac{1}{2}}} dw, \tag{3.3}$$

where $t,\ w \in [0,1],\ p \in \mathbb{R}_+$ and the partially ordered set $(C[0,1], \delta, \leq)$. Here we have

$$d = 1,\ \alpha = \frac{1}{2},\ A(t,w,p) = \frac{t^2 w^{\frac{1}{2}} p}{8},\ B(t,w,p) = \frac{tw^2}{8}.$$

We observe that both $A(.,.,p),\ B(.,.,p)$ are nondecreasing on $\mathbb{R}_+$ and continuous on $[0,1] \times [0,1] \times \mathbb{R}_+$.

Let any $p,\ \hat{p},\ v,\ \hat{v}$ with

$$v(t) \leq \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{t^2 w^{\frac{1}{2}} p(w) - tw^2}{8(t-w)^{\frac{1}{2}}} dw$$

and

$$\hat{v}(t) \geq \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{t^2 w^{\frac{1}{2}} \hat{p}(w) - tw^2}{8(t-w)^{\frac{1}{2}}} dw.$$

Now,

$$\begin{aligned}
&|A(t,w,p(w)) - B(t,w,\hat{p}(w)) - A(t,w,v(w)) + B(t,w,\hat{v}(w))| \\
&= \left| \frac{1}{8} t^2 w^{\frac{1}{2}} p(w) - \frac{1}{8} tw^2 - \frac{1}{8} t^2 w^{\frac{1}{2}} v(w) + \frac{1}{8} tw^2 \right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left|p(w)-v(w)\right| \\
&= \frac{1}{4}\hat{\phi}\left(\left|p(w)-v(w)\right|\right) \\
&= \lambda\hat{\phi}\left(\left|p(w)-v(w)\right|\right),
\end{aligned}
$$

where $\hat{\phi}(t)=\frac{t}{2}$ and $\lambda=\frac{1}{4}$ such that $\frac{\lambda}{\Gamma(\frac{3}{2})}<1$.

Since all assumptions of Theorem 3.1 are satisfied by equation (3.3), equation (3.3) has a unique solution in $C[0,1]$.

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References

[1] M. Abbas, M.A. Khan and S. Radenović, *Common coupled fixed point theorems in cone metric spaces for w-compatible mappings*, Appl. Math. Comput., **217** (2010), 195–202.

[2] H. Aydi, E. Karapinar and B. Samet, *Remarks on some recent fixed point theorems*, Fixed Point Theory Appl., **2012** (76) (2012).

[3] A. Amini Harandi, *Coupled and tripled fixed point theory in partially ordered metric spaces with applications to initial value problem*, Math. Comput. Model., **57** (2011), 2343-2348, http://dx.doi.org/10.1016/ j.mcm.2011.12.006.

[4] A. Bataihah, W. Shatanawi and A. Tallafha, *Fixed point results with simulation functions,* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 13-23, doi.org/10.22771/nfaa.2020.25.01.02.

[5] V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal.: TMA, **74** (2011), 7347–7355.

[6] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal.: TMA, **65**(7) (2006), 1379–1393.

[7] M. Borcut, *Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces*, Appl. Math. Comput., **218** (2012), 7339–7346.

[8] D. Dorić, Z. Kadelburg, S. Radenović and P. Kumam, *A note on fixed point results without monotone property in partially ordered metric spaces*, RACSAM., **108** (2014), 503–510.

[9] Y. Fan, C. Zhu and Z. Wu, *Some ϕ-coupled fixed point results via modified F-control function's concept in metric spaces and its applications*, J. Comput. Appl. Math., **349** (2019), 70–81.

[10] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal.: TMA, **11** (1987), 623–632.

[11] B. Hazarika, H.M. Srivastava, R. Arab and M. Rabbani, *Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution*, Appl. Math. Comput., **360** (2019), 131–146.

[12] B. Hazarika, R. Arab and P. Kumam, *Coupled fixed point theorems in partially ordered metric spaces via mixed g-monotone property*, J. Fixed Point Theory Appl., **29**(1) (2019), 1–19.
[13] Ding Hui-Sheng, L. Li and S. Radenović, *Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces*, Fixed Point Theory Appl., **2012**(96) (2012).
[14] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., **194** (1995), 293–303.
[15] M. Jleli, B. Samet and C. Vetro, *Fixed point theory in partial metric spaces via ϕ-fixed point's concept in metric spaces*, J. Inequal. Appl., **2014**(426) (2014).
[16] A. Karami, S. Sedghi, N. Shobe and H.G. Hyun, *Fixed point results on s-metric space via simulation functions* , Nonlinear Funct. Anal. Appl., **24**(4) (2019), 665-674, doi.org/10.22771/nfaa.2019.24.04.02
[17] S. Matthews, *Partial metric topology*, Ann. New York Acad. Sci, (1994), 183–197.
[18] H. K. Nashine and I. Altun, *New fixed point results for maps satisfying implicit relations on ordered metric spaces and application*, Appl. Math. Comput., **240** (2014), 259–272.
[19] V. I. Opoitsev, *Heterogenic and combined-concave operators*, Syber. Math. J., **16** (1975), 781–792 (in Russian).
[20] V. I. Opoitsev, *Dynamics of collective behavior. III. Heterogenic system*, Avtomat. i Telemekh., **36** (1975), 124–138 (in Russian).
[21] A. Padcharoen and J.K. Kim, *Berinde type resukts via simulation functions in metric spaces*, Nonlinear Funct. Anal. Appl., **25**(3) (2020), 665-674, doi.org/10.22771/nfaa.2020.25.03.07.
[22] S. Radenović, *Remarks on some coupled coincidence point results in partially ordered metric spaces*, Arab J Math Sci., **20**(1) (2014), 29-39.
[23] S. Radenović, *Some coupled coincidence points results of monotone mappings in partially ordered metric spaces*, Int. J. Nonlinear Anal. Appl., **5**(2) (2014), 174–184.
[24] S. Radenović, *Coupled fixed point theorems for monotone mappings in partially ordered metric spaces*, Kragujev. J. Math., **38**(2) (2014), 249–257.
[25] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443.
[26] A. Razani and V. Parvaneh, *Coupled coincidence point results for (ψ, α, β)-weak contractions in partially ordered metric spaces*, J. Appl. Math., **2012** (2012). Article ID 496103.