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SOME RADIUS RESULTS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE SRIVASTAVA-ATTIYA OPERATOR

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Abstract. The main object of the present paper is to investigate some radius results of the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n (|z| < 1)$ with $|a_n| \leq n$ for all $n \in \mathbb{N}$. Some applications for certain operator defined through convolution are also considered.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in the unit disk $\mathbb{U}$.

Here and in the following, let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive integers, and non-positive integers, respectively. Further we let $\mathcal{P}(\alpha)$

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denote the subclass of $\mathcal{S}$ consisting of those functions satisfying

$$\mathrm{Re}\{f'(z)\} > \alpha \qquad (0 \leq \alpha < 1). \tag{1.2}$$

Here the class $\mathcal{P}(\alpha)$ was introduced by Hallenbeck [6]. In particular, the class $\mathcal{P}(0) = \mathcal{P}$ was investigated by MacGregor [11]. Here we recall Noshiro-Warschawski theorem (see, e.g., [3, Theorem 2.16]): *If f is analytic in a convex domain D and* $\mathrm{Re}\{f'(z)\} > 0$ *there, then f is univalent in D.*

In view of this theorem, we find that the class of functions in $\mathcal{A}$ which satisfy (1.2) become a subclass of $\mathcal{S}$. In the regard, the condition "$\mathcal{P}(\alpha)$ is a subclass of $\mathcal{S}$" is redundant.

Let α_j $(j = 1, \cdots, p)$ and β_j $(j = 1, \cdots, q)$ be complex numbers with $\beta_j \notin \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}$. Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q+1), \end{aligned} \tag{1.3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$\begin{aligned} (\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \qquad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ &= \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \end{aligned}$$

it being understood conventionally that $(0)_0 = 1$. We note that the series ${}_pF_q(z)$ in (1.3) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathbb{U}$ if $p = q+1$. The condition $p \leq q+1$ stated with the definition (1.3) will be assumed to hold true throughout this paper.

For functions $f_j(z) \in \mathcal{A}$, given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j}\ z^n \qquad (j = 1, 2),$$

we define the *Hadamard product (or convolution)* of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1}\ a_{n,2}\ z^n = (f_2 * f_1)(z) \qquad (z \in \mathbb{U}).$$

With a view to defining the Srivastava-Attiya transform, we recall here a general Hurwitz-Lerch zeta function, which is defined in [15] by the following series:

$$\Phi(z,\lambda,\delta) := \frac{1}{\delta^\lambda} + \sum_{k=1}^{\infty} \frac{z^k}{(k+\delta)^\lambda}$$

$$(\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-; \lambda \in \mathbb{C} \text{ when } z \in \mathcal{U}; \mathrm{Re}(\lambda) > 1 \text{ when } |z| = 1).$$

For further interesting properties and characteristics of the Hurwitz-Lerch Zeta function and other related special functions, see for example [4], [10] and [16].

In the literature on special functions one often finds applications of operational techniques involving various integral transforms (see [7], [13] and [17]). Recently, Srivastava and Attiya [14] have introduced the linear operator $\mathcal{L}_{\lambda,\delta} : \mathcal{A} \to \mathcal{A}$, defined in terms of the Hadamard product by

$$\mathcal{L}_{\lambda,\delta} f(z) = \mathcal{G}_{\lambda,\delta}(z) * f(z) \qquad (\delta \in \mathbb{C} \setminus \mathbb{Z}_0^-; \lambda \in \mathbb{C}; z \in \mathbb{U}), \tag{1.4}$$

where

$$\mathcal{G}_{\lambda,\delta}(z) = (1+\delta)^\lambda \left[\Phi(z,\lambda,\delta) - \delta^{-\lambda}\right] \qquad (z \in \mathbb{U}). \tag{1.5}$$

The operator $\mathcal{L}_{\lambda,\delta}$ is now popularly known in the literature as the Srivastava-Attiya operator. Various class-mapping properties of the operator $\mathcal{L}_{\lambda,\delta}$ (and its variants) are discussed in the recent works of Srivastava and Attiya [14], Liu [9], Murugusundaramoorthy [12], Yuan and Liu [18] and others.

It is easy to observe from (1.1) and (1.4) that

$$\mathcal{L}_{\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+\delta}{k+\delta}\right)^\lambda a_k z^k. \tag{1.6}$$

We note that:

(i) $\mathcal{L}_{0,\delta} f(z) = f(z)$;
(ii) $\mathcal{L}_{1,0} f(z) = \mathcal{L} f(z) = \int_0^z \frac{f(t)}{t} dt \quad (f \in \mathcal{A})$ (see Alexander [1]);
(iii) $\mathcal{L}_{m,1} f(z) = \mathcal{I}^m f(z) \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,3,\cdots\})$ (see Flett [5]);
(iv) $\mathcal{L}_{\gamma,1} f(z) = \mathcal{Q}^\gamma f(z) \quad (\gamma > 0)$ (see Jung et al. [7]);
(v) $\mathcal{L}_{m,0} f(z) = \mathcal{L}^m f(z) \quad (m \in \mathbb{N}_0)$ (see Sălăgean [13]).

In this article, we determine some radii for normalized analytic functions $f(z)$ defined by (1.1) with $|a_n| \leq n$ $(n = 2,3,4\cdots)$ in order to satisfy such inequalities as $\mathrm{Re}\{f'(z)\} > \alpha$ $(0 \leq \alpha < 1)$. Moreover, some interesting consequences for certain operator defined through convolution are also pointed out.

2. Main results

We begin by proving the following theorem.

Theorem 2.1. *Let the function* $f(z)$ *be defined by* (1.1) *and suppose that* $|a_n| \leq n$ *for* $n = 2, 3, 4, \cdots$. *Then we have*

$$f(z) \in \mathcal{P}(\alpha) \quad for\ |z| < r_0, \tag{2.1}$$

where $0 \leq \alpha < 1$ *and* r_0 *is the unique solution of the equation*

$$(2-\alpha)r^3 - 3(2-\alpha)r^2 + (7-3\alpha)r - 1 + \alpha = 0.$$

Furthermore, the radius r_0 *is sharp.*

Proof. From the hypothesis, we obtain

$$\mathrm{Re}\left\{f'(z)\right\} \geq 1 - \sum_{n=2}^{\infty} n^2 r^{n-1} \qquad (|z| = r).$$

Since

$$\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \quad \text{and} \quad \frac{1+z}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^{n-1}, \tag{2.2}$$

we readily have

$$\begin{aligned} \mathrm{Re}\{f'(z) - \alpha\} &\geq 2 - \frac{1+r}{(1-r)^3} - \alpha \\ &= \frac{-(2-\alpha)r^3 + 3(2-\alpha)r^2 - (7-3\alpha)r + 1 - \alpha}{(1-r)^3}. \end{aligned} \tag{2.3}$$

If we put $g(r) = (2-\alpha)r^3 - 3(2-\alpha)r^2 + (7-3\alpha)r - 1 + \alpha$, it is easily seen that there exist the unique solution r_0 of the equation $g(r) = 0$. Since $g(0) = -1+\alpha$ and $g(1) = 2$,

$$r_0 \in (0, 1).$$

By applying (2.2) and (2.3), we see that

$$\mathrm{Re}\{f'(z) - \alpha\} > 0 \quad \text{for } |z| < r_0,$$

which implies that $f(z) \in \mathcal{P}(\alpha)$ for $|z| < r_0$.

Also, if we take

$$f(z) = 2z - \frac{z}{(1-z)^2} \qquad (|z| < 1),$$

then from (2.2) we have

$$\mathrm{Re}\{f'(z) - \alpha\} \geq 0 \qquad |z| \leq r_0 < 1,$$

and $f'(r_0) = \alpha$. Hence the radius r_0 is sharp. This evidently completes the proof of Theorem 2.1. □

By the theorem of de Branges [2], we can restate Theorem 2.1 as follows:

Corollary 2.2. *If $f(z) \in \mathcal{S}$, then*

$$f(z) \in \mathcal{P}(\alpha) \qquad (|z| < r_0),$$

where $0 \leq \alpha < 1$ and r_0 is defined in Theorem 2.1.

Corollary 2.3. *Let the function $f(z)$ be defined by (1.1) and suppose that $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$. If we put*

$$F(z) = \frac{1}{z}\int_0^z f(t)dt, \tag{2.4}$$

then

$$F(z) \in \mathcal{P}(\alpha) \quad for \ \ |z| < r_1,$$

where $0 \leq \alpha < 1$ and r_1 is the positive root of the equation

$$\frac{1}{(1-r)^2} - \frac{1}{r(1-r)} - \frac{1}{r^2}\log(1-r) + \alpha = 1.$$

Proof. From (2.4) we obtain

$$\begin{aligned} \operatorname{Re}\{F'(z) - \alpha\} &\geq \frac{1}{2} - \sum_{n=2}^{\infty} \frac{n^2}{n+1} r^{n-1} - \alpha \quad (|z| = r) \\ &= 1 - \alpha - \sum_{n=1}^{\infty} \frac{n^2}{n+1} r^{n-1}. \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2}{n+1} z^{n-1} &= \left(\frac{1}{z}\int_0^z \frac{t}{(1-t)^2}dt\right)' \\ &= \frac{1}{(1-z)^2} - \frac{1}{z(1-z)} - \frac{1}{z^2}\log(1-z), \end{aligned}$$

it follows from (2.5) that $\operatorname{Re}\{F'(z) - \alpha\} > 0$ for $|z| < r_1$. Hence we conclude that $F(z) \in \mathcal{P}(\alpha)$ for $|z| < r_1$, which completes the proof of Corollary 2.3. □

Next, we can prove the following theorem.

Theorem 2.4. *Let the function $f(z)$ be defined by (1.1) and suppose that $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$. Then*

$$\mathcal{L}_{m,\delta} f(z) \in \mathcal{P}(\alpha) \quad for \ \ |z| < r_2, \tag{2.6}$$

where $m \in \mathbb{N}$, $\delta > 0$, $0 \leq \alpha < 1$ and r_2 is the solution of the equation

$${}_{m+2}F_{m+1}(1+\delta, \cdots, 1+\delta, 2, 2; 1, 2+\delta, \cdots, 2+\delta; r) + \alpha = 2.$$

Proof. From (1.6) we have

$$\begin{aligned}
&\operatorname{Re}\left\{(\mathcal{L}_{m,\delta}f(z))' - \alpha\right\} \\
&\geq 1 - \sum_{n=2}^{\infty}\left(\frac{1+\delta}{n+\delta}\right)^m n^2 r^{n-1} - \alpha \quad (|z| = r) \\
&= 2 - \sum_{n=0}^{\infty}\left(\frac{1+\delta}{n+1+\delta}\right)^m (n+1)^2 r^n - \alpha \\
&= 2 - \alpha - {}_{m+2}F_{m+1}(1+\delta, \cdots, 1+\delta, 2, 2; 1, 2+\delta, \cdots, 2+\delta; r).
\end{aligned}$$

Hence, if $|z| < r_2$, then we obtain (2.6). This completes the proof of Theorem 2.4. □

Taking $\alpha = 0$ in Theorem 2.4, we have the following.

Corollary 2.5. *Let the function $f(z)$ be defined by* (1.1) *and suppose that $|a_n| \leq n$ for $n = 2, 3, 4, \cdots$. Then*

$$\mathcal{L}_{m,\delta}f(z) \in \mathcal{P} \quad for\ |z| < r_3,$$

where $m \in \mathbb{N}$, $\delta > 0$ and r_3 is the solution of the equation

$${}_{m+2}F_{m+1}(1+\delta, \cdots, 1+\delta, 2, 2; 1, 2+\delta, \cdots, 2+\delta; r) = 2.$$

Remark 2.6. If we put $\delta = 1$ in Corollary 2.5, then it would immediately yield the result due to Kim and Nunokawa [8, Theorem 3].

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