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SOME L^q INEQUALITIES FOR POLYNOMIAL

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Abstract. Let $p(z)$be a polynomial of degree n. Then Bernstein's inequality [12,18] is

$$\max_{|z|=1}|p^{'}(z)| \leq n\max_{|z|=1}|(z)|.$$

For $q > 0$, we denote

$$\|p\|_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},$$

and a well-known fact from analysis [17] gives

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} = \max_{|z|=1}|p(z)|.$$

Above Bernstein's inequality was extended by Zygmund [19] into L^q norm by proving

$$\|p^{'}\|_q \leq n\|p\|_q, \ \ q \geq 1.$$

Let $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having no zero in $|z| < k, k \geq 1$. Then for $0 < r \leq R \leq k$, Aziz and Zargar [4] proved

$$\max_{|z|=R}|p^{'}(z)| \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\max_{|z|=r}|p(z)|.$$

In this paper, we obtain the L^q version of the above inequality for $q > 0$. Further, we extend a result of Aziz and Shah [3] into L^q analogue for $q > 0$. Our results not only extend some known polynomial inequalities, but also reduce to some interesting results as particular cases.

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1. Introduction

Let $p(z)$ be a polynomial of degree n. We define

$$\|p\|_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}},\ \ 0<q<\infty. \tag{1.1}$$

If we let $q\to\infty$ in the above equality and make use of the well-known fact from analysis [17] that

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}=\max_{|z|=1}|p(z)|,$$

we can suitably denote

$$\|p\|_\infty=\max_{|z|=1}|p(z)|.$$

Similarly, we can define $\|p\|_0=\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}log|p(e^{i\theta})|d\theta\right\}$ and show that $\lim\limits_{q\to0^+}\|p\|_q=\|p\|_0$. It would be of further interest that by taking limits as $\lim\limits_{q\to0^+}$ that the stated result holding for $q>0$, holds for $q=0$ as well.

For $r>0$, we denote by $M(p,r)=\max\limits_{|z|=r}|p(z)|$.

A famous result due to Bernstein [12 or also see 18] states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty\le n\|p\|_\infty. \tag{1.2}$$

Inequality (1.2) can be obtained by letting $q\to\infty$ in the inequality

$$\|p'\|_q\le n\|p\|_q,\ \ q>0. \tag{1.3}$$

Inequality (1.3) for $q\ge1$ is due to Zygmund [19]. Arestov [1] proved that (1.3) remains valid for $0<q<1$ as well. If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then inequality (1.2) and (1.3) can be respectively improved by

$$\|p'\|_\infty\le\frac{n}{2}\|p\|_\infty \tag{1.4}$$

and

$$\|p'\|_q\le\frac{n}{\|1+z\|_q}\|p\|_q,\ \ q>0. \tag{1.5}$$

Inequality (1.4) was conjectured by Erdös and later verified by Lax [10], whereas, inequality (1.5) was proved by de-Bruijn [6] for $q\ge1$. Rahman and Schmeisser [15] showed that (1.5) remains true for $0<q<1$.

As a generalization of (1.4), Malik [11] proved that if $p(z)$ does not vanish in $|z|<k$, $k\ge1$, then

$$\|p'\|_\infty\le\frac{n}{1+k}\|p\|_\infty. \tag{1.6}$$

Under the same hypotheses of the polynomial $p(z)$, Govil and Rahman [9] extended inequality (1.6) to L^q norm by showing that

$$\|p'\|_q \leq \frac{n}{\|k+z\|_q}\|p\|_q, \quad q \geq 1. \tag{1.7}$$

It was shown by Gardner and Weems [8] and independently by Rather [16] that (1.7) also holds for $0 < q < 1$. Further, as a generalization of (1.6) Bidkham and Dewan [5] proved that if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then

$$\|p'(rz)\|_\infty \leq \frac{n(r+k)^{n-1}}{(1+k)^n}\|p\|_\infty \quad \text{for } 1 \leq r \leq k. \tag{1.8}$$

As a generalization of (1.8), Aziz and Zargar [4] proved the following theorem.

Theorem 1.1. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, has no zeros in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq k$,*

$$\|p'(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\|p(rz)\|_\infty. \tag{1.9}$$

The result is best possible and equality in (1.9) holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Further, as an improvement and generalization of (1.8), Aziz and Shah [3] proved the following theorem.

Theorem 1.2. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in the disk $|z| < k$, $k \geq 0$, then for $0 < r \leq R \leq k$,*

$$\|p'(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\left\{\|p(rz)\|_\infty - m\right\}, \tag{1.10}$$

where

$$m = \min_{|z|=k} |p(z)|.$$

The result is best possible and equality in (1.10) holds for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

Lemma 2.1. ([14]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \le \mu \le n,$ *having no zeros in* $|z| < k,\ k \ge 1,$ *then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Lemma 2.2. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \le \mu \le n,$ *has no zeros in* $|z| < k,$ $k > 0$ *then for* $0 < r \le R \le k,$

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt \tag{2.2}$$

and

$$M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt \le \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} M(p,r). \tag{2.3}$$

Proof. Since $p(z) \neq 0$ for $|z| < 1$, $p(tz) \neq 0$ for $|z| < \frac{1}{t}$ and so by Lemma 2.1,

$$\max_{|z|=1} t|p'(tz)| \le \frac{n}{k^\mu + t^{-\mu}} \max |p(tz)|,$$

this gives

$$M(p',t) \le \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t). \tag{2.4}$$

Now, for $0 \le r < R \le k$ and $\theta \in [0, 2\pi)$ we have,

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})|dt,$$

which implies

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R M(p',t)dt. \tag{2.5}$$

Using (2.4) in (2.5) we obtain

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt. \tag{2.6}$$

Which completes the first inequality (2.2).

Further, taking maximum over θ in (2.6), we have

$$M(p,R) \le M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt. \tag{2.7}$$

Now let us denote the right hand side of inequality (2.7) by $\phi(R)$. Then

$$\phi^{'}(R) \leq \frac{nR^{\mu-1}}{R^\mu + k^\mu}\phi(R)$$

or

$$\phi^{'}(R) - \frac{nR^{\mu-1}}{R^\mu + k^\mu}\phi(R) \leq 0. \tag{2.8}$$

Multiplying both side of (2.8) by $(R^\mu + k^\mu)^{\frac{-n}{\mu}}$, we obtain

$$\frac{d}{dR}(R^\mu + k^\mu)^{\frac{-n}{\mu}}\phi(R) \leq 0,$$

which implies that $(R^\mu + k^\mu)^{\frac{-n}{\mu}}\phi(R)$ is a nonincreasing function of R in $(0, k]$. Thus for $0 < r \leq R \leq k$,

$$\phi(r) \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}} \phi(R). \tag{2.9}$$

Since $\phi(r) = M(p, r)$ and using the value of $\phi(R)$ in (2.9), we get

$$M(p,r) \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}} \left[M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu}M(p,t)dt\right].$$

This completes the proof of inequality (2.3). □

Lemma 2.3. ([14]) *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then on $|z| = 1$*

$$|q^{'}(z)| \geq k^\mu |p^{'}(z)|, \ \text{where } q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}. \tag{2.10}$$

Lemma 2.4. ([2]) *If $p(z)$ is a polynomial of degree n and $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, then for each α, $0 \leq \alpha < 2\pi$ and $q > 0$,*

$$\int_0^{2\pi}\int_0^{2\pi} |q^{'}(e^{i\theta}) + e^{i\alpha}p^{'}(e^{i\theta})|^q d\theta d\alpha \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta. \tag{2.11}$$

Lemma 2.5. ([7]) *Let z be complex and independent of α, where α is real, then for $q > 0$,*

$$\int_0^{2\pi} |1 + ze^{i\alpha}|^q d\alpha = \int_0^{2\pi} |e^{i\alpha} + |z||^q d\alpha. \tag{2.12}$$

Lemma 2.6. ([13]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *which does not vanish in* $|z| < k$, $k \ge 1$, *then*

$$\max_{|z|=1} |p^{'}(z)| \le \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{2.13}$$

Lemma 2.7. *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zeros in* $|z| < k$, $k > 0$, *then for* $0 < r \le R \le k$,

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + n \left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt \right] \tag{2.14}$$

and

$$\begin{aligned} M(p,r) &+ \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} m dt \\ &\le \left[M(p,r) - \left\{ 1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \right\} m \right] \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}}, \end{aligned} \tag{2.15}$$

where $m = \min_{|z|=k} |p(z)|$.

Proof. By hypotheses, $p(z)$ has no zeros in $|z| < k$, therefore, the polynomial $F(z) = p(tz)$ has no zeros in $|z| < \frac{k}{t}$, $\frac{k}{t} \ge 1$, where $0 < t \le k$. Since $\frac{k}{t} \ge 1$, by applying Lemma 2.6 to $F(z)$, it follows that

$$\max_{|z|=1} |F^{'}(z)| \le \frac{n}{1 + \frac{k^\mu}{t^\mu}} \left\{ \max_{|z|=1} |F(z)| - \min_{|z|=\frac{k}{t}} |F(z)| \right\},$$

this gives

$$\max_{|z|=t} |p^{'}(z)| \le \frac{nt^{\mu-1}}{t^\mu + k^\mu} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{2.16}$$

Now, for $0 < r \le R \le k$, and $0 \le \theta < 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| = \left| \int_r^R e^{i\theta} p^{'}(te^{i\theta}) dt \right| \le \int_r^R |p^{'}(te^{i\theta})| dt,$$

from which it follows

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R |p^{'}(te^{i\theta})| dt,$$

which implies

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + \int_r^R M(p^{'}, t) dt. \tag{2.17}$$

Using (2.16) in (2.17), we obtain

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt\right], \quad (2.18)$$

which is the first inequality of Lemma 2.7.

Further, taking maximum over θ in (2.18), we have

$$M(p,R) \leq M(p,r) + n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt\right]. \quad (2.19)$$

Now let us denote the right hand side of inequality (2.19) by $\phi(R)$. Then

$$\phi^{'}(R) = \frac{nR^{\mu-1}}{R^\mu + k^\mu} M(p,R) - \frac{nR^{\mu-1}}{R^\mu + k^\mu} m. \quad (2.20)$$

Using $M(p,R) \leq \phi(R)$, equality (2.20) can be written as

$$\phi^{'}(R) - \frac{nR^{\mu-1}}{R^\mu + k^\mu}\{\phi(R) - m\} \leq 0. \quad (2.21)$$

Multiplying both sides of (2.21) by $(R^\mu + k^\mu)^{\frac{-n}{\mu}}$, we get

$$\phi^{'}(R)(R^\mu + k^\mu)^{\frac{-n}{\mu}} - n(\phi(R) - m)(R^\mu + k^\mu)^{\frac{-n}{\mu}-1} R^{\mu-1} \leq 0,$$

which implies

$$\frac{d}{dR}\left\{(\phi(R) - m)(R^\mu + k^\mu)^{\frac{-n}{\mu}}\right\} \leq 0. \quad (2.22)$$

From (2.22) we conclude that the function,

$$\{\phi(R) - m\}(R^\mu + k^\mu)^{\frac{-n}{\mu}}$$

is a nonincreasing function of R in $(0,k]$. Hence for $0 < r \leq R \leq k$,

$$\phi(r) \geq \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}} \phi(R) + \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}\right\} m. \quad (2.23)$$

Since $\phi(r) = M(p,r)$ and using the value of $\phi(R)$ in (2.23), we get

$$M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} m dt$$
$$\leq \left[M(p,r) - \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}\right\} m\right]\left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}.$$

This completes the proof of inequality (2.15) of Lemma 2.7. $\square$

3. Main results

In this paper, first we extend Theorem 1.1 into L^q norm with the value of $k>0$ instead of just $k\geq 1$. More precisely, we prove:

Theorem 3.1. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu, 1\leq\mu\leq n,$ *has no zeros in* $|z|<k$, $k>0$, *then for* $0<r\leq R\leq k$, *and* $q>0$,

$$\left\{\int_0^{2\pi}|p^{'}(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\leq\frac{n}{R}T_q\left\{\int_0^{2\pi}\left||p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^\mu+k^\mu}M(p,t)dt\right|^q d\theta\right\}^{\frac{1}{q}}, \tag{3.1}$$

where

$$T_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^\mu+e^{i\alpha}\right|^q d\alpha\right\}^{\frac{-1}{q}}.$$

Letting $q\to\infty$ *on both sides of* (3.1), *we obtain inequality* (1.9) *of Theorem* 1.1.

Proof. By hypothesis the polynomial $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$ has no zero in $|z|<k$, $k>0$, therefore the polynomial $P(z)=p(Rz)$ has no zero in $|z|<\frac{k}{R}$, $\frac{k}{R}\geq 1$. By applying Lemma 2.3 to the polynomial $P(z)$, we have

$$A|P^{'}(z)|\leq|Q^{'}(z)| \text{ for } |z|=1, \text{ where } Q(z)=z^n\overline{P(\frac{1}{\overline{z}})} \tag{3.2}$$

and

$$A=\left(\frac{k}{R}\right)^\mu\geq 1. \tag{3.3}$$

We can easily verify that for every real number α and $R\geq r\geq 1$,

$$|R+e^{i\alpha}|\geq|r+e^{i\alpha}|.$$

This implies for each $q>0$,

$$\int_0^{2\pi}|R+e^{i\alpha}|^q d\alpha\geq\int_0^{2\pi}|r+e^{i\alpha}|^q d\alpha. \tag{3.4}$$

For point $e^{i\theta}$, $0\leq\theta\leq 2\pi$, for which $P^{'}(e^{i\theta})\neq 0$, we denote

$$R=\left|\frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})}\right|,$$

and $r=A$, then from (3.2) and (3.3), $R\geq r\geq 1$.

Now, for each $q>0$, by Lemma 2.5 and (3.4), we have

$$\begin{aligned}\int_0^{2\pi}|Q'(e^{i\theta})+e^{i\alpha}P'(e^{i\theta})|^q d\alpha &= |P'(e^{i\theta})|^q\int_0^{2\pi}\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}+e^{i\alpha}\right|^q d\alpha\\ &= |P'(e^{i\theta})|^q\int_0^{2\pi}\left|\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right|+e^{i\alpha}\right|^q d\alpha\\ &\geq |P'(e^{i\theta})|^q\int_0^{2\pi}|A+e^{i\alpha}|^q d\alpha. \qquad (3.5)\end{aligned}$$

For points $e^{i\theta}$, $0\leq\theta<2\pi$, for which $P'(e^{i\theta})=0$, inequality (3.5) trivially holds.

Now using (3.5) in Lemma 2.4, we obtain for each $q>0$,

$$\int_0^{2\pi}|A+e^{i\alpha}|^q d\alpha\int_0^{2\pi}|P'(e^{i\theta})|^q d\theta\leq 2\pi n^q\int_0^{2\pi}|P(e^{i\theta})|^q d\theta. \qquad (3.6)$$

Since $P(z)=p(Rz)$,

$$P'(z)=Rp'(Rz).$$

Thus inequality (3.6) can be written as

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^\mu+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp'(Re^{i\theta})|^q d\theta\leq 2\pi n^q\int_0^{2\pi}|p(Re^{i\theta})|^q d\theta. \qquad (3.7)$$

Now applying inequality (2.2) of Lemma 2.2 in (3.7), we have

$$\begin{aligned}&\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^\mu+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp'(Re^{i\theta})|^q d\theta\\ &\leq 2\pi n^q\int_0^{2\pi}\left\{|p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^\mu+k^\mu}M(p,t)dt\right\}^q d\theta \qquad (3.8)\end{aligned}$$

or equivalently

$$\begin{aligned}&\left\{\int_0^{2\pi}|Rp'(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}\\ &\leq\frac{nT_q}{R}\left[\int_0^{2\pi}\left\{|p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^\mu+k^\mu}M(p,t)dt\right\}^q d\theta\right]^{\frac{1}{q}}.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Both the ordinary inequalities (1.9) and (1.10) of Theorems 1.1 and 1.2 are best possible for the polynomial $p(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ. It may be expected that inequality (3.1) of Theorem 3.1 is sharp for this polynomial. We discuss it as follows:

For $p(z)=(z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ, inequality (3.1) of Theorem 3.1 equivalently takes

$$\left\{\int_0^{2\pi}\left|k^\mu+R^\mu e^{i\alpha}\right|^q d\alpha\right\}\left\{\int_0^{2\pi}\left|R^\mu e^{i\theta\mu}+k^\mu\right|^{q(\frac{n}{\mu}-1)} d\theta\right\}$$
$$\leq\left[\int_0^{2\pi}\left\{\left|r^\mu e^{i\theta\mu}+k^\mu\right|^{\frac{n}{\mu}}+(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}\right\}^q d\theta\right]. \tag{3.9}$$

In particular, if we set $k=R=r$, and $\mu=1$, then inequality (3.9) assumes

$$\left\{\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha\right\}\left\{\int_0^{2\pi}\left|e^{i\theta}+1\right|^{q(n-1)} d\theta\right\}\leq\left\{\int_0^{2\pi}\left|e^{i\theta}+1\right|^{nq} d\theta\right\}. \tag{3.10}$$

Now, we have for $p>-1$,

$$\int_0^{\frac{\pi}{2}}\cos^p\theta d\theta=\frac{\sqrt{\pi}\Gamma(\frac{p}{2}+\frac{1}{2})}{2\Gamma(\frac{p}{2}+1)}. \tag{3.11}$$

For $q>0$, by a simple calculation, we have

$$\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha=2^{q+2}\int_0^{\frac{\pi}{2}}\cos^q\alpha d\alpha,$$

which on using (3.11) gives

$$\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha=2^{q+1}\sqrt{\pi}\frac{\Gamma(\frac{q}{2}+\frac{1}{2})}{\Gamma(\frac{q}{2}+1)}. \tag{3.12}$$

Applying equality (3.12) in inequality (3.10), we have

$$2^{q(n-1)+1}\sqrt{\pi}\frac{\Gamma(\frac{q(n-1)}{2}+\frac{1}{2})}{\Gamma(\frac{q(n-1)}{2}+1)}\times 2^{q+1}\sqrt{\pi}\frac{\Gamma(\frac{q}{2}+\frac{1}{2})}{\Gamma(\frac{q}{2}+1)}\leq 2^{nq+1}\sqrt{\pi}\frac{\Gamma(\frac{nq}{2}+\frac{1}{2})}{\Gamma(\frac{nq}{2}+1)},$$

that is,

$$2\sqrt{\pi}\frac{\Gamma(\frac{q(n-1)}{2}+\frac{1}{2})}{\Gamma(\frac{q(n-1)}{2}+1)}\times\frac{\Gamma(\frac{q}{2}+\frac{1}{2})}{\Gamma(\frac{q}{2}+1)}\leq\frac{\Gamma(\frac{nq}{2}+\frac{1}{2})}{\Gamma(\frac{nq}{2}+1)}. \tag{3.13}$$

Further, when $n=3, q=4$, inequality (3.13) becomes

$$2\sqrt{\pi}\frac{\Gamma(4+\frac{1}{2})}{\Gamma(5)}\times\frac{\Gamma(2+\frac{1}{2})}{\Gamma(3)}\leq\frac{\Gamma(6+\frac{1}{2})}{\Gamma(7)}$$

which on simplification gives

$$10\pi\leq 11,$$

which is absurd. This shows that inequality (3.1) of Theorem 3.1 is not sharp.

Remark 3.3. Using $|p(re^{i\theta})|\leq M(p,r)$ in Theorem 3.1, we have the following result.

Corollary 3.4. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \le \mu \le n$, *has no zeros in* $|z| < k$, $k > 0$ *then for* $0 < r \le R \le k$, *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \le \frac{n}{R} T_q \left| M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt \right|, \tag{3.14}$$

where T_q *is as defined in Theorem* 3.1.

Further, using inequality (2.3) of Lemma 2.2 in the inequality (3.11) of Corollary 3.4, we have the L^q version of Theorem 1.1, which has some interesting consequences as discussed below.

Corollary 3.5. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \le \mu \le n$, *has no zeros in* $|z| < k$, $k > 0$ *then for* $0 < r \le R \le k$ *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \le \frac{n}{R} T_q \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu} \right)^{\frac{n}{\mu}} M(p,r), \tag{3.15}$$

where T_q *is as defined in Theorem* 3.1.

Letting $q \to \infty$ in inequality (3.15) we get inequality (1.9) of Theorem 1.1. Further, if we let $\mu = 1$ and $r = 1$ in Corollary 3.5, it matches the L^q analogue of inequality (1.8) proved by Bidkham and Dewan [5].

In addition to the above, when $\mu = 1 = R = r$, Corollary 3.5 gives inequality (1.7) which is the L^q inequality of the famous inequality (1.6) due to Malik [11].

Further, we extend inequality (1.10) of Theorem 1.2 due to Aziz and Shah [3] to integral mean inequality. In fact, we obtain the following theorem.

Theorem 3.6. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \le \mu \le n$, *is a polynomial of degree* n *having no zeros in the disk* $|z| < k$, $k > 0$, *then for* $0 < r \le R \le k$, *and* $q > 0$,

$$\left\{ \int_0^{2\pi} |p^{'}(Re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \tag{3.16}$$

$$\le \frac{n}{R} T_q \left\{ \int_0^{2\pi} \left| |p(re^{i\theta})| + n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} mdt \right] - m \right|^q d\theta \right\}^{\frac{1}{q}},$$

where T_q *is as in Theorem* 3.1 *and* $m = \min_{|z|=k} |p(z)|$. *Letting* $q \to \infty$ *on both sides of* (3.16), *we obtain inequality* (1.10) *of Theorem* 1.2.

Proof. Since the polynomial $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ has no zero in $|z| < k$, $k > 0$, the polynomial $p(Rz)$ has no zero in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$.

Take $P(z) = p(Rz) + \alpha m$ where $|\alpha| < 1$ and $m = \min_{|z|=k} |p(z)$. By applying Lemma 2.3 to the polynomial $P(z)$, we have

$$A|P^{'}(z)| \leq |Q^{'}(z)| \text{ for } |z| = 1, \text{ where } Q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \tag{3.17}$$

and

$$A = \left(\frac{k}{R}\right)^{\mu} \geq 1. \tag{3.18}$$

We can easily verify that for every real number α and $R \geq r \geq 1$,

$$|R + e^{i\alpha}| \geq |r + e^{i\alpha}|.$$

This implies for each $q > 0$,

$$\int_0^{2\pi} |R + e^{i\alpha}|^q d\alpha \geq \int_0^{2\pi} |r + e^{i\alpha}|^q d\alpha. \tag{3.19}$$

For point $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, for which $P^{'}(e^{i\theta}) \neq 0$, we denote

$$R = \left|\frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})}\right|$$

and $r = A$, then from (3.17) and (3.18), $R \geq r \geq 1$.

Now, for each $q > 0$, by Lemma 2.5 and (3.19), we have

$$\begin{aligned}\int_0^{2\pi} |Q^{'}(e^{i\theta}) + e^{i\alpha} P^{'}(e^{i\theta})|^q d\alpha &= |P^{'}(e^{i\theta})|^q \int_0^{2\pi} \left|\frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} + e^{i\alpha}\right|^q d\alpha \\ &= |P^{'}(e^{i\theta})|^q \int_0^{2\pi} \left|\left|\frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})}\right| + e^{i\alpha}\right|^q d\alpha \\ &\geq |P^{'}(e^{i\theta})|^q \int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha. \end{aligned} \tag{3.20}$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P^{'}(e^{i\theta}) = 0$, inequality (3.20) trivially holds.

Now using (3.20) in Lemma 2.4, we obtain for each $q > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \int_0^{2\pi} |P^{'}(e^{i\theta})|^q d\theta \leq 2\pi n^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{3.21}$$

Since $P(z) = p(Rz) + \alpha m$,

$$P^{'}(z) = R(p^{'}(Rz)).$$

Thus inequality (3.21) can be written as

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^{\mu}+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta$$
$$\leq 2\pi n^q\int_0^{2\pi}|p(Re^{i\theta})+\alpha m|^q d\theta. \tag{3.22}$$

Now, in $|p(Re^{i\theta})+\alpha m|$, if we choose suitable argument of α, we have

$$|p(Re^{i\theta})+\alpha m|=|p(Re^{i\theta})|-|\alpha|m.$$

By letting $|\alpha|\to 1$, we obtain

$$|p(Re^{i\theta})+\alpha m|=|p(Re^{i\theta})|-m. \tag{3.23}$$

Using (3.23) in (3.22), we have

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^{\mu}+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta$$
$$\leq 2\pi n^q\int_0^{2\pi}||p(Re^{i\theta})|-m|^q d\theta. \tag{3.24}$$

Now applying inequality (2.14) of Lemma 2.7 in (3.24), we have

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^{\mu}+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta$$
$$\leq 2\pi n^q\int_0^{2\pi}\left||p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}M(p,t)dt-\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}mdt-m\right|^q d\theta. \tag{3.25}$$

or equivalently

$$\left\{\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}$$
$$\leq\frac{nT_q}{R}\left\{\int_0^{2\pi}\left||p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}M(p,t)dt-\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}mdt-m\right|^q d\theta\right\}^{\frac{1}{q}}.$$

This completes the proof of Theorem 3.6. □

Remark 3.7. As is noticed earlier that inequality (1.10) of Theorem 1.2 is sharp for $p(z)=(z^{\mu}+k^{\mu})^{\frac{n}{\mu}}$ where n is a multiple of μ, we examine the sharpness of inequality (3.16) of Theorem 3.6 for this polynomial.

It is obvious that for $p(z) = (z^{\mu} + k^{\mu})^{\frac{n}{\mu}}$ where n is a multiple of μ,

$$m = \min_{|z|=k} |p(z)| = 0$$

and hence by Remark 3.2, inequality (3.16) of Theorem 3.6 is not sharp.

Remark 3.8. Using $|p(re^{i\theta})| \leq M(p,r)$ in Theorem 3.6, we have the following result.

Corollary 3.9. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zeros in the disk* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$ *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \leq \frac{n}{R} T_q \left\{ \left| M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}} m dt - m \right| \right\}, \quad (3.26)$$

where T_q *is as in Theorem* 3.1 *and* $m = \min_{|z|=k} |p(z)|$.

Further, using inequality (2.15) of Lemma 2.7 in the inequality (3.26) of Corollary 3.9, we have, the L^q version of Theorem 1.2:

Corollary 3.10. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zeros in the disk* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$ *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \leq \frac{n}{R} T_q \left(\frac{k^{\mu}+R^{\mu}}{k^{\mu}+r^{\mu}} \right)^{\frac{n}{\mu}} \{M(p,r) - m\}. \quad (3.27)$$

where T_q *is as in Theorem* 3.1 *and* $m = \min_{|z|=k} |p(z)|$.

Letting $q \to \infty$ in inequality (3.27), we get inequality (1.10) of Theorem 1.2. Further, if we let $\mu = 1$ and $r = 1$ in Corollary 3.10, we obtain an improvement in L^q version of inequality (1.8) proved by Bidkham and Dewan [5].

Also, when $\mu = 1 = R = r$ in Corollary 3.10, it gives an improvement of L^q inequality (1.7) due to Govil and Rahman [9] of the ordinary inequality (1.6) proved by Malik [11].

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References

[1] V.V. Arestov, *On inequalities for trigonometric polynomials and their derivative*, IZV. Akad. Nauk. SSSR. Ser. Math., **45** (1981), 3-22.

[2] A. Aziz and N.A. Rather, *Some Zygmund type L^p inequalities for polynomials,* J. Math. Anal. Appl., **289** (2004), 14-29.

[3] A. Aziz and W.M. Shah, *Inequalities for a polynomial and its derivative,* Math. Ineq. Appl., **3** (2004), 379-391.

[4] A. Aziz and B.A. Zargar, *Inequalities for a polynomial and its derivative,* Math. Ineq. Appl., **4** (1998), 543-550.

[5] M. Bidkham and K.K. Dewan, *Inequalities for polynomial and its derivative,* J. Math. Anal. Appl., **166** (1992), 319-324.

[6] N.G. de-Bruijn, *Inequalities concerning polynomials in the complex domain,* Ned-erl. Akad. Wetench. Proc. Ser. A, **50** (1947), 1265-1272; Indag. Math 9(1947), 591-598.

[7] R.B. Gardner and N.K. Govil, *An L^p inequality for a polynomial and its derivative,* J. Math. Anal. Appl., **194** (1995), 720-726.

[8] R.B. Gardner and A. Weems, *A Bernstein-type of L^p inequality for a certain class of polynomials,* J. Math. Anal. Appl., **219** (1998), 472-478.

[9] N.K. Govil and Q.I. Rahman, *Functions of exponential type not vanishing in a half-plane and related polynomials,* Trans. Amer. Math. Soc., **137** (1969), 501-517.

[10] P.D. Lax, *Proof of a conjecture of P.Erdös on the derivative of a polynomial,* Bull. Amer. Math. Soc., **50** (1944), 509-513.

[11] M.A. Malik, *On the derivative of a polynomial,* J. London Math. Soc., **1** (1969), 57-60. https://doi.org/10.1112/jlms/s2-1.1.57

[12] G.V. Milovanović, D.S. Mitrinović and Th. Rassias, *Topics in polynomials: External problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.

[13] M.S. Pukhta, *Ph.D Thesis, Submitted to the Jamia Millia Islamia,* New Delhi, 1995.

[14] M.A. QAZI, *On the maximum modulus of polynomials,* Proc. Amer. Math. Soc., **115** (1992), 337-343.

[15] Q.I. Rahman and G. Schmeisser, *L^p inequalities for polynomials,* J. Approx. Theory, **53** (1988), 26-32.

[16] N.A. Rather, *Extremal properties and location of the zeros of polynomials,* Ph.D Thesis, University of Kashmir, 1998.

[17] W. Rudin, *Real and Complex Analysis,* Tata McGraw-Hill Publishing Company(reprinted in India), 1977.

[18] A.C. Schaeffer,*Inequalities of A. Markoff and S. N. Bernstein for polynomials and related functions,* Bull. Amer. Math. Soc., **47** (1941), 565-579.

[19] A. Zygmund, *A remark on conjugate series,* Proc. London Math. Soc., **34** (1932), 392-400.