Nonlinear Functional Analysis and Applications
Vol. 15, No. 4 (2010), pp. 657-668

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

STRONG MAXIMUM PRINCIPLES FOR INFINITE IMPLICIT SYSTEMS OF PARABOLIC DIFFERENTIAL FUNCTIONAL INEQUALITIES WITH NONLOCAL INEQUALITIES WITH SUMS

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Abstract. The aim of the paper is to give strong maximum principles for infinite implicit systems of parabolic differential-functional inequalities with nonlocal inequalities together with sums in relatively arbitrary $(n+1)-$dimensional time space sets more general than the cylindrical domain.

1. Introduction

In this paper we consider infinite implicit diagonal systems of nonlinear parabolic functional differential inequalities of the form

$$\begin{aligned} &F_i(x,t,u^i(x,t),u^i_t(x,t),u^i_x(x,t),u^i_{xx}(x,t),u) \\ &\geq F_i(x,t,v^i(x,t),v^i_t(x,t),v^i_x(x,t),v^i_{xx}(x,t),v) \ (i\in\mathbb{N}) \end{aligned} \tag{1.1}$$

for $(x,t)=(x_1,...,x_n,t)\in D$, where $D\subset\mathbb{R}^n\times(t_0,t_0+T]$ is one of five relatively arbitrary sets more general than the cylindrical domain $D_0\times(t_0,t_0+T]\subset\mathbb{R}^{n+1}$.

The symbol $w(=u$ or $v)$ denotes the mapping

$$w:\mathbb{N}\times\tilde{D}\ni(i,x,t)\to w^i(x,t)\in\mathbb{R},$$

[0]Received September 30, 2009. Revised October 28, 2009.
[0]2000 Mathematics Subject Classicification: 35B50, 35R45, 35K20, 35K99.
[0]Keywords: Strong maximum principles, infinite implicit systems, parabolic problems, nonlocal inequalities.

where $\tilde{D}$ is an arbitrary set such that

$$\bar{D} \subset \tilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T];$$

F_i $(i \in \mathbb{N})$ are functionals of $w, w^i_x = grad_x w^i(x,t)$ and $w^i_{xx}(x,t)$ denote the matrices of second order derivatives with respect to x of $w^i(x,t)$ $(i \in \mathbb{N})$. We give a theorem on strong maximum principles for problems with inequalities (1.1) and with nonlocal inequalities together with sums.

Results obtained are based on those by Besala [1], Brandys [2], [3], Chabrowski [9], Redheffer and Walter [15], Szarski [16]-[18], Walter [19], Yoshida [21] and the author [6]-[8].

Some infinite and finite, parabolic and hyperbolic systems were considered by Brzychczy [4], [5], Jaruszewska-Walczak [10], Kamont [11], [12], Lakshmikantham and Leela [13], Pudełko [14] and Zabawa [22].

Infinite parabolic systems have physical application. For this purpose please see the publication [20] by Wrzosek.

2. Preliminaries

The notation, definitions and assumptions given in this section are applied throughout the paper.

We shall use the following notation: $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_- = (-\infty, 0], \mathbb{N} = \{1, 2, ..., \}, x = (x_1, .., x_n) \in \mathbb{R}^n$ $(n \in \mathbb{N})$.

By ℓ^∞ we denote the Banach space of real sequences $\xi = (\xi^1, \xi^2, ...)$ such that

$$\sup \{| \xi^j |: j = 1, 2, ...\} < \infty$$

and

$$\| \xi \|_{\ell^\infty} = \sup \{| \xi^j |: j = 1, 2, ...\}.$$

For $\xi = (\xi^1, \xi^2, ...), \eta = (\eta^1, \eta^2, ...) \in \ell^\infty$ we write $\xi \leq \eta$ in the sense $\xi^j \leq \eta^i$ $(i \in \mathbb{N})$.

Let t_0 be a real finite number and let $T \in (0, \infty)$.

A set $D \subset \{(x,t) : x \in \mathbb{R}^n,\ t_0 < t \leq t_0 + T\}$ is called *a set of type* (P) if

(a) The projection of the interior of set D on the t-axis is the interval $(t_0, t_0 + T)$.
(b) For every $(\tilde{x}, \tilde{t}) \in D$, there exists a positive number $r = r(\tilde{x}, \tilde{t})$ such that

$$\{(x,t) : \sum_{i=1}^n (x_i - \tilde{x}_i)^2 + (t - \tilde{t})^2 < r, t < \tilde{t}\} \subset D.$$

(c) All the boundary points $(\tilde{x}, \tilde{t})$ of D for which there is a positive number $r = r(\tilde{x}, \tilde{t})$ such that

$$\{(x,t) : \sum_{i=1}^{n} (x_i - \tilde{x}_i)^2 + (t - \tilde{t})^2 < r,\ t \le \tilde{t}\} \subset D.$$

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} int\{x \in \mathbb{R}^n : (x, t_0) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in \mathbb{R}^n : (x,t) \in D\} & \text{for } t \ne t_0, \end{cases}$$

$$\sigma_t = \begin{cases} int[\bar{D} \cap (\mathbb{R}^n \times \{t_0\})] & \text{for } t = t_0, \\ D \cap (\mathbb{R}^n \times \{t\}) & \text{for } t \ne t_0. \end{cases}$$

Let $\tilde{D}$ be an arbitrary set such that

$$\bar{D} \subset \tilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T].$$

We introduce the following sets:

$$\partial_p D := \tilde{D} \setminus D \text{ and } \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{x}, \tilde{t}) \in D$, we denote by $S^-(\tilde{x}, \tilde{t})$ the set of points $(x,t) \in D$, that can be joined to $(\tilde{x}, \tilde{t})$ by a polygonal line contained in D along which the t-coordinate is weakly increasing from (x,t) to $(\tilde{x}, \tilde{t})$.

Let $Z_\infty(\tilde{D})$ denote the linear space of mappings

$$w : \mathbb{N} \times \tilde{D} \ni (i, x, t) \longrightarrow w^i(x,t) \in \mathbb{R},$$

where functions

$$w^i : \tilde{D} \ni (x,t) \longrightarrow w^i(x,t) \in \mathbb{R}$$

are continuous in $\bar{D}$ and

$$\sup\{|w^i(x,t)| : (x,t) \in \tilde{D},\ i \in \mathbb{N}\} < \infty.$$

For $w, \tilde{w} \in Z_\infty(\tilde{D})$ we write $w \le \tilde{w}$ in the sense $w^i \le \tilde{w}^i$ $(i \in \mathbb{N})$.

In the set of mappings w belonging to $Z_\infty(\tilde{D})$ we define the functional $[\cdot]_{t,\infty}$ by the formula

$$[w]_{t,\infty} = \sup\{0, w^i(x, \tilde{t}) : (x, \tilde{t}) \in \tilde{D},\ \tilde{t} \le t,\ i \in \mathbb{N}\},$$

where $t \le t_0 + T$.

By $Z_\infty^{2,1}(\tilde{D})$ we denote the linear subspace of $Z_\infty(\tilde{D})$. A mapping w belongs to $Z_\infty^{2,1}(\tilde{D})$ if $w_t^i, w_x^i = (w_{x_1}^i, ..., w_{x_n}^i)$, $w_{xx}^i = [w_{x_jx_k}^i]_{n\times n}$ $(i \in \mathbb{N})$ are continuous in D.

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symmetric matrices $r = [r_{jk}]_{n\times n}$.

For $r \in M_{n\times n}(\mathbb{R})$ we write $r \geq 0$ if $\sum_{j,k=1}^{n} r_{jk}\lambda_j\lambda_k \geq 0$ for all $(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$.

Let the mappings

$$F_i : D \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times M_{n\times n}(\mathbb{R}) \times Z_\infty(\tilde{D}) \ni (x,t,z,p,q,r,w) \longrightarrow F_i(x,t,z,p,q,r,w) \in \mathbb{R} \ \ (i \in \mathbb{N})$$

be given and let for an arbitrary function $w \in Z_\infty^{2,1}(\tilde{D})$

$$F_i[x,t,w] := F_i(x,t,w^i(x,t), w_t^i(x,t), w_x^i(x,t), w_{xx}^i(x,t), w), \quad (x,t) \in D \ \ (i \in \mathbb{N}).$$

For a given subset $E \subset D$ and a given mapping $w \in Z_\infty^{2,1}(\tilde{D})$, and a fixed index $i \in \mathbb{N}$ the function F_i is called *uniformly parabolic* with respect to w in E if there is a constant $\kappa > 0$ (depending on E) such that for any two matrices $r = [r_{jk}] \in M_{n\times n}(\mathbb{R}), \tilde{r} = [\tilde{r}_{jk}] \in M_{n\times n}(\mathbb{R})$ with $r \leq \tilde{r}$ and for $(x,t) \in E$, we have

$$\begin{aligned} &F_i(x,t,w^i(x,t), w_t^i(x,t), w_x^i(x,t), \tilde{r}, w) \\ &-F_i(x,t,w^i(x,t), w_t^i(x,t), w_x^i(x,t), r, w) \\ &\geq \kappa \sum_{j=1}^{n} (\tilde{r}_{jj} - r_{jj}). \end{aligned} \tag{2.1}$$

If (2.1) is satisfied for $\kappa = 0$ and $r = w_{xx}^i(x,t)$, where $(x,t) \in D$, and for $\tilde{r} = w_{xx}^i(x,t) + \hat{r}$, where $(x,t) \in E$ and $\hat{r} \geq 0$, then F_i is called *parabolic* with respect to w in E.

Two functions $u, v \in Z_\infty^{2,1}(\tilde{D})$ are called *solutions* of the system

$$F_i[x,t,u] \geq F_i[x,t,v] \ \ (i \in \mathbb{N}) \tag{2.2}$$

in D, if they satisfy (2.2) for $(x,t) \in D$.

Assumption (L): There are constants $L_i > 0$ $(i = 1,2)$ such that

$$F_i(x,t,z,p,q,r,w) - F_i(x,t,z,\tilde{p},q,r,w) \leq L_0(\tilde{p} - p) \ (i \in \mathbb{N})$$

for all $(x,t) \in D, z \in \mathbb{R}, p > \tilde{p}, q \in \mathbb{R}^n, r \in M_{n\times n}(\mathbb{R}), w \in Z_\infty(\tilde{D})$ and

$$
\begin{aligned}
&F_i(x,t,z,p,q,r,w) - F_i(x,t,\tilde{z},\tilde{p},\tilde{q},\tilde{r},\tilde{w})\\
&\le L\Big(|z-\tilde{z}| + |p-\tilde{p}| + |x|\sum_{j=1}^{n}|q_j-\tilde{q}_j|\\
&\quad +|x|^2\sum_{j,k=1}^{n}|r_{jk}-\tilde{r}_{jk}| + [w-\tilde{w}]_{t,\infty}\Big)\ (i\in\mathbb{N})
\end{aligned}
$$

for all $(x,t)\in D, z,\tilde{z}\in\mathbb{R}, p,\tilde{p}\in\mathbb{R}, q,\tilde{q}\in\mathbb{R}^n, r,\tilde{r}\in M_{n\times n}(\mathbb{R}), w,\tilde{w}\in Z_\infty(\tilde{D})$.

For every set $A\subset\tilde{D}$ and for each function $w\in Z_\infty(\tilde{D})$ we apply the notation:

$$
\max_{(x,t)\in A} w(x,t) := \Big(\max_{(x,t)\in A} w^1(x,t), \max_{(x,t)\in A} w^2(x,t), ...\Big).
$$

Let $I=\mathbb{N}$ or I is a finite set of mutually different natural numbers.

Define the set

$$
Z = \bigcup_{i\in I}\sigma_{T_i},
$$

where, in the case if $I=\mathbb{N}$, the following conditions are satisfied:

(i) $t_0 < T_i \le t_0 + T$ for $i\in I$ and $T_i\neq T_j$ for $i,j\in I, i\neq j$;

(ii) $T_0 := \inf_{i\in I} T_i > t_0$;

(iii) $S_{t_0}\subset S_{T_i}$ for $i\in I$;

(iv) $S_{t_0}\subset S_t$ for every $t\in[T_0, t_0+T]$,

and in the case if I is a finite set of mutually different natural numbers, the conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called *a set of type* $(P_{Z\Gamma})$, if

(a) $Z\neq\phi$,

(b) $\Gamma\cap\bar{\sigma}_{t_0}\neq\phi$.

Let Z_* denote a nonempty subset of Z. We define the following set

$$
I_* = \{i\in I : \sigma_{T_i}\subset Z_*\}.
$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type $(P_{Z\Gamma})$ is called *a set of type* (P_{ZB}).

An unbounded set D of type (P) satisfying condition (b) of the definition of a set of type $(P_{Z\Gamma})$ is called *a set of type* (P_Γ).

A bounded set D of type (P) is called *a set of type* (P_B).

It is easy to see that if D is a set of type (P_{ZB}) or (P_B) then D satisfies condition (b) of the definition of a set of type $(P_{Z\Gamma})$. Moreover, it is obvious that if D_0 is a bounded subset [D_0 is an unbounded essential subset] of $\mathbb{R}^n$ then $D = (t_0, t_0 + T] \times D_0$ is a set of type (P_{ZB}) and (P_B)[$(P_{Z\Gamma})$ and (P_Γ), respectively].

3. Strong maximum principles

Theorem 3.1. *Assume that:*

(1) *D is a set of type $(P_{Z\Gamma})$ or (P_{ZB}).*
(2) *The functions F_i $(i \in \mathbb{N})$ satisfy Assumption (L).*
(3) *$u \in Z^{2,1}_\infty(\tilde{D})$ and the maximum of function u on Γ is attained. Moreover,*

$$K^i := \max_{(x,t)\in\Gamma} u^i(x,t) \ (i \in \mathbb{N}) \tag{3.1}$$

and $K \in \ell^\infty$ is defined by formulae

$$K : \mathbb{N} \times \tilde{D} \ni (i,x,t) \longrightarrow K^i.$$

(4) *The following inequalities hold*

$$(u^j(x,t_0) - K^j) + \sum_{i\in I_*} h_i(x)(u^j(x,T_i) - K^j) \le 0, \tag{3.2}$$

for $x \in S_{t_0}$ $(j \in \mathbb{N})$, where $h_i : S_{t_0} \longrightarrow \mathbb{R}_-$ $(i \in I_*)$ are given functions such that

$$-1 \le \sum_{i\in I_*} h_i(x) \le 0 \ \text{ for } \ x \in S_{t_0}$$

and, additionally, if $card I_* = \aleph_0$, then the series $\sum\limits_{i\in I_*} h_i(x)u^j(x,T_i)$ $(j \in \mathbb{N})$ are convergent for $x \in S_{t_0}$.

(5) *There exists a point $(x^*,t^*) \in \tilde{D}$ such that*

$$u(x^*,t^*) = \max_{(x,t)\in\tilde{D}} u(x,t).$$

Moreover,

$$M^i := u^i(x^*,t^*) \ (i \in \mathbb{N}) \tag{3.3}$$

and $M \in \ell^\infty$ is defined by

$$M : \mathbb{N} \times \tilde{D} \ni (i,x,t) \longrightarrow M^i.$$

(6) *u and $v = M$ are solutions of system (2.2) in D.*
(7) *F_i $(i \in \mathbb{N})$ are parabolic with respect to u in D and uniformly parabolic with respect to M in any compact subset of D.*

Then

$$\max_{(x,t)\in\tilde{D}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t). \tag{3.4}$$

Moreover, if there is a point $(\tilde{x},\tilde{t}) \in D$ such that

$$u(\tilde{x},\tilde{t}) = \max_{(x,t)\in\tilde{D}} u(x,t),$$

then

$$u(x,t) = \max_{(x,t)\in\Gamma} u(x,t) \ \text{ for } \ (x,t) \in S^-(\tilde{x},\tilde{t}).$$

Proof. We shall prove Theorem 3.1 for a set of type $(P_{Z\Gamma})$ only, since the proof of this theorem for a set of type (P_{ZB}) is analogous.

We shall argue by contradiction. Suppose that the contrary of (3.4) holds, that is,

$$M \neq K.$$

Next, (3.1) and (3.3) implies inequalities

$$K^i \leq M^i \ (i \in \mathbb{N}).$$

Consequently, there is $\ell \in \mathbb{N}$ such that

$$K^\ell < M^\ell. \tag{3.5}$$

Observe, from assumption (5), that there is a point $(x^*,t^*) \in \tilde{D}$ such that

$$u(x^*,t^*) = M := \max_{(x,t)\in\tilde{D}} u(x,t). \tag{3.6}$$

By (3.6), assumption (3) and (3.5), we have

$$(x^*,t^*) \in \tilde{D} \setminus \Gamma = D \cup \sigma_{t_0}. \tag{3.7}$$

Assume that

$$(x^*,t^*) \in D. \tag{3.8}$$

From assumption (6) and (3.6), we get

$$\left.\begin{array}{l} u \in Z^{2,1}_\infty(\tilde{D}), \\ F_i[x,t,u] \geq F_i[x,t,M] \ \text{ for } \ (x,t) \in D \ (i \in \mathbb{N}), \\ u(x,t) \leq M \ \text{ for } \ (x,t) \in \tilde{D}, \\ u(x^*,t^*) = M. \end{array}\right\} \tag{3.9}$$

The assumption that D is a set of type (P), Assumption (L), relations (3.8) and (3.9), and assumption (7) imply, by Theorem 4.1 from [8], the equation

$$u(x,t) = M \text{ for } (x,t) \in S^-(x^*,t^*). \tag{3.10}$$

On the other hand, from the definition of a set of type $(P_{Z\Gamma})$, there is a polygonal line $\gamma \subset S^-(x^*,t^*)$ such that

$$\bar{\gamma} \cap \Gamma \neq \phi. \tag{3.11}$$

Since $u^i \in C(\bar{D})$ $(i \in \mathbb{N})$, we have a contradiction of formulae (3.10) and (3.11) with formulae (3.1) and (3.5). Therefore, $(x^*,t^*) \notin D$ and, consequently, from (3.7),

$$(x^*,t^*) \in \sigma_{t_0}. \tag{3.12}$$

Consider now two possible cases:

$$(I) \quad \sum_{i \in I_*} h_i(x) = 0, \qquad (II) \quad -1 \leq \sum_{i \in I_*} h_i < 0.$$

In case (I) condition (3.12) leads, by (3.5), to a contradiction of (3.2) with (3.6). From this contradiction the proof of (3.4) is complete in case (I).

In case (II), by the definition of sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e, without loss of generality, there is a number $p \in \mathbb{N}$ such that $I_* = \{1,...,p\}$.

(B) $card\ I_* = \aleph_0$.

First we shall consider case (A). By (3.2) and the inequalities

$$u(x^*,T_i) < u(x^*,t_0) \ (i = 1,...,p),$$

being a consequence of (3.6) and (3.12) and of conditions (a)(i), (a)(iii) of the definition of a set of type $(P_{Z\Gamma})$, we have

$$\begin{aligned} 0 &\geq (u^j(x^*,t_0) - K^j) + \sum_{i=1}^{p} h_i(x^*)(u^j(x^*,T_i) - K^j) \\ &\geq (u^j(x^*,t_0) - K^j)\Big(1 + \sum_{i=1}^{p} h_i(x^*)\Big) \ (j \in \mathbb{N}). \end{aligned}$$

From the last inequalities we have

$$u(x^*,t_0) \leq K \text{ if } 1 + \sum_{i=1}^{p} h_i(x^*) > 0. \tag{3.13}$$

Hence, from (3.5) and (3.12), we obtain a contradiction of (3.13) with (3.6).

Assume now that

$$\sum_{i=1}^{p} h_i(x^*) = -1.$$

Observe that for every $j \in \mathbb{N}$, there is a number $\ell_j \in \{1, ..., p\}$ such that

$$u^j(x^*, T_{\ell_j}) = \max_{i=1,...,p} u^j(x^*, T_i).$$

Hence, by (3.2), we have

$$\begin{aligned} u^j(x^*, t_0) - u^j(x^*, T_{\ell_j}) &= (u^j(x^*, t_0) - K^j) - (u^j(x^*, T_{\ell_j}) - K^j) \\ &= (u^j(x^*, t_0) - K^j) \\ &\quad + \sum_{i=1}^{p} h_i(x^*)(u^j(x^*, T_{\ell_j}) - K^j) \\ &\leq (u^j(x^*, t_0) - K^j) \\ &\quad + \sum_{i=1}^{p} h_i(x^*)(u^j(x^*, T_j) - K^j) \\ &\leq 0 \ \ (j \in \mathbb{N}). \end{aligned}$$

Consequently,

$$u^j(x^*, t_0) \leq u^j(x^*, T_{\ell_j}) \ \ (j \in \mathbb{N}) \ \text{ if } \sum_{i=1}^{p} h_i(x^*) = -1. \tag{3.14}$$

Since, condition (a)(i) of the definition of a set of type $(P_{Z\Gamma})$ implies inequalities $T_{\ell_j} > t_0$ $(j \in \mathbb{N})$, from (3.12), we see that (3.14) contradits (3.6). This completes the proof of formula (3.4) in case (A).

It remains to investigate case (B). Analogously as in the proof of (3.4) in case (A), by assumption (4) and the inequalities

$$u(x^*, T_i) < u(x^*, t_0) \ (i \in I_*),$$

we have

$$\begin{aligned} 0 &\geq (u^j(x^*, t_0) - K^j) + \sum_{i \in I_*} h_i(x^*)(u^j(x^*, T_i) - K^j) \\ &\geq (u^j(x^*, t_0) - K^j)\Big(1 + \sum_{i \in I_*} h_i(x^*)\Big) \ \ (j \in \mathbb{N}). \end{aligned}$$

Hence

$$u(x^*, t_0) \leq K \ \text{ if } \ 1 + \sum_{i \in I_*} h_i(x^*) > 0. \tag{3.15}$$

Then, from (3.5) and (3.12), we obtain a contradiction of (3.15) with (3.6).

Assume now that $\sum\limits_{i\in I_*} h_i(x^*) = -1$ and let

$$T_0^* := \inf_{i\in I_*} T_i.$$

Since $u^i \in C(\bar{D})$ ($i \in \mathbb{N}$) and since, by (a)(iv) of the definition of a set of type $(P_{Z\Gamma})$, $x^* \in S_t$ for every $t \in [T_0, t_0+T]$, if $card I = \aleph_0$, it follows that for every $j \in \mathbb{N}$ there is $\hat{t}_j \in [T_0^*, t_0+T]$ such that

$$u^j(x^*, \hat{t}_j) = \max_{t\in[T_0^*, t_0+T]} u^j(x^*, t).$$

Consequently, by assumption (4), we obtain

$$\begin{aligned} u^j(x^*, t_0) - u^j(x^*, \hat{t}_j) &= (u^j(x^*, t_0) - K^j) - (u^j(x^*, \hat{t}_j) - K^j) \\ &= (u^j(x^*, t_0) - K^j) \\ &\quad + \sum_{i\in I_*} h_i(x^*)(u^j(x^*, \hat{t}_j) - K^j) \\ &\leq (u^j(x^*, t_0) - K^j) \\ &\quad + \sum_{i\in I_*} h_i(x^*)(u^j(x^*, T_i) - K^j) \\ &\leq 0 \quad (j \in \mathbb{N}). \end{aligned}$$

Hence

$$u^j(x^*, t_0) \leq u^j(x^*, \hat{t}_j) \ (j \in \mathbb{N}) \ \text{ if } \sum_{i\in I_*} h_i(x^*) = -1. \tag{3.16}$$

Since, condition (a)(ii) of the definition of a set of type $(P_{Z\Gamma})$ implies inequalities $\hat{t}_j > t_0$ ($j \in \mathbb{N}$), we see from (3.12) that (3.16) contradicts (3.6). This completes the proof of formula (3.4).

The second part of the thesis of Theorem 3.1 is a consequence of (3.4) and of Theorem 4.1 from [8]. Therefore, the proof of Theorem 3.1 is complete. □

Remark 3.2. It is easy to see, from the proof of Theorem 3.1, that if the functions h_i ($i \in I_*$) from assumption (4) of Theorem 3.1 satisfy the condition

$$\Big[\sum_{i\in I_*} h_i(x) = 0\Big] \ -1 < \sum_{i\in I_*} h_i(x) < 0 \ \text{ for } \ x \in S_{t_0}$$

then it is sufficient in that theorem to assume that [D is an unbounded set of type (P) satisfying condition (b) of the definition of a set of type $(P_{Z\Gamma})$ or D is a bounded set of type (P), i.e. D is a set of type (P_Γ) or (P_B), respectively] D is an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii), and (b) of the definition of a set of type $(P_{Z\Gamma})$ or D is a bounded set of type (P)

satisfying conditions (a)(i) and (a)(iii) of the definition of a set of type $(P_{Z\Gamma})$. Moreover, if I_* is a finite set and

$$-1 \le \sum_{i \in I_*} h_i(x) < 0 \ \text{ for } \ x \in S_{t_0},$$

then it is sufficient in Theorem 3.1 to assume that D is an unbounded set of type (P) satisfying conditions (a)(i), (a)(iii), and (b), or D is a bounded set of type (P) satisfying conditions (a)(i) and (a)(iii).

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