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# ON AN INEQUALITY OF S. BERNSTEIN

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**Abstract.** If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros on |z| = k,  $k \leq 1$ , then Govil [3] proved that

$$\max_{|z|=1} |p'(z)| \le \frac{n}{k^n + k^{n-1}} \max_{|z|=1} |p(z)|.$$

In this paper, by involving certain coefficients of p(z), we not only improve the above inequality but also improve a result proved by Dewan and Mir [2].

#### 1. Introduction

For a polynomial p(z) of degree n, let  $M(p,r) = \max_{|z|=r} |p(z)|$  and  $q(z) = z^n \overline{p(1/\overline{z})}$  be the reciprocal polynomial of p(z). Bernstein [1] proved that

$$M(p',1) \le nM(p,1).$$
 (1.1)

Equality in (1.1) is attained for the polynomial  $p(z) = \alpha z^n$ ,  $\alpha \neq 0$ .

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If we restrict ourselves to the class of polynomials p(z) of degree n having no zero in |z| < 1, then Erdös conjectured and Lax [8] later proved that

$$M(p',1) \le \frac{n}{2}M(p,1).$$
 (1.2)

Inequality (1.2) is best possible and the extremal polynomial is  $p(z) = \alpha + \beta z^n$  with  $|\alpha| = |\beta|$ .

The renowned mathematician Boas asked that if p(z) is a polynomial of degree n not vanishing in |z| < k, k > 0, then how large can  $\frac{M(p',1)}{M(p,1)}$  be? A partial answer to this problem was given by Malik [7] by proving a generalization of (1.2).

**Theorem 1.1.** ([7]) If p(z) is a polynomial of degree n having no zero in  $|z| < k, k \ge 1$ , then

$$M(p',1) \le \frac{n}{1+k}M(p,1).$$
 (1.3)

Equality in (1.3) occurs for  $p(z) = (z + k)^n$ .

The question as to what happens to the inequality (1.3) if k < 1 remains unanswered. For quite sometime, it was believed that the inequality analogous to (1.3) for k < 1 would be

$$M(p',1) \le \frac{n}{1+k^n}M(p,1)$$
 (1.4)

until Professor E.B. Saff countered the belief with the example  $p(z) = (z - \frac{1}{2}) \times (z + \frac{1}{3})$ . Though, Govil [4] proved the validity of (1.4), it was achieved with additional condition that both |p'(z)| and |q'(z)| attain their maxima at the same point on |z| = 1. Further in this quest, Govil [3] could only prove.

**Theorem 1.2.** ([3]) If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros on |z| = k,  $k \le 1$ , then

$$M(p',1) \le \frac{n}{k^n + k^{n-1}} M(p,1).$$
 (1.5)

Inequality (1.5) was further improved by Dewan and Mir [2] by involving certain coefficients of p(z).

**Theorem 1.3.** If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros on |z| = k,  $k \le 1$ , then

$$M(p',1) \le \frac{n}{k^n} \left\{ \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n|(1+k^2) + 2|a_{n-1}|} \right\} M(p,1). \tag{1.6}$$

## 2. Lemmas

We need the following lemmas to prove our result.

**Lemma 2.1.** ([3, Lemma 3]) If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having no zero in |z| < k,  $k \le 1$  and  $q(z) = z^{n} p\left(\frac{1}{z}\right)$ , then  $k^{n} \max_{|z|=1} |p'(z)| \le \max_{|z|=1} |q'(z)|. \tag{2.1}$ 

**Lemma 2.2.** ([5, page 511, inequality (3.2)]) If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n and  $q(z) = z^{n} \overline{p\left(\frac{1}{\overline{z}}\right)}$ , then on |z| = 1,  $|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{2.2}$ 

**Lemma 2.3.** ([6, page 327, Proof of Theorem 1]) If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros in  $|z| \geq k$ ,  $k \geq 1$  and  $q(z) = z^{n} p\left(\frac{1}{z}\right)$ , then for |z| = 1

$$|p'(z)| \le \frac{1}{k} \frac{(1-|\lambda|)(1+k^2|\lambda|) + k(n-1)|\mu - \lambda^2|}{(1-|\lambda|)(|\lambda|+k^2) + k(n-1)|\mu - \lambda^2|} |q'(z)|, \tag{2.3}$$
where  $\lambda = \frac{k}{n} \frac{a_1}{a_0}$  and  $\mu = \frac{2k^2}{n(n-1)} \frac{a_2}{a_0}$ .

**Lemma 2.4.** If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros on |z| = k,  $k \le 1$  and  $q(z) = z^{n} \overline{p\left(\frac{1}{\overline{z}}\right)}$ , then for |z| = 1  $|p'(z)| \ge \frac{1}{k} \frac{(1 - |\tau|)(1 + k^{2}|\tau|) + k(n-1)|\sigma - \tau^{2}|}{(1 - |\tau|)(|\tau| + k^{2}) + k(n-1)|\sigma - \tau^{2}|} |q'(z)|, \tag{2.4}$ 

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$$\tau = \frac{1}{kn} \frac{\overline{a}_{n-1}}{\overline{a}_n} \text{ and } \sigma = \frac{2}{k^2 n(n-1)} \frac{\overline{a}_{n-2}}{\overline{a}_n}.$$

*Proof.* Since p(z) has all its zeros on |z| = k,  $q(z) = \sum_{\nu=0}^{n} \overline{a}_{n-\nu} z^{\nu}$  has all its zeros on  $|z| = \frac{1}{k}$ , where  $\frac{1}{k} \ge 1$ . Thus, q(z) has no zeros in  $|z| < \frac{1}{k}$ ,  $\frac{1}{k} \ge 1$ . Applying Lemma 2.3 to the polynomial q(z), the desired conclusion of the lemma follows.

**Lemma 2.5.** ([6, page 320, inequality (9)]) If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having no zero in |z| < k,  $k \ge 1$ , then

$$\frac{k}{n} \left| \frac{a_1}{a_0} \right| \le 1 \tag{2.5}$$

and

$$(n-1)\left|\frac{2k^2}{n(n-1)}\frac{a_2}{a_0} - \frac{k^2}{n^2}\left(\frac{a_1}{a_0}\right)^2\right| \le 1 - \frac{k^2}{n^2}\left|\frac{a_1}{a_0}\right|^2.$$
 (2.6)

**Lemma 2.6.** If  $p(z) = \sum_{\nu}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros in  $|z| \leq k$ ,  $k \leq 1$ , then

$$\frac{1}{kn} \left| \frac{a_{n-1}}{a_n} \right| \le 1 \tag{2.7}$$

and

$$(n-1) \left| \frac{2}{k^2 n(n-1)} \frac{\overline{a}_{n-2}}{\overline{a}_n} - \frac{1}{k^2 n^2} \left( \frac{\overline{a}_{n-1}}{\overline{a}_n} \right)^2 \right| \le 1 - \frac{1}{k^2 n^2} \left| \frac{a_{n-1}}{a_n} \right|^2. \tag{2.8}$$

*Proof.* Since p(z) has all its zeros in  $|z| \le k$ , then  $q(z) = z^n \overline{p\left(\frac{1}{z}\right)}$  has no zeros in  $|z| < \frac{1}{k}, \frac{1}{k} \ge 1$ . Thus, applying Lemma 2.5 to q(z), we obtain inequalities (2.7) and (2.8) respectively from (2.5) and (2.6).

**Lemma 2.7.** Let a, b, c, d > 0 be real numbers such that  $c \le d$ . If  $a \le b$ , then

$$\frac{a+c}{b+c} \le \frac{a+d}{b+d}. (2.9)$$

*Proof.* Suppose that c < d. Then (2.9) follows easily as

$$a \le b$$
 implies  $a(d-c) \le b(d-c)$ ,

which simplifies to

$$(a+c)(b+d) \le (a+d)(b+c),$$

giving the desired conclusion of the lemma. For c = d, the equality in (2.9) holds trivially.

**Lemma 2.8.** If  $p(z) = \sum_{\nu}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros in  $|z| \le k$ ,  $k \le 1$ , then

$$\frac{n}{k^{n} + k^{n-1}} \left( \frac{(1 - |\tau|)(k^{2} + |\tau|) + k(n-1)|\sigma - \tau^{2}|}{(1 - |\tau|)(1 - k + k^{2} + k|\tau|) + k(n-1)|\sigma - \tau^{2}|} \right) 
\leq \frac{n}{k^{n-1}} \left( \frac{k + |\tau|}{1 + k^{2} + 2k|\tau|} \right),$$
(2.10)

where

$$\tau = \frac{1}{kn} \frac{\overline{a}_{n-1}}{\overline{a}_n}$$
 and  $\sigma = \frac{2}{k^2 n(n-1)} \frac{\overline{a}_{n-2}}{\overline{a}_n}$ .

*Proof.* Let  $a=(1-|\tau|)(k^2+|\tau|)$ ,  $b=(1-|\tau|)(1-k+k^2+k|\tau|)$ ,  $c=k(n-1)|\sigma-\tau^2|$  and  $d=k(1-|\tau|^2)$ . Then, by inequality (2.8) of Lemma 2.6, we have  $c\leq d$ . It is also easy to verify that  $a\leq b$  as  $k\leq 1$  and  $|\tau|\leq 1$  by inequality (2.7) of Lemma 2.6. Thus, by Lemma 2.7, we have

$$\begin{split} &\frac{n}{k^n+k^{n-1}}\left(\frac{(1-|\tau|)(k^2+|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|)+k(n-1)|\sigma-\tau^2|}\right)\\ &\leq \frac{n}{k^n+k^{n-1}}\left(\frac{(1-|\tau|)(k^2+|\tau|)+k(1-|\tau|^2)}{(1-|\tau|)(1-k+k^2+k|\tau|)+k(1-|\tau|^2)}\right)\\ &=\frac{n}{k^{n-1}(k+1)}\left(\frac{k^2+|\tau|+k+k|\tau|}{1-k+k^2+k+k|\tau|}\right)\\ &=\frac{n}{k^{n-1}}\left(\frac{k+|\tau|}{1+k^2+2k|\tau|}\right). \end{split}$$

#### 3. Main results

In this section, we prove an improvement of (1.5) due to Govil [3]. Moreover, our result also improves (1.6) proved by Dewan and Mir [2].

**Theorem 3.1.** If  $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$  is a polynomial of degree n having all its zeros on |z| = k,  $k \le 1$ , then

$$M(p',1) \le \frac{n}{k^n + k^{n-1}} \left( \frac{(1-|\tau|)(k^2+|\tau|) + k(n-1)|\sigma - \tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma - \tau^2|} \right) \times M(p,1), \tag{3.1}$$

where

$$\tau = \frac{1}{kn} \frac{\overline{a}_{n-1}}{\overline{a}_n} \quad and \quad \sigma = \frac{2}{k^2 n(n-1)} \frac{\overline{a}_{n-2}}{\overline{a}_n}.$$
 (3.2)

*Proof.* Let  $z_0$  be a point on |z| = 1 such that  $|q'(z_0)| = M(q', 1)$ . Then, by Lemma 2.2, we have

$$|p'(z_0)| + M(q', 1) \le nM(p, 1), \tag{3.3}$$

which on combining with inequality (2.4) of Lemma 2.4, we get

$$\frac{1}{k} \left( \frac{(1-|\tau|)(1+k^2|\tau|) + k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(|\tau|+k^2) + k(n-1)|\sigma-\tau^2|} \right) |q'(z_0)| + M(q',1) 
\leq nM(p,1),$$
(3.4)

which simplifies to

$$\left(\frac{(1-|\tau|)(1-k+k^2+k|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(|\tau|+k^2)+k(n-1)|\sigma-\tau^2|}\right)M(q',1) 
\leq \frac{kn}{k+1}M(p,1).$$
(3.5)

Inequality (3.5), in conjunction with Lemma 2.1 yields

$$k^{n} \left( \frac{(1-|\tau|)(1-k+k^{2}+k|\tau|)+k(n-1)|\sigma-\tau^{2}|}{(1-|\tau|)(|\tau|+k^{2})+k(n-1)|\sigma-\tau^{2}|} \right) M(p',1)$$

$$\leq \frac{kn}{k+1} M(p,1), \tag{3.6}$$

which is equivalent to

$$\begin{split} M(p',1) \leq & \frac{n}{k^n + k^{n-1}} \\ & \times \left( \frac{(1-|\tau|)(|\tau| + k^2) + k(n-1)|\sigma - \tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|) + k(n-1)|\sigma - \tau^2|} \right) M(p,1), \end{split}$$

which proves Theorem 3.1 completely.

**Remark 3.2.** To prove that the bound given by (3.1) improves over the bound given by (1.5) of Theorem 1.2 proved by Govil [3], we show that

$$\frac{n}{k^n + k^{n-1}} \left( \frac{(1 - |\tau|)(k^2 + |\tau|) + k(n-1)|\sigma - \tau^2|}{(1 - |\tau|)(1 - k + k^2 + k|\tau|) + k(n-1)|\sigma - \tau^2|} \right) \le \frac{n}{k^n + k^{n-1}},$$

for which it is sufficient to show that

$$\frac{(1-|\tau|)(k^2+|\tau|)+k(n-1)|\sigma-\tau^2|}{(1-|\tau|)(1-k+k^2+k|\tau|)+k(n-1)|\sigma-\tau^2|} \le 1$$

or

$$(k^2 + |\tau|) \le (1 - k + k^2 + k|\tau|),$$

which clearly holds as  $k \le 1$  and  $|\tau| \le 1$  by (2.7) of Lemma 2.6.

**Remark 3.3.** Inequality (3.1) is also an improvement of inequality (1.6) of Theorem 1.3 due to Dewan and Mir [2]. It is enough to show that

$$\frac{n}{k^{n} + k^{n-1}} \left( \frac{(1 - |\tau|)(k^{2} + |\tau|) + k(n-1)|\sigma - \tau^{2}|}{(1 - |\tau|)(1 - k + k^{2} + k|\tau|) + k(n-1)|\sigma - \tau^{2}|} \right) 
\leq \frac{n}{k^{n}} \left\{ \frac{n|a_{n}|k^{2} + |a_{n-1}|}{n|a_{n}|(1 + k^{2}) + 2|a_{n-1}|} \right\}.$$
(3.7)

Dividing both numerator and denominator of  $\frac{n}{k^n} \left\{ \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n|(1+k^2) + 2|a_{n-1}|} \right\}$  by  $kn|a_n|$ , we get, from (3.2)

$$\frac{n}{k^n} \left( \frac{k + \frac{1}{kn} \frac{|a_{n-1}|}{|a_n|}}{\frac{1+k^2}{k} + \frac{2}{kn} \frac{|a_{n-1}|}{|a_n|}} \right) = \frac{n}{k^{n-1}} \left( \frac{k + |\tau|}{1 + k^2 + 2k|\tau|} \right)$$

which, in conjunction with inequality (2.10) of Lemma 2.8 proves (3.7).

As discussed earlier, Theorem 3.1 improves both Theorem 1.2 and Theorem 1.3. We illustrate this by means of the following example.

**Example 3.4.** Let  $p(z) = z^3 - \frac{3}{50}z^2 - \frac{3}{500}z + \frac{1}{1000}$ ,  $k = \frac{1}{10}$ . By Theorem 1.2 and Theorem 1.3, we have

$$M(p',1) \le 272.727 \ M(p,1)$$

and

$$M(p',1) \le 85.7143 \ M(p,1)$$

respectively, while by Theorem 3.1, we obtain

$$M(p', 1) \le 74.3802 \ M(p, 1),$$

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which shows improvement in the bound of about 72.73% and 13.22% over bounds given by Theorem 1.2 and Theorem 1.3, respectively.

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