Nonlinear Functional Analysis and Applications
Vol. 26, No. 2 (2021), pp. 381-392
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.02.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

A TYPE OF FRACTIONAL KINETIC EQUATIONS ASSOCIATED WITH THE (p, q)-EXTENDED τ-HYPERGEOMETRIC AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Abstract. During the last several decades, a great variety of fractional kinetic equations involving diverse special functions have been broadly and usefully employed in describing and solving several important problems of physics and astrophysics. In this paper, we aim to find solutions of a type of fractional kinetic equations associated with the (p, q)-extended τ-hypergeometric function and the (p, q)-extended τ-confluent hypergeometric function, by mainly using the Laplace transform. It is noted that the main employed techniques for this chosen type of fractional kinetic equations are Laplace transform, Sumudu transform, Laplace and Sumudu transforms, Laplace and Fourier transforms, P_χ-transform, and an alternative method.

[0]Received October 2, 2020. Revised October 15, 2020. Accepted November 1, 2020.
[0]2010 Mathematics Subject Classification: 26A33, 33A10, 33E20, 44A10.
[0]Keywords: Fractional calculus, fractional kinetic equations, (generalized) Mittag-Leffler functions, H-function, $\aleph$-function, generalized M-series, (p, q)-extended τ-hypergeometric function, (p, q)-extended τ-confluent hypergeometric function, Laplace transform, Sumudu transform.
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1. Introduction and preliminaries

The fractional-order calculus, which deals with differentiation and integration of an arbitrary real or complex order including, of course, the integer one, is popularly believed to be initiated in a letter from Leibniz to L'Hospital in 1695 (see, e.g., [24]). The fractional calculus has been developed and used in different areas of science and engineering. During the last several decades, fractional kinetic equations of various forms have been broadly and usefully employed in describing and solving several important problems of physics and astrophysics (see, e.g., [2, 3, 4, 7, 13, 17, 20, 25, 28, 33, 34, 35, 36, 37]; see also [26] and the references therein).

By integrating a standard kinetic equation, Haubold and Mathai [22, Eq. (21)] introduced the following fractional kinetic equation (see also [33, Eq. (11)]):

$$N(t) - N_0 = -c^{\nu} \cdot {}_0D_t^{-\nu}\, N(t), \tag{1.1}$$

where ${}_0D_t^{-\nu}$ is the familiar Riemann-Liouville fractional integral operator (see, e.g., [24]) defined by

$${}_0D_t^{-\nu}\, f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1}\, f(u)\, \mathrm{d}u \quad (\Re(\nu) > 0), \tag{1.2}$$

and Γ being the well-known Gamma function (see, e.g., [39, Section 1.1]). Here $N(t)$ is an arbitrary reaction characterized by a time-dependent quantity, $N_0 := N(t = 0)$ is the quantity at time $t = 0$, and c is a positive constant. The solutions of the equation (1.1) were given in terms of the Fox's H-function (see, e.g., [27]).

The solutions of (1.1) are also expressed as (see [33, Eq. (13)])

$$N(t) = N_0\, E_{\nu}\left(-c^{\nu}\, t^{\nu}\right) \quad \left(\nu \in \mathbb{R}^+\right), \tag{1.3}$$

where $E_{\alpha}(z)$ is the Mittag-Leffler function defined by

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\Re(\alpha) > 0) \tag{1.4}$$

(see, e.g., [47]). Saxena et al. [33] investigated solutions of three generalized forms of (1.1) in terms of the following generalized Mittag-Leffler function (see, e.g., [47])

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\Re(\alpha) > 0,\, \beta \in \mathbb{C}). \tag{1.5}$$

Prabhakar [31] introduced the following extension of the generalized Mittag-Leffler function (1.5)

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_n \, z^n}{\Gamma(\alpha n+\beta)\, n!} \quad (\Re(\alpha)>0,\ \beta,\ \gamma \in \mathbb{C}), \tag{1.6}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for λ, $\nu \in \mathbb{C}$) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases} \tag{1.7}$$

it being accepted conventionally that $(0)_0 := 1$.

In the sequel, the fractional kinetic equation (1.1) have been solved by using various extensions of the Mittag-Leffler function (1.4) (see, e.g., [2, 13, 31, 33, 35, 47]).

Saxena et al. [34, Theorem 1] investigated the following unified fractional kinetic equation:

$$N(t) - N_0\, f(t) = -c^\nu \cdot {}_0D_t^{-\nu}\, N(t) \quad \left(\nu,\, c \in \mathbb{R}^+\right), \tag{1.8}$$

$f(t)$ being any integrable function on the finite interval $[0,b]$, whose solution was given by

$$N(t) = c\, N_0 \int_0^t H^{1,1}_{1,2}\left[c^\nu\,(t-\tau)^\nu \,\middle|\, \begin{matrix} (-\frac{1}{\nu},1) \\ (-\frac{1}{\nu},1),\,(0,\nu) \end{matrix} \right] f(\tau)\,\mathrm{d}\tau, \tag{1.9}$$

where $H^{1,1}_{1,2}(\cdot)$ is the H-function. Saxena et al. [36, Theorem 1] presented a solution of the following more generalized fractional kinetic equation than that in (1.8):

$$N(t) - N_0\, f(t) = -\sum_{j=1}^{n} a_j\, {}_0D_t^{-\nu_j}\, N(t) \tag{1.10}$$

$$\left(n\in\mathbb{N},\, \Re(\nu_j)>0,\, a_j\in\mathbb{R}^+,\, j=1,\ldots,n\right),$$

where $f(t)$ is a given function $(0,\infty)$.

Kumar et al. [25] gave solutions of the fractional kinetic equations with $f(t)$ in (1.8) replaced by certain forms of the generalized Mittag-Leffler function.

Saxena and Kalla [32] used an alternative method which has been employed by Al-Saqabi and Tuan [5] for solving differintegral equations and applying the operator $(-c^\nu)^m\, D_t^{-m\nu}$ to both sides of (1.8). We have

$$\begin{aligned} &(-c^\nu)^m\, {}_0D_t^{-m\nu} N(t) - (-c^\nu)^{m+1}\, {}_0D_t^{-\nu(m+1)} N(t) \\ &= N_0 (-c^\nu)^m\, {}_0D_t^{-m\nu}\, f(t), \end{aligned} \tag{1.11}$$

where $\nu \in \mathbb{R}^+$ and $m \in \mathbb{N}_0$. Summing each side of (1.11) over $m \in \mathbb{N}_0$, we get

$$\begin{aligned}&\sum_{m=0}^{\infty}(-c^{\nu})^m\,{}_0D_t^{-m\nu}N(t)-\sum_{m=0}^{\infty}(-c^{\nu})^{m+1}\,{}_0D_t^{-\nu(m+1)}N(t)\\&=N_0\sum_{m=0}^{\infty}(-c^{\nu})^m\,{}_0D_t^{-m\nu}f(t).\end{aligned}\tag{1.12}$$

By telescoping the sums on the left member of (1.12), we obtain

$$N(t)=N_0\sum_{m=0}^{\infty}(-c^{\nu})^m\,{}_0D_t^{-m\nu}f(t).\tag{1.13}$$

By using Sumudu transformation introduced by Watugala [45, 46] (see also [6, 10, 11, 17]), Gupta et al. [21, Theorem 1] gave a solution for a generalized fractional kinetic equation involving the generalized Lauricella confluent hypergeometric functions of several variables (see, e.g., [41, p. 34]).

By using Sumudu transformation on the fractional kinetic equations with $f(t)$ in (1.8) replaced by certain forms of generalized Mittag-Leffler function and the G-function, Saxena et al. [37, Theorems 1, 2 and 3] presented their corresponding solutions.

By using Sumudu transformation on the fractional kinetic equations with $f(t)$ in (1.8) replaced by certain forms of the generalized M-series and the $\aleph$-function, Choi and Kumar [17, Theorems 1, 2 and 3] gave their corresponding solutions.

Also, Laplace and Sumudu transforms [25], Laplace and Fourier transforms [12], and P_{χ}-transform [1] have been used to solve the generalized fractional kinetic equation (1.10).

Extensions, generalizations and unifications of a variety of special functions, especially hypergeometric functions of one and several variables, have been done (see, e.g., [8, 23, 27, 40, 41, 42]). For over the last two decades, extensions of various special functions of p-variant and (p,q)-variant have been investigated broadly along with a class of hypergeometric type special functions (see, e.g., [14, 15, 16, 18, 19, 29, 30]). Among them, Choi et al. [19] introduced and investigated the (p,q)-extended Beta, the (p,q)-extended hypergeometric, and the (p,q)-extended confluent hypergeometric functions, respectively, as follows:

$$B(x,y;p,q)=\int_0^1 t^{x-1}(1-t)^{y-1}\,e^{-\frac{p}{t}-\frac{q}{1-t}}\,dt\tag{1.14}$$

$$\left(\min\{\Re(x),\,\Re(y)\}>0;\,\min\{\Re(p),\,\Re(q)\}\geq 0\right),$$

$$F_{p,q}(a,b;\, c;\, z) = \sum_{n=0}^{\infty} (a)_n \, \frac{B(b+n,c-b;p,q)}{B(b,c-b)} \, \frac{z^n}{n!} \quad (|z|<1;\, \Re(c)>\Re(b)>0) \tag{1.15}$$

and

$$\Phi_{p,q}(b;\, c;\, z) = \sum_{n=0}^{\infty} \frac{B(b+n,c-b;p,q)}{B(b,c-b)} \, \frac{z^n}{n!} \quad (\Re(c)>\Re(b)>0)\,. \tag{1.16}$$

Here $B(\alpha,\beta)$ is the beta function defined by (see, e.g., [39, Section 1.1])

$$B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt & (\min\{\Re(\alpha),\, \Re(\beta)\}>0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\, \beta \in \mathbb{C}\setminus \mathbb{Z}_0^-\right). \end{cases} \tag{1.17}$$

Also, τ-extensions of hypergeometric and confluent hypergeometric functions have been introduced and investigated (see [29, 43, 44]). Motivated by certain recent (p,q)-variant and τ-extensions of special functions, Parmar et al. [30] used the (p,q)-extended beta function to introduce and investigate the following (p,q)-extended τ-hypergeometric function and (p,q)-extended τ-confluent hypergeometric function, respectively, as follows:

$$R^{\tau}_{p,q}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B(b+\tau n,c-b;p,q)}{B(b,c-b)} \frac{z^n}{n!} \tag{1.18}$$

$$\left(\min\{\Re(p),\Re(q)\}>0,\, \tau\in\mathbb{R}_0^+,\, |z|<1;\, p=0=q,\, \Re(c)>\Re(b)>0\right)$$

and

$$\Phi^{\tau}_{p,q}(b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+\tau n,c-b;p,q)}{B(b,c-b)} \frac{z^n}{n!} \tag{1.19}$$

$$\left(\min\{\Re(p),\Re(q)\}>0,\, \tau\in\mathbb{R}_0^+;\, p=0=q,\, \Re(c)>\Re(b)>0\right).$$

The case $p=0=q$ of (1.18) and (1.19) reduces to, respectively, the τ-hypergeometric function and the τ-confluent hypergeometric function (see [44]):

$${}_2R^{\tau}_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} (a)_n \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \tag{1.20}$$

$$\left(\tau\in\mathbb{R}^+,\, \Re(a)>0,\, \Re(c)>\Re(b)>0,\, |z|<1\right)$$

and

$${}_1\Phi^{\tau}_1(b;c;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \quad \left(\tau\in\mathbb{R}^+,\, \Re(c)>\Re(b)>0\right). \tag{1.21}$$

In this paper, we aim to investigate solutions of the unified fractional kinetic equations (1.10) with replaced $f(t)$ by several forms of the (p,q)-extended

τ-hypergeometric function (1.18) and the (p, q)-extended τ-confluent hypergeometric function (1.19), by mainly using the Laplace transform. The results presented here, being general, are also pointed to reduce to fractional kinetic equations involving simpler special functions.

Suppose that $f(t)$ is a real(or complex)-valued function of the (time) variable $t > 0$ and s is a real or complex parameter. The Laplace transform of the function $f(t)$ is defined by

$$\begin{aligned} F(s) = \mathcal{L}\{f(t) : s\} &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \lim_{\kappa\to\infty} \int_0^\kappa e^{-st} f(t)\, dt, \end{aligned} \tag{1.22}$$

whenever the limit exits (as a finite number). The convolution of two functions $f(t)$ and $g(t)$, which are defined for $t > 0$, plays an important role in a number of different physical applications. The Laplace convolution of the functions $f(t)$ and $g(t)$ is given by the following integral:

$$(f * g)(t) = \int_0^t f(u)\, g(t-u)\, du = (g * f)(t), \tag{1.23}$$

which exists if the functions f and g are at least piecewise continuous. One of the very significant properties possessed by the convolution in connection with the Laplace transform is that the Laplace transform of the convolution of two functions is the product of their transforms (see, e.g., [25, 38]).

The Laplace Convolution Theorem. *If f and g are piecewise continuous on $[0, \infty)$ and of exponential order α when $t \to \infty$, then*

$$\mathcal{L}\{(f * g)(t) : s\} = \mathcal{L}\{f(t) : s\} \cdot \mathcal{L}\{g(t) : s\} \qquad \big(\Re(s) > \alpha\big). \tag{1.24}$$

We find

$$\begin{aligned} \mathcal{L}\left\{{}_0D_t^{-\nu} f(t) : s\right\} &= \frac{1}{\Gamma(\nu)}\, \mathcal{L}\left\{t^{\nu-1} * f(t) : s\right\} \\ &= \frac{1}{\Gamma(\nu)}\, \mathcal{L}\left\{t^{\nu-1} : s\right\} \mathcal{L}\{f(t) : s\} = \frac{1}{s^\nu}\, \mathcal{L}\{f(t) : s\} \end{aligned} \tag{1.25}$$

by using the following well-known identity:

$$\begin{aligned} &\mathcal{L}\{t^\nu : s\} = \frac{\Gamma(\nu+1)}{s^{\nu+1}} \\ &\iff \mathcal{L}^{-1}\left(\frac{1}{s^{\nu+1}}\right) = \frac{t^\nu}{\Gamma(\nu+1)} \quad \big(\Re(\nu) > -1;\, \Re(s) > 0\big), \end{aligned} \tag{1.26}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

2. Solutions of fractional kinetic equations

We find solutions of certain generalized fractional kinetic equations involving the (p,q)-extended τ-hypergeometric function (1.18) and the (p,q)-extended τ-confluent hypergeometric function (1.19) by applying the Laplace transform technique.

Theorem 2.1. *Let* $\mu,\nu,d \in \mathbb{R}^+$, $h \in \mathbb{C}$ *with* $d \neq h$ *and* $\tau \in \mathbb{R}_0^+$. *Also let* $\min\{\Re(p),\Re(q)\} > 0$ *while* $\Re(c) > \Re(b) > 0$ *when* $p = 0 = q$. *Then the solution of the following generalized fractional kinetic equation:*

$$N(t) - N_0\, t^{\mu-1} R_{p,q}^{\tau}(a,b;c;ht^{\nu}) = -d^{\nu}\,{}_0D_t^{-\nu} N(t) \tag{2.1}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{n=0}^{\infty} (a)_n \frac{B(b+\tau n, c-b;p,q)}{B(b,c-b)} \frac{\Gamma(\mu+\nu n)\,(ht^{\nu})^n}{n!} E_{\nu,\mu+\nu n}\left(-d^{\nu}\, t^{\nu}\right), \tag{2.2}$$

where $E_{\alpha,\beta}(\cdot)$ *is the generalized Mittag-Leffler function* (1.5).

Proof. Taking the Laplace transform on both sides of (2.1) and using (1.18), (1.25) and (1.26), we obtain

$$\mathcal{N}(s) = \frac{1}{1+\left(\frac{d}{s}\right)^{\nu}} N_0 \sum_{n=0}^{\infty} (a)_n \frac{B(b+\tau n, c-b;p,q)}{B(b,c-b)} \frac{h^n}{n!} \frac{\Gamma(\mu+\nu n)}{s^{\mu+\nu n}}, \tag{2.3}$$

where $\mathcal{N}(s) = \mathcal{L}\{N(t):s\}$. Using the geometric series

$$\frac{1}{1+\left(\frac{d}{s}\right)^{\nu}} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{d}{s}\right)^{\nu k} \qquad \left(\left|\frac{d}{s}\right| < 1\right)$$

in (2.3), we get

$$\mathcal{N}(s) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B(b+\tau n, c-b;p,q)}{B(b,c-b)} \frac{h^n}{n!} \Gamma(\mu+\nu n) \sum_{k=0}^{\infty} \frac{(-1)^k\, d^{\nu k}}{s^{\mu+\nu n+\nu k}}. \tag{2.4}$$

By inverting the Laplace transform on each side of (2.4) with the aid of (1.26), we find

$$\begin{aligned} N(t) = N_0\, t^{\mu-1} &\sum_{n=0}^{\infty} (a)_n \frac{B(b+\tau n, c-b;p,q)}{B(b,c-b)} \frac{\Gamma(\mu+\nu n)\,(ht^{\nu})^n}{n!} \\ &\times \sum_{k=0}^{\infty} \frac{\left(-d^{\nu}\, t^{\nu}\right)^k}{\Gamma(\nu k+\mu+\nu n)}, \end{aligned} \tag{2.5}$$

which, upon expressing the inner summation in terms of the generalized Mittag-Leffler function (1.5), yields the desired solution (2.2). □

Theorem 2.2. *Let* $\mu, \nu, d \in \mathbb{R}^+$, γ, $h \in \mathbb{C}$ *with* $d \neq h$ *and* $\tau \in \mathbb{R}_0^+$. *Also let* $\min\{\Re(p), \Re(q)\} > 0$ *while* $\Re(c) > \Re(b) > 0$ *when* $p = 0 = q$. *Then the solution of the following generalized fractional kinetic equation:*

$$N(t) - N_0\, t^{\mu-1} R_{p,q}^{\tau}(a, b; c; ht^{\nu}) = -\left\{ \sum_{j=1}^{\infty} \binom{\gamma}{j} d^{\nu j}\, {}_0D_t^{-\nu j} \right\} N(t) \tag{2.6}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{n=0}^{\infty} (a)_n \frac{B(b+\tau n, c-b; p, q)}{B(b, c-b)} \frac{\Gamma(\mu+\nu n)\,(ht^{\nu})^n}{n!} E_{\nu,\mu+\nu n}^{\gamma}\left(-d^{\nu}\, t^{\nu}\right), \tag{2.7}$$

where $E_{\alpha,\beta}^{\gamma}(\cdot)$ *is the generalized Mittag-Leffler function* (1.6).

Proof. The proof here would run parallel to that of Theorem 2.1. Here we use the following binomial expansions:

$$(1+z)^{\gamma} = \sum_{k=0}^{\infty} \binom{\gamma}{k} z^k \quad (\gamma \in \mathbb{C},\ |z| < 1), \tag{2.8}$$

where $\binom{\gamma}{k}$ $(\gamma \in \mathbb{C},\ k \in \mathbb{N}_0)$ is the generalized binomial coefficients defined by

$$\binom{\gamma}{k} = \begin{cases} 1 & (k = 0) \\ \frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!} & (k \in \mathbb{N}) \end{cases} \tag{2.9}$$

and

$$(1-z)^{-\gamma} = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} z^k \quad (\gamma \in \mathbb{C},\ |z| < 1). \tag{2.10}$$

We omit its details. □

In the same process of analysis as in Theorems 2.1 and 2.2, we can find solutions of the generalized fractional kinetic equations involving the (p, q)-extended τ-confluent hypergeometric function (1.19), which are given in the following two theorems, without their proofs.

Theorem 2.3. *Let* $\mu, \nu, d \in \mathbb{R}^+$, $h \in \mathbb{C}$ *with* $d \neq h$ *and* $\tau \in \mathbb{R}_0^+$. *Also let* $\min\{\Re(p), \Re(q)\} > 0$ *while* $\Re(c) > \Re(b) > 0$ *when* $p = 0 = q$. *Then the solution of the following generalized fractional kinetic equation:*

$$N(t) - N_0\, t^{\mu-1} \Phi_{p,q}^{\tau}(b; c; ht^{\nu}) = -d^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{2.11}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{n=0}^{\infty} \frac{B(b+\tau n, c-b; p, q)}{B(b, c-b)} \frac{\Gamma(\mu+\nu n)\,(ht^{\nu})^n}{n!} E_{\nu,\mu+\nu n}\left(-d^{\nu}\, t^{\nu}\right), \tag{2.12}$$

where $E_{\alpha,\beta}(\cdot)$ *is the generalized Mittag-Leffler function* (1.5).

Theorem 2.4. *Let* $\mu, \nu, d \in \mathbb{R}^+$, γ, $h \in \mathbb{C}$ *with* $d \neq h$ *and* $\tau \in \mathbb{R}_0^+$. *Also let* $\min\{\Re(p), \Re(q)\} > 0$ *while* $\Re(c) > \Re(b) > 0$ *when* $p = 0 = q$. *Then the solution of the following generalized fractional kinetic equation:*

$$N(t) - N_0\, t^{\mu-1} \Phi_{p,q}^{\tau}(b; c; ht^{\nu}) = -\left\{ \sum_{j=1}^{\infty} \binom{\gamma}{j} d^{\nu j}\, {}_0D_t^{-\nu j} \right\} N(t) \tag{2.13}$$

is given by

$$N(t) = N_0\, t^{\mu-1} \sum_{n=0}^{\infty} \frac{B(b+\tau n, c-b; p, q)}{B(b, c-b)} \frac{\Gamma(\mu+\nu n)\,(ht^{\nu})^n}{n!} E_{\nu,\mu+\nu n}^{\gamma}\left(-d^{\nu}\, t^{\nu}\right), \tag{2.14}$$

where $E_{\alpha,\beta}^{\gamma}(\cdot)$ *is the generalized Mittag-Leffler function* (1.6).

3. Concluding remarks

Without depending on Laplace transform and Sumudu transform, which have been used among most of the numerous papers to solve the fractional kinetic equations (1.1) and (1.10), Saxena and Kalla [32] chose an alternative method, which had been employed by Al-Saqabi and Tuan [5].

The results in Theorems 2.2 and 2.4 when $\gamma = 1$ yield, respectively, those in Theorems 2.1 and 2.3. The results in Theorems 2.1, 2.2, 2.3, and 2.4 are, easily, found to reduce to those corresponding ones for the τ-hypergeometric function (1.20) and the τ-confluent hypergeometric function (1.21) when $p = 0 = q$, and for the hypergeometric series ${}_2F_1$ and the confluent hypergeometric series ${}_1F_1$ when $p = 0 = q$ and $\tau = 1$.

Batalov and Batalova [9, Eq. (8)] investigated the following generalized fractional kinetic equation:

$$\zeta\, \partial_t \varphi + {}_0^cD_t^{\alpha} \varphi = \lambda \left(b\, \Delta\varphi - (-\Delta)^{\sigma/2}\varphi - \mathsf{g}\, \varphi^m/m! - \tau\varphi \right) + F, \tag{3.1}$$

where $0 < \alpha \leq 1$ and $0 < \sigma \leq 2$. Here parameters ζ and b represent parts of the Markovian property and short-range interaction property, respectively, m is the power of non-linearity; ${}_0^cD_t^{\alpha}$ is the Caputo fractional derivative (see, e.g., [24]); $\varphi(\mathbf{x}, t)$ is the real-valued field depending on the d-dimensional position vector $\mathbf{x}$ and time t; $\lambda \in \mathbb{R}^+$ is the kinetic coefficient, $\mathsf{g} \in \mathbb{R}^+$ is the coupling constant, which characterizes a vertex of the fluctuation interaction, $\tau \in \mathbb{R}_0^+$ is the deviation from the critical temperature $\tau \sim T - T_c$, and F is a random external force. For many other types of kinetic equations, one may be referred, for example, to [26] and the references therein.

Acknowledgements: The authors would like express their deep-felt thanks for the reviewers' favorable and constructive comments. The third-named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2020R111A1A01052440).

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