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A HILBERT-TYPE INTEGRAL INEQUALITY IN THE FINITE INTERVAL

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Abstract. In this paper, by using the methods of real analysis and functional analysis, a Hilbert-type integral inequality in the finite interval $(0, b)$ $(0 < b < \infty)$ with the homogeneous kernel of $(-\lambda)$-degree and a best constant factor is given. We also consider its operator expression. A few improved results, the equivalent forms and some new inequalities with the particular kernels are obtained.

1. Introduction

Let $f, g(\geq 0) \in L^2(0,\infty), ||f|| = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}$ and $||g|| = \{\int_0^\infty g^2(x)dx\}^{\frac{1}{2}}$. Then we have the following Hilbert's integral inequality [1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \pi ||f|| \cdot ||g||, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is important in analysis and its applications ([1,2]). Define an integral operator $T : L^2(0,\infty) \to L^2(0,\infty)$, for $f(\geq 0) \in L^2(0,\infty)$,

$$T(f)(y) := \int_0^\infty \frac{f(x)}{x+y} dx (y \in (0,\infty)). \tag{1.2}$$

Then inequality (1.1) is rewritten to

$$(Tf, g) \leq \pi ||f|| \cdot ||g||,$$

where

$$(Tf, g) := \int_0^\infty (\int_0^\infty \frac{f(x)}{x+y} dx) g(y) dy$$

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is the inner product of Tf and g. We named of T Hilbert integral operator. By (1.1), we can prove that the equivalent form is

$$||Tf|| \leq \pi ||f||,$$

and conclude that [3]

$$||T|| = \pi.$$

If we replace $\frac{1}{x+y}$ by a bilinear function $k(x,y)(\geq 0)$ in (1.1), then the problem is how to make sure the conditions of $k(x,y)$ for giving an integral operator T as (1.2) and the inequality with a best constant factor as (1.1).

In recent years, Yang [4,5] considered the case of $k(x,y)$ being continuous and symmetric in the function space $L^p(0,\infty)$, Yang [6,7,8] considered the same case of $k(x,y)$ in the disperse space l^p, and Zhong et al. [9] considered the case of $k(x,y)$ in $L^p(R_+^n)$. But their given conditions are not quite simple.

In 1998, by introducing $\lambda \in (0,1]$ and the Beta function $B(u,v)$ as [10]:

$$B(u,v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{-u+1} dt (u,v > 0), \tag{1.3}$$

Yang [11] gave an extension of (1.1) in the subinterval $(0,b)(0 < b < \infty)$ as:

$$\int_0^b \int_0^b \frac{f(x)g(y)dxdy}{(x+y)^\lambda} \leq k_\lambda \{\int_0^b \sigma(x)x^{1-\lambda} f^2(x)dx \int_0^b \sigma(x)x^{1-\lambda} g^2(x)dx\}^{\frac{1}{2}}, \tag{1.4}$$

where $k_\lambda = B(\frac{\lambda}{2}, \frac{\lambda}{2})$ and $\sigma(x) = 1 - \frac{1}{2}(\frac{x}{b})^{\frac{\lambda}{2}}$. When $\lambda = 1, b \to \infty$, inequality (1.4) deduces to (1.1). In recent years, a number of papers studied some improvements and extensions of (1.4) (cf. [12-15]).

In this paper, a simple condition of the homogeneous kernel $k_\lambda(x,y)$ with $(-\lambda)$-degree $(\lambda > 0)$ is considered. By using the methods of real analysis and functional analysis, a Hilbert-type integral inequality in the finite interval $(0,b)$ with the homogeneous kernel of $(-\lambda)$-degree and a best constant factor and its operator expression are given. A few improved results, the equivalent forms and some new inequalities of the particular kernels are obtained.

2. Lemmas and main results

If $\lambda > 0$, the function $k_\lambda(x,y)$ is non-negative measurable in $(0,\infty)\times(0,\infty)$, satisfying $k_\lambda(ux,uy) = u^{-\lambda}k_\lambda(x,y)$ for any $u,x,y > 0$, then we call $k_\lambda(x,y)$ the homogeneous function of $(-\lambda)$-degree. If for any $x,y > 0, k_\lambda(x,y) = k_\lambda(y,x)$, then we call $k_\lambda(x,y)$ the symmetric homogeneous function. Assume that $r > 1, \frac{1}{r} + \frac{1}{s} = 1$. Setting $k_\lambda(r)$ and $\widetilde{k}_\lambda(s)$ as

$$k_\lambda(r) := \int_0^\infty k_\lambda(u,1)u^{\frac{\lambda}{r}-1}du, \ \ \widetilde{k}_\lambda(s) := \int_0^\infty k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du, \tag{2.1}$$

then it follows $k_\lambda(r) = \widetilde{k}_\lambda(s)$. In fact, setting $v = \frac{1}{u}$, we obtain

$$\widetilde{k}_\lambda(s) = \int_0^\infty k_\lambda(1, \frac{1}{v}) v^{\frac{-\lambda}{s}+1} \frac{dv}{v^2} = \int_0^\infty k_\lambda(v, 1) v^{\frac{\lambda}{r}-1} dv = k_\lambda(r).$$

Suppose that $k_\lambda(r)$ is a positive number. For $0 < b < \infty, x, y \in (0, b)$, define the weight functions $\omega_\lambda(r, y, b)$ and $\varpi_\lambda(s, x, b)$ as

$$\omega_\lambda(r, y, b) := \int_0^b k_\lambda(x, y) \frac{y^{\frac{\lambda}{s}}}{x^{1-\frac{\lambda}{r}}} dx, \quad \varpi_\lambda(s, x, b) := \int_0^b k_\lambda(x, y) \frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{s}}} dy. \tag{2.2}$$

Setting $u = y/x$ in the integral $\omega_\lambda(r, y, b)$, for $y \in (0, b)$, we find

$$\omega_\lambda(r, y, b) = \int_{\frac{y}{b}}^\infty k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \leq \int_0^\infty k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du = \widetilde{k}_\lambda(s). \tag{2.3}$$

Similarly, $\varpi_\lambda(s, x, b) \leq k_\lambda(r)$ $(x \in (0, b))$. Setting $\theta_\lambda(r)$ and $\widetilde{\theta}_\lambda(s)$ as

$$\theta_\lambda(r) := \int_0^1 k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du, \quad \widetilde{\theta}_\lambda(s) := \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du. \tag{2.4}$$

If $\theta_\lambda(r), \widetilde{\theta}_\lambda(s) > 0$, then for $0 < y < b$, we find

$$\omega_\lambda(r, y, b) = \int_0^{\frac{b}{y}} k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \geq \int_0^1 k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du = \theta_\lambda(r) > 0.$$

Similarly, $\varpi_\lambda(s, x, a) \geq \widetilde{\theta}_\lambda(s) > 0$ $(0 < x < b)$. By (2.2), for fixed $0 < y < b, k_\lambda(x, y) > 0$ *a.e.* in $(0, b)$, and for fixed $0 < x < b, k_\lambda(x, y) > 0$ *a.e.* in $(0, b)$.

Lemma 2.1. If both $k_\lambda(1, u), k_\lambda(u, 1) \geq l_\lambda > 0, u \in (0, 1]$, then we have

$$\omega_\lambda(r, y, b) \leq k_\lambda(r)[1 - \frac{s l_\lambda}{\lambda k_\lambda(r)} (\frac{y}{b})^{\frac{\lambda}{s}}] \ (y \in (0, b)); \tag{2.5}$$

$$\varpi_\lambda(s, x, b) \leq k_\lambda(r)[1 - \frac{r l_\lambda}{\lambda k_\lambda(r)} (\frac{x}{b})^{\frac{\lambda}{r}}] \ (x \in (0, b)). \tag{2.6}$$

Proof. As (2.3), we find

$$\begin{aligned} \omega_\lambda(r, y, b) &= \int_{\frac{y}{b}}^\infty k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \\ &= k_\lambda(r) - \int_0^{\frac{y}{b}} k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \\ &\leq k_\lambda(r) - l_\lambda \int_0^{\frac{y}{b}} u^{\frac{\lambda}{s}-1} du \\ &= k_\lambda(r) - \frac{s l_\lambda}{\lambda} (\frac{y}{b})^{\frac{\lambda}{s}} \ (y \in (0, b)). \end{aligned}$$

Hence we have (2.5). Similarly we have (2.6). This completes the proof. □

Lemma 2.2. If both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing function in $(0,1]$, then we have

$$\begin{aligned} \omega_\lambda(r,y,b) &\leq k_\lambda(r)[1-\frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}]\ (y\in(0,b)); && (2.7)\\ \varpi_\lambda(s,x,b) &\leq k_\lambda(r)[1-\frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}]\ (x\in(0,b)). && (2.8)\end{aligned}$$

In particular, if $k_\lambda(x,y)$ is symmetric, setting $k_\lambda := k_\lambda(2)$, then

$$\begin{aligned} \omega_\lambda(2,y,b) &\leq k_\lambda[1-\frac{1}{2}(\frac{y}{b})^{\frac{\lambda}{2}}], \varpi_\lambda(2,x,b)\\ &\leq k_\lambda[1-\frac{1}{2}(\frac{x}{b})^{\frac{\lambda}{2}}]. && (2.9)\end{aligned}$$

Proof. Since $k'_\lambda(1,u)\leq 0, u\in(0,1)$, for $y\in(0,1)$, we obtain

$$\begin{aligned} \frac{d}{dy}[y^{\frac{-\lambda}{s}}\int_0^y k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du] &= \frac{-\lambda}{s}y^{\frac{-\lambda}{s}}\int_0^y k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du + k_\lambda(1,y)y^{-1}\\ &= -y^{\frac{-\lambda}{s}}\int_0^y k_\lambda(1,u)du^{\frac{\lambda}{s}} + k_\lambda(1,y)y^{-1}\\ &= y^{\frac{-\lambda}{s}}\int_0^y k'_\lambda(1,u)u^{\frac{\lambda}{s}}du\\ &\leq 0\end{aligned}$$

and

$$y^{\frac{-\lambda}{s}}\int_0^y k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du \geq \int_0^1 k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du = \widetilde{\theta}_\lambda(s).$$

Hence, we find

$$\begin{aligned} \omega_\lambda(r,y,b) &= k_\lambda(r) - [(\frac{y}{b})^{\frac{-\lambda}{s}}\int_0^{\frac{y}{b}} k_\lambda(1,u)u^{\frac{\lambda}{s}-1}du](\frac{y}{b})^{\frac{\lambda}{s}}\\ &\leq k_\lambda(r) - \widetilde{\theta}_\lambda(s)(\frac{y}{b})^{\frac{\lambda}{s}}\ (y\in(0,b)).\end{aligned}$$

Hence we obtain (2.7). Similarly, we obtain (2.8). If $k_\lambda(x,y)$ is symmetric, then we find $\theta_\lambda(2) = \widetilde{\theta}_\lambda(2)$ and

$$\begin{aligned} k_\lambda &= \theta_\lambda(2) + \int_1^\infty k_\lambda(1,u)u^{\frac{\lambda}{2}-1}du\\ &= \theta_\lambda(2) + \int_0^1 k_\lambda(v,1)v^{\frac{\lambda}{2}-1}dv\\ &= 2\theta_\lambda(2).\end{aligned}$$

Then by (2.7) and (2.8), we have (2.9). The Lemma is proved. □

For a measurable function $\varphi(x) > 0$, we set the function spaces as

$$L_{\varphi}^{\rho}(0,b) := \{h \geq 0; ||h||_{\rho,\varphi} := \{\int_0^b \varphi(x)h^{\rho}(x)dx\}^{\frac{1}{\rho}} < \infty\}(\rho = p, q).$$

Theorem 2.3. Assume that $p, r > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $\lambda > 0$, $k_{\lambda}(x,y)$ is a homogeneous function of $(-\lambda)$-degree in $(0,\infty)\times(0,\infty)$, satisfying $k_{\lambda}(r), \theta_{\lambda}(r)$ and $\widetilde{\theta}_{\lambda}(s)$ are positive numbers. For $0 < b < \infty$, there exist measurable functions $\kappa(y)$ and $\widetilde{\mu}(x)$, such that $0 < \kappa(y), \widetilde{\mu}(x) \leq 1$ and

$$\begin{aligned} \omega_{\lambda}(r,y,b) &\leq k_{\lambda}(r)\kappa(y), \varpi_{\lambda}(s,x,b) \\ &\leq k_{\lambda}(r)\widetilde{\mu}(x)(x,y \in (0,b)). \end{aligned} \tag{2.10}$$

If $\phi_r(x) := x^{p(1-\frac{\lambda}{r})-1}, \psi_s(x) := x^{q(1-\frac{\lambda}{s})-1}(x \in (0,b))$, $f \in L_{\phi_r}^{p}(0,b), g \in L_{\psi_s}^{q}(0,b), ||f||_{p,\phi_r}, ||g||_{q,\psi_s} > 0$, then we have the equivalent inequalities as

$$\begin{aligned} I_{\lambda}(b) : &= \int_0^b \int_0^b k_{\lambda}(x,y)f(x)g(y)dxdy \\ &< k_{\lambda}(r)||f||_{p,\widetilde{\mu}\cdot\phi_r}||g||_{q,\kappa\cdot\psi_s}; \end{aligned} \tag{2.11}$$

$$\begin{aligned} J_{\lambda}(b) : &= \int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_0^b k_{\lambda}(x,y)f(x)dx)^p dy \\ &< k_{\lambda}^{p}(r)||f||_{p,\widetilde{\mu}\cdot\phi_r}^{p}, \end{aligned} \tag{2.12}$$

where the constant factors $k_{\lambda}(r)$ and $k_{\lambda}^{p}(r)$ are the best possible. In particular, for $\kappa(y) = \widetilde{\mu}(x) = 1$, we have the equivalent inequalities as

$$I_{\lambda}(b) < k_{\lambda}(r)\{\int_0^b x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\}^{\frac{1}{q}}; \tag{2.13}$$

$$\int_0^b y^{\frac{p\lambda}{s}-1}(\int_0^b k_{\lambda}(x,y)f(x)dx)^p dy < k_{\lambda}^{p}(r)\int_0^b x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx. \tag{2.14}$$

Proof. Since $0 < \theta_{\lambda}(r)/k_{\lambda}(r) \leq \kappa(y) \leq k_{\lambda}(r), 0 < \widetilde{\theta}_{\lambda}(s)/k_{\lambda}(r) \leq \widetilde{\mu}(x) \leq k_{\lambda}(r)(x,y \in (0,b))$, it is obvious that the condition

$$0 < ||f||_{p,\phi_r}, ||g||_{q,\psi_s} < \infty$$

is equivalent to the condition

$$0 < ||f||_{p,\widetilde{\mu}\cdot\phi_r}, ||g||_{q,\kappa\cdot\psi_s} < \infty.$$

By H$\ddot{o}$lder's inequality [16], in view of (2.2) and Fubini's theorem [17], we find

$$\begin{aligned} I_\lambda(b) &= \int_0^b \int_0^b k_\lambda(x,y)[\frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}} f(x)][\frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}} g(y)]dxdy \\ &\leq \{\int_0^b \int_0^b k_\lambda(x,y)\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}} f^p(x)dxdy\}^{\frac{1}{p}} \qquad (2.15) \\ &\quad \times\{\int_0^b \int_0^b k_\lambda(x,y)\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}} g^q(y)dxdy\}^{\frac{1}{q}} \\ &= \{\int_0^b \varpi_\lambda(s,x,b)\phi_r(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b \omega_\lambda(r,y,b)\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}. \end{aligned}$$

If inequality (2.15) keeps the form of equality, then, there exist constants A and B, such that they are not all zero and

$$A\frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}} f^p(x) = B\frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}} g^q(y) \ \ a.e. \text{ in } (0,b)\times(0,b).$$

It follows $Ax^{p(1-\frac{\lambda}{r})}f^p(x) = By^{q(1-\frac{\lambda}{s})}g^q(y)$ $a.e.$ in $(0,b)\times(0,b)$. Assuming that $A \neq 0$, there exists $0 < y < b$, such that

$$x^{p(1-\frac{\lambda}{r})-1}f^p(x) = [By^{q(1-\frac{\lambda}{s})}g^q(y)]\frac{1}{Ax}, a.e. \text{ in } x \in (0,b).$$

This contradicts the fact that $0 < ||f||_{p,\phi_r} < \infty$. Then inequality (2.15) keeps the strict form and inequality (2.11) is valid by using (2.10).

For $x \in (0,b)$, setting a bounded measurable function $[f(x)]_n$ as

$$[f(x)]_n := \min\{f(x), n\} = \begin{cases} f(x), \text{ for } f(x) \leq n \\ n, \text{ for } f(x) > n, \end{cases}$$

then, since $||f||_{p,\phi_r} > 0$, there exists $n_0 \in N$, such that

$$\int_{\frac{1}{n}}^b \phi_r(x)[f(x)]_n^p dx > 0 (n \geq n_0),$$

and then

$$\int_{\frac{1}{n}}^b \widetilde{\mu}(x)\phi_r(x)[f(x)]_n^p dx > 0.$$

Setting $\widetilde{g}_n(y)$ as

$$\widetilde{g}_n(y) := \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_{\frac{1}{n}}^b k_\lambda(x,y)[f(x)]_n dx)^{p-1} \, (y \in (\frac{1}{n}, b); n \geq n_0),$$

then by (2.11), we have

$$\begin{aligned} 0 &< \int_{\frac{1}{n}}^{b} \kappa(y)\psi_s(y)\widetilde{g}_n^q(y)dy \qquad (2.16)\\ &= \int_{\frac{1}{n}}^{b} \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_{\frac{1}{n}}^{b} k_\lambda(x,y)[f(x)]_n dx)^p dy \\ &= \int_{\frac{1}{n}}^{b}\int_{\frac{1}{n}}^{b} k_\lambda(x,y)[f(x)]_n \widetilde{g}_n(y)dxdy \\ &< k_\lambda(r)\{\int_{\frac{1}{n}}^{b} \widetilde{\mu}(x)\phi_r(x)[f(x)]_n^p dx\}^{\frac{1}{p}}\{\int_{\frac{1}{n}}^{b} \kappa(y)\psi_s(y)\widetilde{g}_n^q(y)dy\}^{\frac{1}{q}} \\ &< \infty \end{aligned}$$

and

$$\begin{aligned} 0 &< \int_{\frac{1}{n}}^{b} \kappa(y)\psi_s(y)\widetilde{g}_n^q(y)dy \qquad (2.17)\\ &< k_\lambda^p(r)\int_0^b \widetilde{\mu}(x)\phi_r(x)f^p(x)dx \\ &< \infty. \end{aligned}$$

It follows $0 < ||g||_{q,\kappa\cdot\psi_s} < \infty$ and then $0 < ||g||_{q,\psi_s} < \infty$. For $n \to \infty$, by (2.11), both (2.16) and (2.17) still keep the forms of strict inequality. Hence we have (2.12).

On the other-hand, suppose that (2.12) is valid. By Hölder's inequality,

$$\begin{aligned} J_\lambda(b) &= \int_0^b [y^{\frac{-1}{p}+\frac{\lambda}{s}}\kappa^{\frac{-1}{q}}(y)\int_0^b k_\lambda(x,y)f(x)dx][\kappa^{\frac{1}{q}}(y)y^{\frac{1}{p}-\frac{\lambda}{s}}g(y)]dy \qquad (2.18)\\ &\le \{\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)}(\int_0^b k_\lambda(x,y)f(x)dx)^p dy\}^{\frac{1}{p}}\{\int_0^b \kappa(y)\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}. \end{aligned}$$

In view of (2.12), we have (2.11). Hence (2.11) is equivalent to (2.12).

For $n \in N, n > \max\{\frac{\lambda}{r}, \frac{\lambda}{s}\}$, setting f_n, g_n as

$$f_n(x) = x^{\frac{\lambda}{r}+\frac{1}{np}-1}$$

and

$$g_n(x) = x^{\frac{\lambda}{s}+\frac{1}{nq}-1},$$

for $x \in (0,b)$, if there exists $0 < K \le k_\lambda(r)$, such that (2.11) is still valid if we replace $k_\lambda(r)$ by K, then we have

$$I_\lambda^{(n)}(b) := \int_0^b \int_0^b k_\lambda(x,y) f_n(x) g_n(y) dx dy \tag{2.19}$$
$$< K||f_n||_{p,\cdot\phi_r}||g_n||_{q,\psi_s}$$
$$= nKb^{\frac{1}{n}}$$

and

$$\begin{aligned} I_\lambda^{(n)}(b) &= \int_0^b [\int_0^b k_\lambda(x,y) x^{\frac{\lambda}{r}+\frac{1}{np}-1} y^{\frac{\lambda}{s}+\frac{1}{nq}-1} dx] dy \\ &= \int_0^b y^{\frac{1}{n}-1} [\int_0^{\frac{b}{y}} k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du] dy (u = x/y) \quad (2.20) \\ &= nb^{\frac{1}{n}} \int_0^1 k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du + \int_0^b y^{\frac{1}{n}-1} \int_1^{\frac{b}{y}} k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du dy \\ &= nb^{\frac{1}{n}} \int_0^1 k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du + \int_1^\infty (\int_0^{\frac{b}{u}} y^{\frac{1}{n}-1} dy) k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du \\ &= nb^{\frac{1}{n}} [\int_0^1 k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du + \int_1^\infty k_\lambda(u,1) u^{\frac{\lambda}{r}-\frac{1}{nq}-1} du]. \end{aligned}$$

Hence by (2.19) and (2.20), we have

$$\int_0^1 k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du + \int_1^\infty k_\lambda(u,1) u^{\frac{\lambda}{r}-\frac{1}{nq}-1} du < K$$

and by Fatou's Lemma [17], it follows

$$\begin{aligned} k_\lambda(r) &= \int_0^1 \lim_{n\to\infty} k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du + \int_1^\infty \lim_{n\to\infty} k_\lambda(u,1) u^{\frac{\lambda}{r}-\frac{1}{nq}-1} du \\ &\le \underline{\lim_{n\to\infty}} [\int_0^1 k_\lambda(u,1) u^{\frac{\lambda}{r}+\frac{1}{np}-1} du + \int_1^\infty k_\lambda(u,1) u^{\frac{\lambda}{r}-\frac{1}{nq}-1} du] \\ &\le K. \end{aligned}$$

Therefor $K = k_\lambda(r)$ is the best constant factor of (2.11). If the constant factor in (2.12) is not the best possible, then by (2.18), we can get a contradiction that the constant factor in (2.11) is not the best possible. This completes the proof. □

Define an operator $T_b : L^p_{\phi_r}(0,b) \to L^p_{\psi_s^{1-p}}(0,b)$ as: for $f \in L^p_{\phi_r}(0,b)$,

$$(T_b f)(y) := \int_0^b k_\lambda(x,y) f(x) dx (y \in (0,b)). \tag{2.21}$$

In view of (2.14), it follows $T_b f \in L^p_{\psi_s^{1-p}}(0,b)$. For $g \in L^q_{\psi_s}(0,b)$, define the formal inner of $T_b f$ and g as

$$(T_b f, g) := \int_0^b \int_0^b k_\lambda(x,y) f(x) g(y) dx dy. \tag{2.22}$$

Hence the equivalent inequalities (2.13) and (2.14) may be rewritten as

$$(T_b f, g) < k_\lambda(r) ||f||_{p,\phi_r} ||g||_{q,\psi_s}; ||T_b f||_{p,\psi_s^{1-p}} < k_\lambda(r) ||f||_{p,\phi_r}, \tag{2.23}$$

where the constant factor $k_\lambda(r)$ is the best possible, T_b is obviously bounded and $||T_b|| = k_\lambda(r)$. We call T_b Hilbert-type integral operator with the homogeneous kernel of $(-\lambda)$-degree in the finite interval $(0,b)$.

Corollary 2.4. As the assumption of Theorem 1, if both $k_\lambda(1,u), k_\lambda(u,1) \geq l_\lambda > 0$, $u \in (0,1]$, we have the following equivalent inequalities:

$$\begin{aligned} (T_b f, g) \quad &< \quad k_\lambda(r) \{\int_0^b [1 - \frac{r l_\lambda}{\lambda k_\lambda(r)} (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx\}^{\frac{1}{p}} \\ &\quad \times \{\int_0^b [1 - \frac{s l_\lambda}{\lambda k_\lambda(r)} (\frac{y}{b})^{\frac{\lambda}{s}}] \psi_s(y) g^q(y) dy\}^{\frac{1}{q}} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{s l_\lambda}{\lambda k_\lambda(r)} (\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}} (\int_0^b k_\lambda(x,y) f(x) dx)^p dy \\ < \quad & k_\lambda^p(r) \int_0^b [1 - \frac{r l_\lambda}{\lambda k_\lambda(r)} (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx, \end{aligned} \tag{2.25}$$

where the constant factors $k_\lambda(r)$ and $k_\lambda^p(r)$ are the best possible. We still have the following two pairs of equivalent inequalities:

$$(T_b f, g) < k_\lambda(r) \{\int_0^b [1 - \frac{r l_\lambda}{\lambda k_\lambda(r)} (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx\}^{\frac{1}{p}} ||g||_{q,\psi_s}, \tag{2.26}$$

$$||T_b f||^p_{p,\psi_s^{1-p}} < k_\lambda^p(r) \int_0^b [1 - \frac{r l_\lambda}{\lambda k_\lambda(r)} (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx, \tag{2.27}$$

$$(T_b f, g) < k_\lambda(r) ||f||_{p,\phi_r} \{\int_0^b [1 - \frac{s l_\lambda}{\lambda k_\lambda(r)} (\frac{y}{b})^{\frac{\lambda}{s}}] \psi_s(y) g^q(y) dy\}^{\frac{1}{q}}, \tag{2.28}$$

$$\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{s l_\lambda}{\lambda k_\lambda(r)} (\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}} (\int_0^b k_\lambda(x,y) f(x) dx)^p dy < k_\lambda^p(r) ||f||^p_{p,\phi_r}. \tag{2.29}$$

Proof. By Lemma 2.1, setting $\kappa(y)=1-\frac{sl_\lambda}{\lambda k_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}$ and $\widetilde{\mu}(x)=1-\frac{rl_\lambda}{\lambda k_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}$ in (2.11) and (2.12), we have (2.24) and (2.25). Since $\kappa(y)\le 1$, by (2.24) and (2.25), we have (2.26) and (2.27). Similarly, since $\widetilde{\mu}(x)\le 1$, we have (2.28) and (2.29). The Corollary is proved. □

Corollary 2.5. As the assumption of Theorem 2.3, if both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing function in $(0,1]$, then we have the following equivalent inequalities with the best constant factors:

$$\begin{aligned}(T_bf,g) &< k_\lambda(r)\{\int_0^b[1-\frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}]\phi_r(x)f^p(x)dx\}^{\frac{1}{p}}\\ &\quad\times\{\int_0^b[1-\frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}]\psi_s(y)g^q(y)dy\}^{\frac{1}{q}} \qquad (2.30)\end{aligned}$$

and

$$\begin{aligned}&\int_0^b\frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}}(\int_0^b k_\lambda(x,y)f(x)dx)^pdy\\ &< k_\lambda^p(r)\int_0^b[1-\frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}]\phi_r(x)f^p(x)dx. \qquad (2.31)\end{aligned}$$

If $k_\lambda(x,y)$ is symmetric and $\sigma(x)=1-\frac{1}{2}(\frac{x}{b})^{\frac{\lambda}{2}}$ (as (1.4)), then we have the equivalent inequalities as

$$I_\lambda(b)<k_\lambda\{\int_0^b\sigma(x)\phi_2(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b\sigma(y)\psi_2(y)g^q(y)dy\}^{\frac{1}{q}} \qquad (2.32)$$

and

$$\int_0^b\frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)}(\int_0^b k_\lambda(x,y)f(x)dx)^pdy<k_\lambda^p\int_0^b\sigma(x)\phi_2(x)f^p(x)dx. \qquad (2.33)$$

Proof. By Lemma 2.2, setting $\kappa(y)=1-\frac{\widetilde{\theta}_\lambda(s)}{k_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}$ and $\widetilde{\mu}(x)=1-\frac{\theta_\lambda(r)}{k_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}$ in (2.11) and (2.12), we have (2.30) and (2.31). For $r=s=2, k_\lambda=k_\lambda(2)$, by (2.9), we have (2.32) and (2.33). The Corollary is proved. □

3. Applications to some particular kernels

In the following, some words that $b,\lambda>0, p,r>1, \frac{1}{p}+\frac{1}{q}=1, \frac{1}{r}+\frac{1}{s}=1$, $\phi_r(x):=x^{p(1-\frac{\lambda}{r})-1}, \psi_s(x):=x^{q(1-\frac{\lambda}{s})-1}$, $\sigma(x)=1-\frac{1}{2}(\frac{x}{b})^{\frac{\lambda}{2}}$, $f,g\ge 0$, $0<||f||_{p,\phi_r}, ||g||_{q,\psi_s}<\infty$ and the constants are the best possible are omitted.

Example 3.1. Let $k_\lambda(x,y) = \frac{1}{(x^\alpha+y^\alpha)^{\lambda/\alpha}}$ $(\alpha > 0)$, which is symmetric. Since both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing in $(0,1]$, and $k_\lambda(u,1) = \frac{1}{(u^\alpha+1)^{\lambda/\alpha}} \geq l_\lambda = \frac{1}{2^{\lambda/\alpha}}$ $(u \in (0,1])$, setting $v = u^\alpha$, by (1.3), we have

$$\widetilde{k}_\lambda(r) := \int_0^\infty \frac{u^{\frac{\lambda}{r}-1}du}{(u^\alpha+1)^{\frac{\lambda}{\alpha}}} = \frac{1}{\alpha}\int_0^\infty \frac{v^{\frac{\lambda}{\alpha r}-1}dv}{(v+1)^{\frac{\lambda}{\alpha}}} = \frac{1}{\alpha}B(\frac{\lambda}{\alpha r},\frac{\lambda}{\alpha s}). \tag{3.1}$$

By (28),(29) and (36), (37), we have two pairs of equivalent inequalities as:

$$\begin{aligned} H: &= \int_0^b\int_0^b \frac{f(x)g(y)dxdy}{(x^\alpha+y^\alpha)^{\frac{\lambda}{\alpha}}} \\ &< \widetilde{k}_\lambda(r)\{\int_0^b [1-\frac{r}{2^{\frac{\lambda}{\alpha}}\lambda\widetilde{k}_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}]\phi_r(x)f^p(x)dx\}^{\frac{1}{p}} \\ &\quad \times\{\int_0^b [1-\frac{s}{2^{\frac{\lambda}{\alpha}}\lambda\widetilde{k}_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}]\psi_s(y)g^q(y)dy\}^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{s}{2^{\lambda/\alpha}\lambda\widetilde{k}_\lambda(r)}(\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}}[\int_0^b \frac{f(x)dx}{(x^\alpha+y^\alpha)^{\frac{\lambda}{\alpha}}}]^p dy \\ &< \widetilde{k}_\lambda^p(r)\int_0^b [1-\frac{r}{2^{\frac{\lambda}{\alpha}}\lambda\widetilde{k}_\lambda(r)}(\frac{x}{b})^{\frac{\lambda}{r}}]\phi_r(x)f^p(x)dx, \end{aligned} \tag{3.3}$$

$$H < \widetilde{k}_\lambda(2)\{\int_0^b \sigma(x)\phi_2(x)f^p(x)dx\}^{\frac{1}{p}}\{\int_0^b \sigma(y)\psi_2(y)g^q(y)dy\}^{\frac{1}{q}}, \tag{3.4}$$

$$\int_0^b \frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)}[\int_0^b \frac{f(x)dx}{(x^\alpha+y^\alpha)^{\frac{\lambda}{\alpha}}}]^p dy < \widetilde{k}_\lambda^p(2)\int_0^b \sigma(x)\phi_2(x)f^p(x)dx. \tag{3.5}$$

Example 3.2. Let $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda-y^\lambda}$, which is symmetric. We find that both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing in $(0,1]$[18], and $k_\lambda(u,1) = \frac{\ln u}{u^\lambda-1} \geq l_\lambda = \frac{1}{\lambda}$ $(u \in (0,1])$. Setting $v = u^\lambda$, we obtain [1]

$$k_\lambda(r) = \int_0^\infty \frac{(\ln u)u^{\frac{\lambda}{r}-1}}{u^\lambda-1}du = \int_0^\infty \frac{(\ln v)v^{\frac{1}{r}-1}}{\lambda^2(v-1)}dv = [\frac{\pi}{\lambda\sin(\frac{\pi}{r})}]^2. \tag{3.6}$$

By (2.24),(2.25) and (2.32), (2.33), we have two pairs of equivalent inequalities as

$$\begin{aligned} H' &:= \int_0^b \int_0^b \frac{\ln(\frac{x}{y})f(x)g(y)}{x^\lambda - y^\lambda} dxdy \\ &< [\frac{\pi}{\lambda \sin(\frac{\pi}{r})}]^2 \{\int_0^b [1 - r[\frac{\sin(\frac{\pi}{r})}{\pi}]^2 (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx\}^{\frac{1}{p}} \\ &\quad \times \{\int_0^b [1 - s[\frac{\sin(\frac{\pi}{r})}{\pi}]^2 (\frac{y}{b})^{\frac{\lambda}{s}}] \psi_s(y) g^q(y) dy\}^{\frac{1}{q}}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{[1 - s[\frac{\sin(\frac{\pi}{r})}{\pi}]^2 (\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}} (\int_0^b \frac{\ln(\frac{x}{y}) f(x)}{x^\lambda - y^\lambda} dx)^p dy \\ &< [\frac{\pi}{\lambda \sin(\frac{\pi}{r})}]^{2p} \int_0^b [1 - r[\frac{\sin(\frac{\pi}{r})}{\pi}]^2 (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx, \end{aligned} \tag{3.8}$$

$$H' < (\frac{\pi}{\lambda})^2 \{\int_0^b \sigma(x) \phi_2(x) f^p(x) dx\}^{\frac{1}{p}} \{\int_0^b \sigma(y) \psi_2(y) g^q(y) dy\}^{\frac{1}{q}}, \tag{3.9}$$

$$\int_0^b \frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)} [\int_0^b \frac{\ln(\frac{x}{y}) f(x) dx}{x^\lambda - y^\lambda}]^p dy < (\frac{\pi}{\lambda})^{2p} \int_0^b \sigma(x) \phi_2(x) f^p(x) dx. \tag{3.10}$$

Example 3.3. Let $k_\lambda(x,y) = \frac{1}{(\max\{x,y\})^\lambda}$, which is symmetric. Since both $k_\lambda(1,u)$ and $k_\lambda(u,1)$ are derivable decreasing in $(0,1]$, and $k_\lambda(u,1) = l_\lambda = 1$ ($u \in (0,1]$), we have

$$\begin{aligned} k_\lambda(r) &= \int_0^\infty \frac{u^{\frac{\lambda}{r}-1}}{(\max\{u,1\})^\lambda} du \\ &= \int_0^1 u^{\frac{\lambda}{r}-1} du + \int_1^\infty \frac{u^{\frac{\lambda}{r}-1}}{u^\lambda} du = \frac{rs}{\lambda}. \end{aligned} \tag{3.11}$$

Then by (2.14) and (2.25), we have two equivalent inequalities as follows:

$$\begin{aligned} \int_0^b \int_0^b \frac{f(x)g(y)dxdy}{(\max\{x,y\})^\lambda} &< \frac{rs}{\lambda} \{\int_0^b [1 - \frac{1}{s}(\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx\}^{\frac{1}{p}} \\ &\quad \times \{\int_0^b [1 - \frac{1}{r}(\frac{y}{b})^{\frac{\lambda}{s}}] \psi_s(y) g^q(y) dy\}^{\frac{1}{q}}, \end{aligned} \tag{3.12}$$

$$\begin{aligned}&\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{[1-\frac{1}{r}(\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}}[\int_0^b \frac{f(x)dx}{(\max\{x,y\})^{\lambda}}]^p dy \\ < \; & (\frac{rs}{\lambda})^p \int_0^b [1-\frac{1}{s}(\frac{x}{b})^{\frac{\lambda}{r}}]\phi_r(x) f^p(x)dx. \qquad (3.13)\end{aligned}$$

Example 3.4. Let $k_\lambda(x,y) = \frac{|\ln(x/y)|}{(\max\{x,y\})^\lambda}$, which is symmetric. We find that

$$\begin{aligned}k_\lambda(r) &= \int_0^\infty \frac{|\ln u| u^{\frac{\lambda}{r}-1}du}{(\max\{u,1\})^\lambda} \\ &= \int_0^1 (-\ln u)u^{\frac{\lambda}{r}-1}du + \int_1^\infty \frac{(\ln u)u^{\frac{\lambda}{r}-1}}{u^\lambda}du \\ &= \frac{r^2+s^2}{\lambda^2},\end{aligned}$$

$$\begin{aligned}\omega_\lambda(r,y,b) &= \int_{\frac{y}{b}}^\infty \frac{|\ln u| u^{\frac{\lambda}{s}-1}}{(\max\{u,1\})^\lambda}du \\ &= \frac{r^2+s^2}{\lambda^2} - \int_0^{\frac{y}{b}} (-\ln u)u^{\frac{\lambda}{s}-1}du \\ &= \frac{r^2+s^2}{\lambda^2} - \frac{s}{\lambda}\int_0^{\frac{y}{b}} (-\ln u)du^{\frac{\lambda}{r}} \\ &\leq \frac{r^2+s^2}{\lambda^2}\kappa(y),\end{aligned}$$

$$\kappa(y) = 1 - \frac{s^2}{r^2+s^2}(\frac{y}{b})^{\frac{\lambda}{s}},$$

$$\begin{aligned}\varpi_\lambda(s,x,b) &\leq \frac{r^2+s^2}{\lambda^2}\widetilde{\mu}(x), \widetilde{\mu}(x) \\ &= 1 - \frac{r^2}{r^2+s^2}(\frac{x}{b})^{\frac{\lambda}{r}}.\end{aligned}$$

Then by (2.11) and (2.12), we have two equivalent inequalities as follows:

$$\begin{aligned}
&\int_0^b \int_0^b \frac{|\ln(\frac{x}{y})| f(x) g(y)}{(\max\{x,y\})^{\lambda}} dx dy \\
< \ & \frac{r^2+s^2}{\lambda^2} \{ \int_0^b [1 - \frac{r^2}{r^2+s^2} (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx \}^{\frac{1}{p}} \\
& \times \{ \int_0^b [1 - \frac{s^2}{r^2+s^2} (\frac{y}{b})^{\frac{\lambda}{s}}] \psi_s(y) g^q(y) dy \}^{\frac{1}{q}}, \qquad (3.14)
\end{aligned}$$

$$\begin{aligned}
&\int_0^b \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{s^2}{r^2+s^2}(\frac{y}{b})^{\frac{\lambda}{s}}]^{p-1}} [\int_0^b \frac{|\ln(\frac{x}{y})| f(x) dx}{(\max\{x,y\})^{\lambda}}]^p dy \\
< \ & (\frac{r^2+s^2}{\lambda^2})^p \int_0^b [1 - \frac{r^2}{r^2+s^2} (\frac{x}{b})^{\frac{\lambda}{r}}] \phi_r(x) f^p(x) dx. \qquad (3.15)
\end{aligned}$$

Remarks.

(i) For $\alpha = 1, p = q = 2$, inequality (3.4) deduces to (1.2).

(ii) Inequality (2.11) is a refinement of (2.13), because of

$$\begin{aligned}
I_\lambda(a) \ &< \ k_\lambda(r) ||f||_{p,\widetilde{\mu}\cdot\phi_r} ||g||_{q,\kappa\cdot\psi_s} \\
&\leq \ k_\lambda(r) ||f||_{p,\phi_r}, ||g||_{q,\psi_s}.
\end{aligned}$$

(iii) When $b \to \infty$, (2.13) deduces to a Hilbert-type integral inequality in $(0, \infty)$ with a best constant factor $k_\lambda(r)$ as

$$I_\lambda(0) \leq k_\lambda(r) \{ \int_0^\infty \phi_r(x) f^p(x) dx \}^{\frac{1}{p}} \{ \int_0^\infty \psi_s(y) g^q(y) dy \}^{\frac{1}{q}}. \qquad (3.16)$$

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