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STABILITY OF PERTURBED ITERATIVE ALGORITHM FOR SOLVING A SYSTEM OF GENERALIZED NONLINEAR EQUATIONS

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Abstract. In this paper, we introduce and study a new system of strongly nonlinear quasi-variational inclusions involving generalized m-accretive mappings in Banach spaces. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang, we prove the existence theorem of the solution for this system of variational inclusions in Banach spaces, and discuss the convergence and stability of a new perturbed iterative algorithm for solving this system of nonlinear variational inclusions in Banach spaces. Our results improve and generalize the corresponding results of [3, 6, 9, 12].

1. Introduction

In this paper, we introduce and study the following new system of strongly nonlinear quasi-variational inclusion involving generalized m-accretive mappings:

Find $(x, y) \in X_1 \times X_2$ such that

$$0 \in N_1(x, y) + M_1(x), \quad 0 \in N_2(x, y) + M_2(y), \tag{1.1}$$

where X_1 and X_2 are two real Banach spaces, $N_1 : X_1 \times X_2 \to X_1$ and $N_2 : X_1 \times X_2 \to X_2$ are single-valued mappings and for $i = 1, 2$, $M_i : X_i \to 2^{X_i}$ is a generalized m-accretive mapping, 2^{X_i} denotes the family of all the nonempty subsets of X_i.

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We remark that for a suitable choice of the mappings $N_1, N_2, \eta_1, \eta_2, M_1, M_2$ and the spaces X_1, X_2, a number of known new classes variational inequalities, variational inclusions and corresponding optimization problems can be obtained as special cases of nonlinear quasi-variational inclusion problem (1.1). Moreover, these classes variational inclusions provide us a general and unified framework for studying a wide range of interesting and important problems arising in mathematics, physics, engineering sciences and economics finance, etc. See for more details [1, 3, 4, 6, 9, 14, 15, 17] and the references therein.

In 2001, Huang and Fang [7] first introduced the concept of a generalized m-accretive mapping, which is a generalization of an m-accretive mapping, and gave the definition and properties of the resolvent operator for the generalized m-accretive mapping in Banach space. Further, Bi et al. [2], Huang [5] and Huang et al. [8] introduced and studied some new class of nonlinear variational inclusions involving generalized m-accretive mappings in Banach spaces, they also obtained some new corresponding existence and convergence results (see, for example, [2, 5, 8], respectively. Moreover, Huang, Lan, Zeng, Wang et al. discussed stability of the iterative sequence generated by the algorithm for solving what they studied (see [6, 9, 16, 17]).

On the other hand, Lan et al. [10, 11] introduces and studied a new system of generalized nonlinear variational inclusions involving generalized m-accretive mappings. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang [7], we also prove the existence theorems of the solution and convergence theorems of the generalized Mann iterative procedures with mixed errors for this system of variational inclusions in q-uniformly smooth Banach spaces.

Motivated and inspired by the above works, the main purpose of this paper is to introduce and study the new system of strongly nonlinear quasi-variational inclusions (1.1) involving generalized m-accretive mapping in Banach spaces. By using the resolvent operator technique for generalized m-accretive mapping due to Huang and Fang, we prove the existence theorem of the solution for this kind of variational inclusions in Banach spaces, and discuss the convergence and stability of a new perturbed iterative algorithm for solving this system of nonlinear variational inclusions in Banach spaces. Our results improve and generalize the corresponding results of [3, 6, 9, 12].

2. Preliminaries

Throughout this paper, let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q,\ \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex (see, for example, [13]). If $X = H$ is a Hilbert space, then J_2 becomes the identity mapping of H. In what follows we shall denote the single-valued generalized duality mapping by j_q.

Definition 2.1. The mapping $N : X \times X \to X$ is said to be

(1) σ-strongly accretive with respect to the first argument, if for any $x, y \in X$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle N(x, \cdot) - N(y, \cdot), j_q(x-y) \rangle \geq \sigma \|x-y\|^q,$$

where $\sigma > 0$ is a constant;

(2) ϵ-Lipschitz continuous with respect to the first argument, if there exists a constant $\epsilon > 0$ such that

$$\|N(x, \cdot) - N(y, \cdot)\| \leq \epsilon \|x-y\|, \quad \forall x, y \in X.$$

Similarly, we can define the strongly accretivity and Lipschitz continuity in the second argument of $N(\cdot, \cdot)$, respectively.

Definition 2.2. ([7]) Let $\eta : X \times X \to X^*$ be a single-valued mapping and $A : X \to 2^X$ be a multi-valued mapping. Then A is said to be

(1) η-accretive if

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad \forall x, y \in X, u \in A(x), v \in A(y);$$

(2) generalized m-accretive if A is η-accretive and $(I + \lambda A)(X) = X$ for all (equivalently, for some) $\lambda > 0$.

Remark 2.3. Huang and Fang gave one example of the generalized m-accretive mapping in [7]. If $X = X^* = H$ is a Hilbert space, then (1) and (2) of Definition 2.2 reduce to the definition of η-monotonicity and maximal η-monotonicity respectively; if X is uniformly smooth and $\eta(x, y) = J_2(x-y)$, then (1) and (2) of Definition 2.2 reduce to the definitions of accretivity and m-accretivity in uniformly smooth Banach spaces, respectively (see [7, 8]).

Definition 2.4. The mapping $\eta : X \times X \to X^*$ is said to be

(1) δ-strongly monotone, if there exists a constant $\delta > 0$ such that

$$\langle x - y, \eta(x, y) \rangle \geq \delta \|x-y\|^2, \quad \forall x, y \in X;$$

(2) τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x-y\|, \quad \forall x, y \in X.$$

The modules of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}\|x+y\| + \|x-y\| - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called *uniformly smooth* if $\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0$ and X is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_X \leq ct^q$, where $q > 1$ is a real number.

It is well known that Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces $W^{m,p}$, $1 < p < \infty$, are all q-uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [13] proved the following result:

Lemma 2.5. *Let $q > 1$ be a given real number and X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$, $j_q(x) \in J_q(x)$, there holds the following inequality*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

In [7], Huang and Fang show that for any $\rho > 0$, inverse mapping $(I+\rho A)^{-1}$ is single-valued, if $\eta : X \times X \to X^*$ is strict monotone and $A : X \to 2^X$ is a generalized m-accretive mapping, where I is the identity mapping. Based on this fact, Huang and Fang [7] gave the following definition:

Definition 2.6. Let $A : X \to 2^X$ be a generalized m-accretive mapping. Then the resolvent operator J_A^ρ for A is defined as follows:

$$J_A^\rho(z) = (I + \rho A)^{-1}(z), \quad \forall z \in X,$$

where $\rho > 0$ is a constant and $\eta : X \times X \to X^*$ is a strictly monotone mapping.

Lemma 2.7 ([7, 8])**.** *Let $\eta : X \times X \to X^*$ be τ-Lipschitz continuous and δ-strongly monotone, and $A : X \to 2^X$ be a generalized m-accretive mapping. Then for any $\rho > 0$, the resolvent operator J_A^ρ for A is $\frac{\tau}{\delta}$-Lipschitz continuous, i.e.,*

$$\|J_A^\rho(x) - J_A^\rho(y)\| \leq \frac{\tau}{\delta}\|x-y\|, \quad \forall x, y \in X.$$

3. Existence Theorem

In this section, we shall give the existence theorems of problem (1.1). The solvability of the problem (1.1) depends on the equivalence between (1.1) and the problem of finding the fixed point of the associated generalized resolvent operator. It follows from Definition 2.6 that we can obtain the following conclusion.

Lemma 3.1. *Let* $M_i : X_i \to 2^{X_i}$ *be generalized* m*-accretive and* $N_i : X_1 \times X_2 \to X_i$ *be any nonlinear mapping for* $i = 1, 2$*. Then the following statements are mutually equivalent:*

(i) *An element* $(x, y) \in X_1 \times X_2$ *is a solution to the problem* (1.1)*.*

(ii) *There is an* $(x, y) \in X_1 \times X_2$ *such that*

$$\begin{aligned} x &= J^{\rho}_{M_1}[x - \rho N_1(x, y)], \\ y &= J^{\lambda}_{M_2}[y - \lambda N_2(x, y)], \end{aligned}$$

where $J^{\rho}_{M_1} = (I + \rho M_1)^{-1}$*,* $J^{\lambda}_{M_2} = (I + \lambda M_2)^{-1}$*, and* $\rho > 0$ *and* $\lambda > 0$ *are two constants.*

(iii) *For any given* $\rho > 0$ *and* $\lambda > 0$*, the map* $F_{\rho,\lambda} : X_1 \times X_2 \to X_1 \times X_2$ *defined by*

$$F_{\rho,\lambda}(u, v) = (P_\rho(u, v), Q_\lambda(u, v)), \quad \forall (u, v) \in X_1 \times X_2$$

has a fixed point $(x, y) \in X_1 \times X_2$*, where maps* $P_\rho : X_1 \times X_2 \to X_1$ *and* $Q_\lambda : X_1 \times X_2 \to X_2$ *defined by*

$$P_\rho(u, v) = J^{\rho}_{M_1}[u - \rho N_1(u, v)], \quad Q_\lambda(u, v) = J^{\lambda}_{M_2}[v - \lambda N_2(u, v)].$$

Theorem 3.2. *Let* X_1 *be a* q_1*-uniformly smooth Banach space with* $q_1 > 1$*,* X_2 *be a* q_2*-uniformly smooth Banach space with* $q_2 > 1$ *and* $\eta_1 : X_1 \times X_1 \to X_1^*$ *be* τ_1*-Lipschitz continuous and* δ_1*-strongly monotone,* $\eta_2 : X_2 \times X_2 \to X_2^*$ *be* τ_2*-Lipschitz continuous and* δ_2*-strongly monotone,. Suppose that and* $M_i : X_i \to 2^{X_i}$ *be generalized* m*-accretive for* $i = 1, 2$*,* $N_1 : X_1 \times X_1 \to X_1$ *is* σ_1*-strongly accretive and* γ_1*-Lipschitz continuous in the first argument and* ς_2*-Lipschitz continuous in the second argument,* $N_2 : X_1 \times X_1 \to X_2$ *is* σ_2*-strongly accretive and* γ_2*-Lipschitz continuous in the second argument and* ς_1*-Lipschitz continuous in the first argument, respectively. If*

$$\begin{cases} \tau_1\delta_2 \sqrt[q_1]{1 - q_1\rho\sigma_1 + c_{q_1}\rho^{q_1}\gamma_1^{q_1}} + \varsigma_1\delta_1\tau_2 < \delta_1\delta_2, \\[2ex] \delta_1\tau_2 \sqrt[q_2]{1 - q_2\lambda\sigma_2 + c_{q_2}\lambda^{q_2}\gamma_2^{q_2}} + \tau_1\varsigma_2\delta_2 < \delta_1\delta_2, \end{cases} \tag{3.1}$$

where c_{q_1}*,* c_{q_2} *are the constants as in Lemma 2.5, then the problem* (1.1) *has a unique solution* (x^*, y^*)*.*

Proof. For any given $\rho > 0$ and $\lambda > 0$, define $P_\rho : X_1 \times X_2 \to X_1$ and $Q_\lambda : X_1 \times X_2 \to X_2$ by

$$P_\rho(u, v) = J^{\rho}_{M_1}[u - \rho N_1(u, v)], \qquad Q_\lambda(u, v) = J^{\lambda}_{M_2}[v - \lambda N_2(u, v)] \tag{3.2}$$

for all $(u, v) \in X_1 \times X_2$. Now define $\|\cdot\|_*$ on $X_1 \times X_2$ by

$$\|(u, v)\|_* = \|u\| + \|v\|, \quad \forall (u, v) \in X_1 \times X_2.$$

It is easy to see that $(X_1 \times X_2, \|\cdot\|_*)$ is a Banach space. By (3.2), for any given $\rho > 0$ and $\lambda > 0$, define $F_{\rho,\lambda} : X_1 \times X_2 \to X_1 \times X_2$ by

$$F_{\rho,\lambda}(u,v) = (P_\rho(u,v), Q_\lambda(u,v)), \quad \forall (u,v) \in X_1 \times X_2.$$

In the sequel, we prove that $F_{\rho,\lambda}$ is a contractive mapping. In fact, for any $(u_1,v_1),(u_2,v_2) \in X_1 \times X_2$, it follows from (3.2) and Lemma 2.7 that

$$\begin{aligned}
&\|P_\rho(u_1,v_1) - P_\rho(u_2,v_2)\| \\
&\le \|J^\rho_{M_1}[u_1 - \rho N_1(u_1,v_1)] - J^\rho_{M_1}[u_2 - \rho N_1(u_2,v_2)]\| \\
&\le \frac{\tau_1}{\delta_1}\|u_1 - u_2 - \rho[N_1(u_1,v_1) - N_1(u_2,v_1)]\| \\
&\quad + \frac{\tau_1}{\delta_1}\|N_1(u_2,v_1) - N_1(u_2,v_2)\|
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\|Q_\lambda(u_1,v_1) - Q_\lambda(u_2,v_2)\| \\
&\le \|J^\lambda_{M_2}[v_1 - \lambda N_2(u_1,v_1)] - J^\lambda_{M_2}[v_2 - \lambda N_2(u_2,v_2)]\| \\
&\le \frac{\tau_2}{\delta_2}\|v_1 - v_2 - \lambda[N_2(u_1,v_1) - N_2(u_1,v_2)]\| \\
&\quad + \frac{\tau_2}{\delta_2}\|N_2(u_1,v_2) - N_2(u_2,v_2)\|
\end{aligned} \tag{3.4}$$

By assumptions and Lemma 2.5, we have

$$\begin{aligned}
&\|u_1 - u_2 - \rho[N_1(u_1,v_1) - N_1(u_2,v_1)]\|^{q_1} \\
&\le \|u_1 - u_2\|^{q_1} - q_1\rho\langle N_1(u_1,v_1) - N_1(u_2,v_1), J_{q_1}(u_1 - u_2)\rangle \\
&\quad + \rho^{q_1} c_{q_1}\|N_1(u_1,v_1) - N_1(u_2,v_1)\|^{q_1} \\
&\le (1 - q_1\rho\sigma_1 + c_{q_1}\rho^{q_1}\gamma_1^{q_1})\|u_1 - u_2\|^{q_1},
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\|v_1 - v_2 - \lambda[N_2(u_1,v_1) - N_2(u_1,v_2)]\|^{q_2} \\
&\le (1 - q_2\lambda\sigma_2 + c_{q_2}\lambda^{q_2}\gamma_2^{q_2})\|v_1 - v_2\|^{q_2}
\end{aligned} \tag{3.6}$$

and

$$\|N_1(u_2,v_1) - N_1(u_2,v_2)\| \le \varsigma_2\|v_1 - v_2\|, \tag{3.7}$$

$$\|N_2(u_1,v_2) - N_2(u_2,v_2)\| \le \varsigma_1\|u_1 - u_2\|. \tag{3.8}$$

From (3.3)-(3.8), we obtain

$$\begin{cases}
\|P_\rho(u_1,v_1) - P_\rho(u_2,v_2)\| \\
\qquad \le \frac{\tau_1}{\delta_1}\sqrt[q_1]{1 - q_1\rho\sigma_1 + c_{q_1}\rho^{q_1}\gamma_1^{q_1}}\|u_1 - u_2\| + \frac{\varsigma_2\tau_1}{\delta_1}\|v_1 - v_2\|, \\
\|Q_\lambda(u_1,v_1) - Q_\lambda(u_2,v_2)\| \\
\qquad \le \frac{\tau_2}{\delta_2}\sqrt[q_2]{1 - q_2\lambda\sigma_2 + c_{q_2}\lambda^{q_2}\gamma_2^{q_2}}\|v_1 - v_2\| + \frac{\varsigma_1\tau_2}{\delta_2}\|u_1 - u_2\|.
\end{cases} \tag{3.9}$$

(3.9) implies that

$$\begin{aligned} &\|P_\rho(u_1,v_1)-P_\rho(u_2,v_2)\|+\|Q_\lambda(u_1,v_1)-Q_\lambda(u_2,v_2)\| \\ &\le k(\|u_1-u_2\|+\|v_1-v_2\|), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} k= \quad & \max\{\frac{\tau_1}{\delta_1}\sqrt[q_1]{1-q_1\rho\sigma_1+c_{q_1}\rho^{q_1}\gamma_1^{q_1}}+\frac{\varsigma_1\tau_2}{\delta_2}, \\ & \frac{\varsigma_2\tau_1}{\delta_1}+\frac{\tau_2}{\delta_2}\sqrt[q_2]{1-q_2\lambda\sigma_2+c_{q_2}\lambda^{q_2}\gamma_2^{q_2}}\}. \end{aligned}$$

By (3.1), we know that $0\le k<1$. It follows from (3.10) that

$$\|F_{\rho,\lambda}(u_1,v_1)-F_{\rho,\lambda}(u_2,v_2)\|_*\le k\|(u_1,v_1)-(u_2,v_2)\|_*.$$

This proves that $F_{\rho,\lambda}: X_1\times X_2\times X_1\times X_2$ is a contraction mapping. Hence, there exists a unique $(x^*,y^*)\in X_1\times X_2$ such that

$$F_{\rho,\lambda}(x^*,y^*)=(x^*,y^*),$$

that is,

$$x^*=J_{M_1}^{\rho}[x^*-\rho N_1(x^*,y^*)] \quad \text{and} \quad y^*=J_{M_2}^{\lambda}[y^*-\lambda N_2(x^*,y^*)].$$

By Lemma 3.1, (x^*,y^*) is the unique solution of problem (1.1). This completes the proof. □

Remark 3.3. If X_1 and X_2 are 2-uniformly smooth Banach space and there exists $\lambda=\rho>0$ such that

$$\begin{cases} h_1=\frac{\delta_1\delta_2-\varsigma_1\delta_1\tau_2}{\tau_1\delta_2}<1, \quad h_2=\frac{\delta_1\delta_2-\tau_1\varsigma_2\delta_2}{\delta_1\tau_2}<1, \\ |\rho-\frac{\sigma_1}{c_2\gamma_1^2}|<\frac{\sqrt{\sigma_1^2-(1-h_1^2)c_2\gamma_1^2}}{c_2\gamma_1^2}, \\ |\rho-\frac{\sigma_2}{c_2\gamma_2^2}|<\frac{\sqrt{\sigma_2^2-(1-h_2^2)c_2\gamma_2^2}}{c_2\gamma_2^2}, \\ \sigma_1^2>(1-h_1^2)c_2\gamma_1^2, \quad \sigma_2^2>(1-h_2^2)c_2\gamma_2^2, \end{cases}$$

then (3.1) holds. We note that Hilbert space and L_p (or l_p) ($2\le p<\infty$) spaces are 2-uniformly Banach spaces.

4. Perturbed Algorithm and Stability

In this section, by using the following definition and lemma, we construct a new perturbed iterative algorithm with mixed errors for solving problem (1.1) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Definition 4.1. Let S be a selfmap of X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and let $\epsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.

Lemma 4.2. *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1} \leq (1 - t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$ $(n \to \infty)$.

Algorithm 4.3. Let $N_i : X_1 \times X_2 \to X_i$ be single-valued mappings and $M_i : X_i \to 2^{X_i}$ be a generalized m-accretive mapping for all $i = 1, 2$. Then for a given $(x_0, y_0) \in X_1 \times X_2$, the perturbed iterative sequence $\{(x_n, y_n)\}$ is defined by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J_{M_1}^{\rho}[x_n - \rho N_1(x_n, y_n)] + \alpha_n u_n + w_n, \\ y_{n+1} = (1 - \alpha_n)y_n + \alpha_n J_{M_2}^{\lambda}[y_n - \lambda N_2(x_n, y_n)] + \alpha_n v_n + e_n, \end{cases} \tag{4.1}$$

where $n \geq 0$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{u_n\}, \{w_n\} \subset X_1$ and $\{v_n\}, \{e_n\} \subset X_2$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

(i) $u_n = u_n' + u_n''$, $\quad v_n = v_n' + v_n''$;

(ii) $\lim_{n\to\infty} \|u_n'\| = 0$, $\quad \lim_{n\to\infty} \|v_n'\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|u_n''\| < \infty$, $\sum_{n=0}^{\infty} \|w_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n''\| < \infty$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Let $\{(z_n, t_n)\}$ be any sequence in $X_1 \times X_2$ and define $\{(\epsilon_n, \varepsilon_n)\}$ by

$$\begin{cases} \epsilon_n = \|z_{n+1} - \{(1 - \alpha_n)z_n + \alpha_n J_{M_1}^{\rho}[z_n - \rho N_1(z_n, t_n)] + \alpha_n u_n + w_n\}\|, \\ \varepsilon_n = \|t_{n+1} - \{(1 - \alpha_n)t_n + \alpha_n J_{M_2}^{\lambda}[t_n - \lambda N_2(z_n, t_n)] + \alpha_n v_n + e_n\}\|. \end{cases} \tag{4.2}$$

Theorem 4.4. *Suppose that $X_1, X_2, \eta_1, \eta_2, N_1, N_2, M_1$ and M_2 are the same as in Theorem* 3.2. *If $\sum_{n=0}^{\infty} \alpha_n = \infty$ and condition* (3.1) *holds, then the perturbed iterative sequence $\{(x_n, y_n)\}$ defined by* (4.1) *converges strongly to the unique solution of problem* (1.1). *Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then $\lim_{n\to\infty}(z_n, t_n) = (x^*, y^*)$ if and only if $\lim_{n\to\infty}(\epsilon_n, \varepsilon_n) = (0, 0)$, where $(\epsilon_n, \varepsilon_n)$ is defined by* (4.2).

Proof. From Theorem 3.2, we know that problem (1.1) has a unique solution $(x^*, y^*) \in X_1 \times X_2$. It follows from (4.1) and the proof of (3.9) in Theorem

3.2 that

$$\begin{aligned}
&\|x_{n+1}-x^*\| \\
&\leq (1-\alpha_n)\|x_n-x^*\|+\alpha_n\{\frac{\tau_1}{\delta_1}\sqrt[q_1]{1-q_1\rho\sigma_1+c_{q_1}\rho^{q_1}\gamma_1^{q_1}}\|x_n-x^*\| \\
&\quad +\frac{\varsigma_2\tau_1}{\delta_1}\|y_n-y^*\|\}+\alpha_n\|u_n'\|+(\|u_n''\|+\|w_n\|), \qquad (4.3)\\
&\|y_{n+1}-y^*\| \\
&\leq (1-\alpha_n)\|y_n-y^*\|+\alpha_n\{\frac{\tau_2}{\delta_2}\sqrt[q_2]{1-q_2\lambda\sigma_2+c_{q_2}\lambda^{q_2}\gamma_2^{q_2}}\|y_n-y^*\| \\
&\quad +\frac{\varsigma_1\tau_2}{\delta_2}\|x_n-x^*\|\}+\alpha_n\|v_n'\|+(\|v_n''\|+\|e_n\|), \qquad (4.4)
\end{aligned}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned}
&\|x_{n+1}-x^*\|+\|y_{n+1}-y^*\| \\
&\leq (1-\alpha_n)(\|x_n-x^*\|+\|y_n-y^*\|) \\
&\quad +\alpha_n\{\frac{\tau_1}{\delta_1}\sqrt[q_1]{1-q_1\rho\sigma_1+c_{q_1}\rho^{q_1}\gamma_1^{q_1}}+\frac{\varsigma_1\tau_2}{\delta_2}\}\|x_n-x^*\| \\
&\quad +\alpha_n\{\frac{\varsigma_2\tau_1}{\delta_1}+\frac{\tau_2}{\delta_2}\sqrt[q_2]{1-q_2\lambda\sigma_2+c_{q_2}\lambda^{q_2}\gamma_2^{q_2}}\}\|y_n-y^*\| \\
&\quad +\alpha_n(\|u_n'\|+\|v_n'\|)+(\|u_n''\|+\|w_n\|+\|v_n''\|+\|e_n\|) \\
&\leq [1-\alpha_n(1-k)](\|x_n-x^*\|+\|y_n-y^*\|) \\
&\quad +\alpha_n(1-k)\cdot\frac{1}{1-k}(\|u_n'\|+\|v_n'\|) \\
&\quad +(\|u_n''\|+\|w_n\|+\|v_n''\|+\|e_n\|), \qquad (4.5)
\end{aligned}$$

where k is the same as in (3.10). Since $\sum_{n=0}^{\infty}\alpha_n=\infty$, it follows from Lemma 4.2, (3.1) and (4.5) that $\|x_n-x^*\|+\|y_n-y^*\|\to 0\,(n\to\infty)$. Hence, we know that the sequence $\{(x_n,y_n)\}$ converges to the unique solution (x^*,y^*) of the problem (1.1).

Now we prove the second conclusion. By (4.2), we know

$$\begin{aligned}
\|z_{n+1}-x^*\| &\leq \|(1-\alpha_n)z_n \\
&\qquad +\alpha_n J_{M_1}^{\rho}[z_n-\rho N_1(z_n,t_n)]+\alpha_n u_n+w_n-x^*\|+\epsilon_n, \\
\|t_{n+1}-y^*\| &\leq \|(1-\alpha_n)t_n \\
&\qquad +\alpha_n J_{M_2}^{\lambda}[t_n-\lambda N_2(z_n,t_n)]+\alpha_n v_n+e_n-y^*\|+\varepsilon_n.
\end{aligned} \qquad (4.6)$$

As the proof of inequality (4.5), we have

$$\begin{aligned}
&\|(1-\alpha_n)z_n + \alpha_n J^{\rho}_{M_1}[z_n - \rho N_1(z_n,t_n)] + \alpha_n u_n + w_n - x^*\| \\
&\quad + \|(1-\alpha_n)t_n + \alpha_n J^{\lambda}_{M_2}[t_n - \lambda N_2(z_n,t_n)] + \alpha_n v_n + e_n - y^*\| \\
&\le [1-\alpha_n(1-k)](\|z_n - x^*\| + \|t_n - y^*\|) \\
&\quad + \alpha_n(1-k)\cdot\frac{1}{1-k}(\|u'_n\| + \|v'_n\|) \\
&\quad + (\|u''_n\| + \|w_n\| + \|v''_n\| + \|e_n\|). \qquad (4.7)
\end{aligned}$$

Since $0 < a \le \alpha_n$, it follows from (4.6) and (4.7) that

$$\begin{aligned}
&\|z_{n+1} - x^*\| + \|t_{n+1} - y^*\| \\
&\le [1-\alpha_n(1-k)](\|z_n - x^*\| + \|t_n - y^*\|) \\
&+ \alpha_n(1-k)\cdot\frac{1}{1-k}(\|u'_n\| + \|v'_n\| + \frac{\epsilon_n + \varepsilon_n}{a}) \\
&\quad + (\|u''_n\| + \|v''_n\| + \|w_n\| + \|e_n\|).
\end{aligned}$$

Suppose that $\lim(\epsilon_n, \varepsilon_n) = (0,0)$. Then from $\sum_{n=0}^{\infty}\alpha_n = \infty$ and Lemma 4.2, we have $\lim(z_n, t_n) = (x^*, y^*)$.

Conversely, if $\lim(z_n, t_n) = (x^*, y^*)$, then we get

$$\begin{aligned}
\epsilon_n &= \|z_{n+1} - \{(1-\alpha_n)z_n + \alpha_n J^{\rho}_{M_1}[z_n - \rho N_1(z_n,t_n)] + \alpha_n u_n + w_n\}\| \\
&\le \|z_{n+1} - x^*\| \\
&\quad + \|(1-\alpha_n)z_n + \alpha_n J^{\rho}_{M_1}[z_n - \rho N_1(z_n,t_n)] + \alpha_n u_n + w_n - x^*\|, \\
\varepsilon_n &= \|t_{n+1} - \{(1-\alpha_n)t_n + \alpha_n J^{\lambda}_{M_2}[t_n - \lambda N_2(z_n,t_n)] + \alpha_n v_n + e_n\}\| \\
&\le \|t_{n+1} - y^*\| \\
&\quad + \|(1-\alpha_n)t_n + \alpha_n J^{\lambda}_{M_2}[t_n - \lambda N_2(z_n,t_n)] + \alpha_n v_n + e_n - y^*\|,
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_n + \varepsilon_n \quad &\le \|z_{n+1} - x^*\| + \|t_{n+1} - y^*\| \\
&\quad + [1-\alpha_n(1-k)](\|z_n - x^*\| + \|t_n - y^*\|) \\
&\quad + \alpha_n(1-k)\cdot\frac{1}{1-k}(\|u'_n\| + \|v'_n\|) \\
&\quad + (\|u''_n\| + \|v''_n\| + \|w_n\| + \|e_n\|) \to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Remark 4.5. If $u_n = 0$ or $v_n = 0$ or $w_n = 0$ or $e_n = 0$ $(n \ge 0)$ in Algorithm 4.3, then the conclusions of Theorem 4.4 also hold. The results of Theorems 3.2 and 4.4 improve and generalize the corresponding results of [3, 6, 9, 12].

References

[1] C. Baiocchi and A. Caopelo, *Variational and Quasivariational Inequalities, Application to Free Boundary Problems*, Wiley, New York, 1984.

[2] Z. S. Bi, Z. Han and Y. P. Fang, *Sensitivity analysis for nonlinear variational inclusions involving generalized m-accretive mappings*, J. Sichuan Univ. **40(2)** (2003), 240–243.

[3] X. P. Ding, *Existence and algorithm of solutions for generalized mixed implicit quasi-variational inequalities*, Appl. Math. Comput. **113** (2000), 67–80.

[4] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum, New York, 1995.

[5] N. J. Huang, *Nonlinear implicit quasi-variational inclusions involving generalized m-accretive mappings*, Arch. Inequal. Appl. **2(4)** (2004), 413–425.

[6] N. J. Huang, M. R. Bai, Y. J. Cho and S. M. Kang, *Generalized nonlinear mixed quasi-variational inequalities*, Comput. Math. Appl. **40** (2000), 205–215.

[7] N. J. Huang and Y. P. Fang, *Generalized m-accretive mappings in Banach spaces*, J. Sichuan Univ. **38(4)** (2001), 591–592.

[8] N. J. Huang, Y. P. Fang and C. X. Deng, *Nonlinear variational inclusions involving generalized m-accretive mappings*, Proceedings of the Bellman Continum: International Workshop on Uncertain Systems and Soft Computing, Beijing, China, July 24-27, 2002, pp. 323–327.

[9] H. Y. Lan, J. K. Kim and N. J. Huang, *On the generalized nonlinear quasi-variational inclusions involving non-monotone set-valued mappings*, Nonlinear Funct. Anal. Appl. **9(3)** (2004), 451–465.

[10] H. Y. Lan, Q. K. Liu and J. Li, *Iterative approximation for a system of nonlinear variational inclusions involving generalized m-accretive mappings*, Nonlinear Anal. Forum **9(1)** (2004), 33–42.

[11] H. Y. Lan, N. J. Huang and Y. J. Cho, *New iterative approximation for a system of generalized nonlinear variational inclusions with set-valued mappings in Banach spaces*, Math. Inequal. Appl. **9(1)** (2006), 175–187.

[12] L. S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

[13] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16(12)** (1991), 1127–1138.

[14] G. X. Z. Yuan, *KKM Theory and Applications*, Marcel Dekker, 1999.

[15] H. L. Zeng, *A fast learning algorithm for solving systems of linear equations and related problems*, J. Adv. Modelling Anal. **29** (1995), 67–71.

[16] H. L. Zeng, *Stability analysis on continuous neural networks with slowly synapse-varying structures*, Proceedings of IEEE: International Sympostium on Circuits and Systems, United States of America, 1992, pp. 443–447.

[17] W. Wang, Z. Q. Liu, C. Feng and S. H. Kang, *Three-step iterative algorithm with errors for generalized strongly nonlinear quasi-variational inequalithies*, Adv. Nonlinear Var. Inequal. **7(2)** (2004), 27–34.