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STUDY OF SOME GENERALIZED h-VARIATIONAL INEQUALITY PROBLEMS IN H-PSEUDOSPACE

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Abstract. The main aim is to define a new class of generalized h-variational inequality problems and its generalized h-variational inequality problems. We define the class of h-η-invex set, h-η-invex function and H-pseudospace. Existence of the solution of the problems are established in H-pseudospace with the help of H-KKM mapping theorem and HC_*-condition of η associated with the function h.

1. Introduction

The theory of variational inequalities have turned out to be very useful application in studying optimization problems, financial problems, physical problems, computational applications, engineering problems and many more.

The concept of variational inequality problem was introduced by Stampacchia [12] in 1964. Later the concept of the vector variational inequality

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problem was developed and studied by Giannessi [6, 7] in the setting of the finite dimensional Euclidean spaces. Since then, a large number of results for the vector variational inequality and vector complementarity problems have been obtained. For details, we refer to Chen [3], Chen and Yang [4], Daniilidis and Hadjisavvas [5] etc. In [2], Behera and Das introduced generalized vector variational inequality problems, generalized vector complementarity problems and several classes of generalized vector F-variational inequality problems, generalized vector F-complementarity problems in ordered topological vector spaces and H-spaces. Giannessi [6, 7], Chen [3, 4] and many other authors have developed the vector variational inequality problems.

Bardaro and Ceppitelli (1988, [1]) have explored minimax inequalities in H-spaces. Later Tarafdar (1990, [13]) has studied the fixed point theorems in H-spaces. He applied the fixed point theorem to analyze the results on variational inequalities on sets with H-convex sections, and also results on minimax inequalities. Behera and Das [2] have studied various type of generalized variational inequalities in H-spaces and H-differentiable manifolds.

1.1. **H-Space and H-KKM Mapping Theorem.** For our need, we recall the notion of H-spaces, H-convex sets, H-compact set and H-KKM mapping.

Definition 1.1. ([1]) Let X be a topological space and $\{\Gamma_A\}$ be a given family of nonempty contractible subsets of X, indexed by finite subsets of X. A pair $(X, \{\Gamma_A\})$ is said to be a H-space if $A \subset B$ then $\Gamma_A \subset \Gamma_B$.

Definition 1.2. ([1]) Let $(X, \{\Gamma_A\})$ be a H-space. A subset $D \subset X$ is said to be

(i) H-convex if $\Gamma_A \subset D$ for every finite subset $A \subset D$,
(ii) weakly H-convex if $\Gamma_A \bigcap D$ is nonempty and contractible for every finite subset $A \subset D$, equivalent to say; the pair $(D, \{\Gamma_A \bigcap D\})$ is a H-space,
(iii) H-compact if there exist a compact and weakly H-convex set $K \subset X$ such that $D \bigcup A \subset K$ for every finite subset $A \subset X$.

Definition 1.3. ([1]) Let $(X, \{\Gamma_A\})$ be a H-space. A multifunction $G : X \to 2^X$ (the set of subsets of X) is said to be H-KKM if for every finite subset $A \subset X$, $\Gamma_A \subset \bigcup_{x \in A} G(x)$.

Bardaro and Ceppitelli [1] have studies the existence of the fixed point theorems in H-spaces.

Theorem 1.4. ([1], Theorem 1, p.486) *Let $(X, \{\Gamma_A\})$ be a H-space and $F : X \to 2^X$ be a H-KKM multifunction such that:*

(a) *for each* $x \in X$, $F(x)$ *is compactly closed, that is,* $B \cap F(x)$ *is closed in* B, *for every compact* $B \subset X$,

(b) *there is a compact set* $L \subset X$ *and a* H*-compact set* $K \subset X$, *such that for each weakly* H*-convex set* D *with* $K \subset D \subset X$, *we have* $\bigcap_{x \in D} \{F(x) \cap D\} \subset L$.

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

Theorem 1.5. ([1], Theorem 2, p.486) Let $(X, \{\Gamma_A\})$ be a H-space and $F, G : X \to 2^X$ be two multifunctions such that:

(a) for each $x \in X$, $G(x)$ is compactly closed and $F(x) \subset G(x)$,

(b) $x \in F(x)$ for every $x \in X$,

(c) for every $x \in X$, the set $X - F^{-1}(y)$ is H-convex where $F^{-1}(y) = \{x \in X : y \in F(x)\}$,

(d) there is a compact set $L \subset X$ and a H-compact set $K \subset X$, such that for each weakly H-convex set D with $K \subset D \subset X$, we have

$$\bigcap_{x \in D} \{G(x) \cap D\} \subset L.$$

Then $\bigcap_{x \in X} G(x) \neq \varnothing$.

As an extension, Verma ([14]) has studied the following fixed point theorems in generalized H-space using the RKKM mapping.

Lemma 1.6. ([14]) *Let* $(X, H, \{f\})$ *be a generalized* H*-space and* $P : X \to 2^X$ *be a RKKM mapping such that:*

(i) $P(x)$ *is compactly closed for all* x *in* X;

(ii) *there exists a compact subset* L *of* X *and a generalized* H*-compact subset* K *of* X *such that for each weakly generalized* H*-convex subset* D *of* X *with* $K \subset D$ *we have*

$$\bigcap_{x \in D} (P(x) \cap D) \subset L.$$

Then $\bigcap_{x \in D} P(x) \neq \varnothing$.

Lemma 1.7. ([15], Theorem 2.1, p.134) *Let* $(X, H, \{f\})$ *be a generalized* H*-space and* $P, N : X \to 2^X$ *be multivalued mappings such that:*

(i) $N(x)$ *is compactly closed for all* x *in* X *with* $P(x) \subset N(x)$;

(ii) P *is RKKM mapping,*

(iii) *there exists a compact subset L of X and a generalized H-compact subset C of X such that for each generalized H-convex subset D of X with $C \subset D$ we have*

$$\bigcap_{x \in D} (P(x) \cap D) \subset L,$$

(iv) *for every generalized H-convex subset D of X with $K \subset D$, we have* $\bigcap_{x \in D} (N(x) \cap D) \neq \varnothing$ *if and only if* $\bigcap_{x \in D} (P(x) \cap D) \neq \varnothing$.

Then $\bigcap_{x \in D} P(x) \neq \varnothing$.

2. The problems and corresponding H-KKM mapping theorems

In 1981, Hanson [8] has introduced the concept of invex function to study the theory of optimization problem as a general approach. Later various types of invex functions are defined and used to study the generalized convex optimization problems as well as generalized variational inequality problems. We study the existence of generalized variational inequality problems in the vector spaces with a new approach h-invex with respect to η which is defined below:

For the nonzero elements set

$$X_{\circledcirc} = X - \{0\} = \{x \in X : x_i \neq 0, i = 1, 2, \cdots\},$$

the reciprocals of the elements set $X_{\circledcirc}$ is

$$[X_{\circledcirc}]^{-1} = \left\{ x^{-1} \in X_{\circledcirc} : x^{-1} = \sum_{i=1}^{\infty} x_i^{-1} e_i \right\}.$$

Throughout this paper X is considered as a separable Banach space with Schauder basis. Assume that $h : X \times X_{\circledcirc} \to X$ is any fractional map defined by $h(v, z) \in X$ for all $v \in X$, $0 \neq z \in X$. Assume that the following componentwise properties in X hold: For $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$ and $z = (z_1, z_2, \cdots)$ in X, we have

$$xy = (x_i y_i) = (x_1 y_1, x_2 y_2, \cdots)$$

and

$$h(z, xy) = h(z_i, x_i y_i) = (h(z_1, x_1 y_1), h(z_2, x_2 y_2), \cdots).$$

Definition 2.1. For any set $K \subset X$, the set $K_h(\eta)$ is said to be a *h-η-invex set* if there exists a vector function $\eta : K \times K \to X$ such that

$$K_h(\eta) = \{h(xy, y + \lambda\eta(x, y)) \in X : x, y \in K,\ \ y + \lambda\eta(x, y) \in K,\ \ \lambda \in [0, 1]\}.$$

Definition 2.2. Let $K \subset X$ be a nonempty set and $f : K \to Y$ be any function.

(i) The set K is said to be *h-invex with respect to η* (or *h-η-invex set*) if $K_h(\eta) = K$.

(ii) The function f is said to be h-invex with respect to η (or *h-η-function*) if there exists a vector function $\eta : K \times K \to X$ such that for all $x, y \in K$ and $\lambda \in [0, 1]$,

$$f \circ h(xy, y + \lambda\eta(x, y)) \leq_Y \lambda f(y) + (1 - \lambda)f(x).$$

Theorem 2.3. *Let X be a Banach space, $\phi : X \to X$ be a r-convex function on K, that is, ϕ satisfies*

$$\phi(y + r\lambda(x - y)) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$

for all $x, y \in X$, $\lambda \in [0, 1]$ and $r > 0$. Let K be η-invex on X. Let $f : X \to \mathbb{R}$ be any map satisfies $f(\phi(x)) = \alpha f(x)$ for some $\alpha \in \mathbb{R}$. Let f be a h-convex function on X that satisfies

$$f \circ h(xy, y + \lambda\eta(x, y)) \leq \alpha_1 f(\phi(y)) + \alpha_2 f(\phi(x))$$

for $\lambda \in [0, 1]$, where $\alpha_1 + \alpha_2 = 1$. Then f is h-convex on X.

Proof. Let $\phi : X \to X$ be a r-convex function on K, that is, ϕ satisfies

$$\phi(y + r\lambda(x - y)) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$

for all $x, y \in X$, $\lambda \in [0, 1]$ and $r > 0$. Since K is η-invex on X, $y + t\eta(x, y) \in X$ for all $x, y \in \mathbb{X}$, $\eta(x, y) \in X$ and $t > 0$. Let $f : X \to \mathbb{R}$ be a convex function on X and satisfies $f(\phi(x)) = \alpha f(x)$ for some $\alpha \in \mathbb{R}$. Assume that

$$f \circ h(xy, y + \lambda\eta(x, y)) \leq \alpha_1 f(\phi(y)) + \alpha_2 f(\phi(x)),$$

where $\alpha_1 + \alpha_2 = 1$. Then, we have

$$\begin{aligned} f \circ h(xy, y + \lambda\eta(x, y)) &\leq \lambda f(\phi(y)) + (1 - \lambda)f(\phi(x)) \\ &\leq \lambda f(y) + (1 - \lambda)f(x) \end{aligned}$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. □

Example 2.4. Let $X = \mathbb{R}$ and $Y = \mathbb{R}$. The set $K = [a, b]$ is η-invex for $a < b$. Let X be an affine space as $y + t\eta(x, y) \in X$ for all $x, y \in \mathbb{R}$, $\eta(x, y) \in X$ and $t \in \mathbb{R}$. Taking $x = y + t\eta(x, y)$, then we have $\eta(x, y) = t^{-1}(x - y)$. We can find a $\lambda \in (0, 1)$ such that $t^{-1}\lambda \in (0, 1)$ and $\lambda x + (1 - \lambda)y = y + t^{-1}\lambda(x - y) \in K$. Again taking $h(xy, y + \lambda\eta(x, y)) = \frac{1}{xy} - \ln(y + t^{-1}\lambda(x - y))$ and $f(x) = x$ for all $x, y) \in X$, $\lambda \in [0, 1]$ and $t > 0$, we have

$$f \circ h(xy, y + \lambda\eta(x, y)) \leq \lambda f(y) + (1 - \lambda)f(x).$$

Remark 2.5. The space X reduces to different type of spaces according to definitions of h.

(1) If $Y = \mathbb{R}$, $\eta(x, y) = x - y$, and $h(v, z) = z$, then X is a convex set for $z = y + \lambda(x - y)$, $\lambda \in [0, 1]$ which is well known.
(2) If $Y = \mathbb{R}$ and $h(v, z) = z$, then X is a invex set with respect to η for $z = y + \lambda\eta(x, y)$, $\lambda \in [0, 1]$ ([8]).
(3) If $Y = \mathbb{R}$, $\eta(x, y) = x - y$ and $h(v, z) = v/z$, then X is a harmonic convex set for $v = 1$ and $z = y + \lambda(x - y)$, $\lambda \in [0, 1]$ ([11]).
(4) If $Y = \mathbb{R}$, $\eta(x, y) = x - y$ and $h(v, z) = v/z$, then X is a harmonic convex set for $v = xy$ and $z = y + \lambda(x - y)$, $\lambda \in [0, 1]$ ([9]).
(5) If $h(v, z) = v/z$, then X is a harmonic invex set with respect to η for $v = xy$ and $z = y + \lambda\eta(x, y)$, $\lambda \in [0, 1]$ ([10]).

We need to define the concept of to study our results.

Definition 2.6. (i) A set $D \subset X$ is said to be *almost convex* of X if there exists a finite set $A \subset D$ such that $\Gamma_A \simeq Ch(D)$, where $Ch(D)$ is convex hull of D and $\simeq$ denotes the homotopical equivalence.
(ii) The space $X = (X, \{\Gamma_A\}; K_h(\eta))$ is said to be a *H-pseudospace* if there exists a vector function $\eta : X \times X \to X$ such that η satisfies condition $HC_\star$ on the h-η-invex set $K_h(\eta)$ and $(X, \{\Gamma_A \cap D\})$ is a H-space for each almost convex set $D \subset K_h(\eta)$.

Remark 2.7. The $X = (X, \{\Gamma_A\}; K_h(\eta))$ is a H-space if $K_h(\eta)$ is a weakly H-convex set.

2.1. **The Problems.** Let $K \subset X$ be a nonempty subset of X, $\eta : K \times K \to X$ be a vector valued map. Assume that $(X, \{\Gamma_A\}; K_h(\eta))$ is a H-pseudospace. Let $T : K \to X^*$ be a nonlinear map. Let $F : K \to \mathbb{R}_\circledcirc$ be any map.

(a) The generalized h-variational inequality problem (h-GVIP) is to find: $y \in K$ such that

$$\langle T(y), h\,(xy, \eta(x, y))\rangle \geq 0 \quad \text{for all } x \in K. \qquad (h\text{-GVIP})$$

(b) The generalized dual h-variational inequality problem (h-GDVIP) is to find: $y \in K$ such that

$$\langle T(x), h\,(xy, \eta(y, x))\rangle \leq 0 \quad \text{for all } x \in K. \qquad (h\text{-GDVIP})$$

The generalized h-variational inequalities of associated with F and $\xi \in \partial F(y)$ the subdifferential of F at $y \in K$ are defined as follows:

(a) The generalized h-variational inequality problem associated with F (h-GVIP$_F$) is to find: $y \in K$ such that

$$\langle T(y), h\,(xy, \eta(x, y))\rangle + F(x) - F(y) \geq 0 \quad \text{for all } x \in K. \qquad (h\text{-GVIP}_F)$$

(b) The generalized dual h-variational inequality problem associated with F (h-GDVIP$_F$) is to find: $y \in K$ such that

$$\langle T(x), h\left(xy, \eta(y,x)\right)\rangle + F(x) - F(y) \leq 0 \quad \text{for all } x \in K. \qquad (h\text{-GDVIP}_F)$$

In the following theorem we prove the H-KKM property of the multivalued function $P : K \to 2^X$ without using weakly H-convex set needed for contradiction. We recall the following result. The results are showing the relation between h-variational inequality problems and dual problem in $K_h(\eta)$,

Definition 2.8. Let $K \subset X_{\circledcirc}$ be a h-η-invex set in $X_{\circledcirc}$. The bifunction $\eta : K \times K \to X$ is said to satisfy $HC_\star$ with respect to h on $K_h(\eta)$ if the following conditions hold:

(a) $\eta(y,x) = -\eta(x,y)$ for all $x, y \in K$, i.e., η is anti-symmetric on K,

(b) for all $x, y \in K$, $y_\lambda = h\left(xy, y + \lambda\eta(x,y)\right) \in K$ for $0 < \lambda \leq 1$, we have

$$h\left(y_\lambda y, \eta\left(y_\lambda, y\right)\right) = h\left(xy, (1-\lambda)\eta(x,y)\right),$$

(c) for all $x, y \in K$, $y_\lambda = h\left(xy, y + \lambda\eta(x,y)\right)$ and $\lambda > 0$, we have

$$h\left(y_\lambda x, \eta\left(y_\lambda, x\right)\right) = -h\left(xy, \lambda\eta(x,y)\right),$$

(d) for all $x, y \in K$, $x_\lambda = h\left(xy, x + \lambda\eta(y,x)\right)$ and $\lambda > 0$, we have

$$h\left(x_\lambda y, \eta\left(x_\lambda, y\right)\right) = -h\left(xy, \lambda\eta(y,x)\right),$$

(e) for all $x, y \in K$, $x_\lambda = h\left(xy, x + \lambda\eta(y,x)\right)$ and $\lambda > 0$, we have

$$h\left(x_\lambda x, \eta\left(x_\lambda, x\right)\right) = h\left(xy, (1-\lambda)\eta(y,x)\right).$$

Proposition 2.9. *If $\eta : K \times K \to X_{\circledcirc}$ satisfies the condition $HC_\star$ on $K_h(\eta)$ and $T : K \to X^*$, then for all $x, y \in K$, for any $\overline{y} = h\left(xy, y + \lambda\eta(x,y)\right) \in K_h(\eta)$ and $\lambda \in (0,1)$, we have the following equalities:*

(A) $\lambda\left\langle T(\overline{y}), h\left(x\overline{y}, \eta(x,\overline{y})\right)\right\rangle + (1-\lambda)\left\langle T(\overline{y}), h\left(y\overline{y}, \eta(y,\overline{y})\right)\right\rangle = 0,$

(B) $\lambda\left\langle T(u), h\left(x\overline{y}, \eta(x,\overline{y})\right)\right\rangle + (1-\lambda)\left\langle T(u), h\left(y\overline{y}, \eta(y,\overline{y})\right)\right\rangle = 0$ *for any* $u \in K_h(\eta)$,

(C) $\lambda\left\langle T(x), h\left(x\overline{y}, \eta(\overline{y},x)\right)\right\rangle + (1-\lambda)\left\langle T(y), h\left(y\overline{y}, \eta(\overline{y},y)\right)\right\rangle$
$= -\left\langle T(x) - T(y), h\left(xy, \eta(x,y)\right)\right\rangle.$

Proof. It is clear that the results (A), (B), (C) and (D) hold by Definition 2.8 because η satisfies the condition $HC_\star$ with respect to h on $K_h(\eta)$ and $T(x)$ is linear for each $x \in K$. □

Theorem 2.10. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map. Assume that for each $x \in K_h(\eta)$, $\langle T(x), h\left(xx, \eta(x,x)\right)\rangle = 0$. Then for each $x \in K_h(\eta)$, the multivalued mapping $P : K \to 2^X$ defined by*

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\left(xy, \eta(x,y)\right)\rangle \geq 0\}$$

is a H-KKM mapping.

Proof. Since for each $x \in K_h(\eta)$, $\langle T(x), h(xx, \eta(x,x))\rangle = 0$, the set $P(x)$ is nonempty. Now to show P is a H-KKM mapping. If not, then there exists a finite set A in $K_h(\eta)$ such that $\Gamma_A \nsubseteq \bigcup_{x\in A} P(x)$. If there exists a $z \in \Gamma_A$ such that $z \notin \bigcup_{x\in A} P(x)$, then $z \notin P(x)$ for all $x \in A$, that is, $\langle T(z), h(xz, \eta(x,z))\rangle < 0$ for all $x \in A$. Similarly, we can say $\langle T(z), h(yz, \eta(y,z))\rangle < 0$ for all $y \in A$. Thus

$$(1-\lambda)\langle T(z), h(xz, \eta(x,z))\rangle + \lambda\langle T(z), h(yz, \eta(y,z))\rangle < 0$$

for all $x, y \in A$ and $z \in \Gamma_A$. Since $K_h(\eta)$ is a h-η-invex set, replacing z by $z = h(xy, y + \lambda\eta(x,y))$, $\lambda \in (0,1)$ and using Proposition 2.9(B), we have

$$(1-\lambda)\langle T(u), h(xz, \eta(x,z))\rangle + \lambda\langle T(u), h(yz, \eta(y,z))\rangle = 0$$

for all $x, y \in A$ and $u \in K_h(\eta)$. Taking $u = z$, we have

$$(1-\lambda)\langle T(z), h(xz, \eta(x,z))\rangle + \lambda\langle T(z), h(yz, \eta(y,z))\rangle = 0$$

for all $x, y \in A$. From the above equation we obtain

$$0 = (1-\lambda)\langle T(z), h(xz, \eta(x,z))\rangle + \lambda\langle T(z), h(yz, \eta(y,z))\rangle < 0$$

which is a contradiction. Thus P is a H-KKM mapping. This completes the proof. □

Theorem 2.11. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map. Let the multivalued maps $P, N : K \to 2^X$ be defined by*

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h(xy, \eta(x,y))\rangle \geq 0,\ x \in K_h(\eta)\}$$

and

$$N(x) = \{y \in K_h(\eta) : \langle T(x), h(xy, \eta(y,x))\rangle \leq 0,\ x \in K_h(\eta)\}.$$

Assume that

(*a*) *for each $x \in K_h(\eta)$, $\langle T(x), h(xx, \eta(x,x))\rangle = 0$,*

(*b*) *T is h-η-monotone on $K_h(\eta)$, that is,*

$$\langle T(x), h(xy, \eta(y,x))\rangle + \langle T(y), h(xy, \eta(x,y))\rangle \leq 0$$

for all $x, y \in K$,

(*c*) *for each $y \in K$, the set*

$$P^{-1}(y) = \{x \in K_h(\eta) : \langle T(y), h(xy, \eta(x,y))\rangle < 0\}$$

is either H-convex or empty.

Then N is a H-KKM mapping.

Proof. By (a), we have $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$ for each $x \in K_h(\eta)$, that is, $x \in P(x) \cap N(x)$. Therefore $P(x)$ and $N(x)$ are nonempty for each $x \in K_h(\eta)$.

Our aim is to prove that N is a H-KKM mapping, that is, to show that there exists a finite set A in $K_h(\eta)$ such that $\Gamma_A \nsubseteq \bigcup_{x\in A} N(x)$. On the contrary, if N is not a H-KKM mapping, then there exists a $z \in \Gamma_A$ such that $z \notin \bigcup_{x\in A} N(x)$, this implies $z \notin N(x)$ for all $x \in A$, that is, $\langle T(x), h\,(xz, \eta(z,x))\rangle > 0$ for all $x \in A$. Since T is h-η-monotone on $K_h(\eta)$, we have

$$\langle T(x), h\,(xy, \eta(y,x))\rangle + \langle T(y), h\,(xy, \eta(x,y))\rangle \leq 0$$

for all $x, y \in A$. At $y = z$, we have

$$\langle T(x), h\,(xz, \eta(z,x))\rangle + \langle T(z), h\,(xz, \eta(x,z))\rangle \leq 0$$

for all $x \in A$, that is,

$$\begin{aligned}\langle T(z), h\,(xz, \eta(x,z))\rangle &\leq -\langle T(x), h\,(xz, \eta(z,x))\rangle \\ &< 0\end{aligned}$$

for all $x \in A$, i.e., $x \in X - P^{-1}(z)$ ($X - Z$ denotes the setminus of Z from X). Thus $A \subset X - P^{-1}(z)$. Since $X - P^{-1}(z)$ is a weakly H-convex set for each $z \in K_h(\eta)$, we have $\Gamma_A \subset X - P^{-1}(z)$. Thus $z \in X - P^{-1}(z)$, i.e., $\langle T(z), h\,(zz, \eta(z,z))\rangle < 0$, which contradicts (a). Hence N is an H-KKM mapping. □

Theorem 2.12. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map and $F : K \to \mathbb{R}_{\circledcirc}$ be any differentiable map. Assume that*

- (a) *for all $x, y \in K_h(\eta)$, $\langle T(y), h\,(xy, \eta(x,y))\rangle \geq 0$,*
- (b) *for each $x \in K_h(\eta)$, $\langle \nabla F(x), h\,(xx, \eta(x,x))\rangle = 0$,*
- (c) *for each $y \in K$, the set*

$$B(y) = \{x \in K_h(\eta) : \langle \nabla F(y), h\,(xy, \eta(y,x))\rangle < 0\}$$

is either H-convex or empty.

Then for each $x \in K_h(\eta)$, the multivalued mapping $P : K \to 2^X$ defined by

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0\}$$

is a H-KKM mapping.

Proof. By (a), we have $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$ for each $y = x \in K_h(\eta)$, that is, $x \in P(x)$. Therefore $P(x)$ is nonempty for each $x \in K_h(\eta)$. Now to show that P is a H-KKM mapping. If not, then there exists a finite set A in $K_h(\eta)$ such that $\Gamma_A \nsubseteq \bigcup_{x\in A} P(x)$. Let there exists a $z \in \Gamma_A$ such that $z \notin \bigcup_{x\in A} P(x)$. Then $z \notin P(x)$ for all $x \in A$, that is, $\langle T(z), h\,(xz, \eta(x,z))\rangle + F(x) - F(z) < 0$

for all $x \in A$, that is, $F(x) - F(z) < -\langle T(z), h\,(xz, \eta(x,z))\rangle \leq 0$ for all $x \in A$. Since $K_h(\eta)$ is a h-η-invex set, replacing x by $h\,(xz, x+\lambda\eta(z,x))$, $\lambda \in (0,1)$, we have $F\,(h\,(xz, x+\lambda\eta(z,x))) - F(z) < 0$ for all $x \in A$. Dividing both sides by λ and letting $\lambda \to 0$ to obtain $\langle \nabla F(z), h\,(xz, \eta(z,x))\rangle < 0$ for all $x \in A$, it implies that $x \in B(z)$. Thus $A \subset B(z)$. Since $B(z)$ is a weakly H-convex set, we have $\Gamma_A \subset B(z)$. Therefore $z \in B(z)$, that is, $\langle \nabla F(z), h\,(zz, \eta(z,z))\rangle < 0$ which contradicts (b). Thus P is a H-KKM mapping. This completes the proof. □

Proposition 2.13. *If $\eta : K \times K \to X_{\circledcirc}$ satisfies condition $HC_\star$ on h-η-invex set $K_h(\eta)$ and $T : K \to X^*$, then for all $x, y \in K$, for any $\overline{y} = h\,(xy, x+\lambda\eta(y,x)) \in K_h(\eta)$ and $\lambda \in [0,1]$, we have*

- (A) $(1-\lambda)\,\langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + \lambda\,\langle T(\overline{y}), h\,(y\overline{y}, \eta(y,\overline{y}))\rangle = 0,$
- (B) $(1-\lambda)\,\langle T(u), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + \lambda\,\langle T(u), h\,(y\overline{y}, \eta(y,\overline{y}))\rangle = 0$ *for any $u \in K_h(\eta)$,*
- (C) $(1-\lambda)\,\langle T(x), h\,(x\overline{y}, \eta(\overline{y},x))\rangle + \lambda\,\langle T(y), h\,(y\overline{y}, \eta(\overline{y},y))\rangle$ $= -\langle T(x) - T(y), h\,(xy, \eta(x,y))\rangle\,.$

Proof. We have η satisfies condition $HC_\star$ with respect to h on $K_h(\eta)$. So the results (A), (B) and (C) can be proved easily because $T(x)$ is linear for each $x \in K$. So the proofs of the results are omitted. □

Theorem 2.14. *Let $(X, \{\Gamma_A\}\,; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map and $F : K \to \mathbb{R} - \{0\}$ be any differentiable map. Assume that*

- (a) *for each $y = x \in K_h(\eta)$, $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$,*
- (b) *for each $y \in K$ and $\lambda \in (0,1)$, the set*

$$B(y) = \{x \in K_h(\eta) : (1-\lambda)F(x) + \lambda F(y) < F(h(xy, x+\lambda\eta(y,x)))\}$$

is either H-convex or empty.

Then for each $x \in K_h(\eta)$, the multivalued mapping $P : K \to 2^X$ defined by

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0\}$$

is a H-KKM mapping.

Proof. By (a), we have $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$ for each $y = x \in K_h(\eta)$, that is, $x \in P(x)$. Therefore $P(x)$ is nonempty for each $x \in K_h(\eta)$. Now to show that P is a H-KKM mapping. If not, then there exists a finite set A in $K_h(\eta)$ such that $\Gamma_A \nsubseteq \bigcup_{x\in A} P(x)$. Let there exists a $z \in \Gamma_A$ such that $z \notin \bigcup_{x\in A} P(x)$, then $z \notin P(x)$ for all $x \in A$, that is,

$$\langle T(z), h\,(xz, \eta(x,z))\rangle + F(x) - F(z) < 0$$

for $x \in A$. Therefore

$$\langle T(z), h\,(yz, \eta(y,z))\rangle + F(y) - F(z) < 0$$

for $x, y \in A$. Thus

$$\begin{aligned}(1-\lambda)\,&[\langle T(z), h\,(xz, \eta(x,z))\rangle + F(x) - F(z)]\\ &+\lambda\,[\langle T(z), h\,(yz, \eta(y,z))\rangle + F(y) - F(z)] < 0\end{aligned}$$

for $x, y \in A$ and $z \in \Gamma_A$, that is,

$$\begin{aligned}(1-\lambda)\,&\langle T(z), h\,(xz, \eta(x,z))\rangle + \lambda\,\langle T(z), h\,(yz, \eta(y,z))\rangle\\ &+(1-\lambda)F(x) + \lambda F(y) - F(z) < 0\end{aligned}$$

for $x, y \in A$ and $z \in \Gamma_A$. Since $K_h(\eta)$ is a h-η-invex set, replacing z by $z = h\,(xy, x + \lambda\eta(y,x))$, $\lambda \in (0,1)$ and using Proposition 2.13(A), we have

$$(1-\lambda)\,\langle T(z), h\,(xz, \eta(x,z))\rangle + \lambda\,\langle T(z), h\,(yz, \eta(y,z))\rangle = 0$$

for $x, y \in A$, which follows that $(1-\lambda)F(x)+\lambda F(y)-F(z) < 0$ for all $x, y \in A$ and $z \in \Gamma_A$, that is,

$$(1-\lambda)F(x) + \lambda F(y) < F(z) = F\,(h\,(xy, x + \lambda\eta(y,x)))$$

for all $x, y \in A$, $\lambda \in (0,1)$ and $z \in \Gamma_A$. At $y = z$, we have

$$(1-\lambda)F(x) + \lambda F(z) < F\,(h\,(xz, x + \lambda\eta(z,x)))$$

for all $x \in A$ and $\lambda \in (0,1)$ which follows that $x \in B(z)$, that is, $A \subset B(z)$. Since $B(z)$ is a weakly H-convex set, we have $\Gamma_A \subset B(z)$. Thus $z \in B(z)$, that is,

$$(1-\lambda)F(z) + \lambda F(z) < F\,(h\,(zz, z + \lambda\eta(z,z))) = F(z).$$

Since $\eta(z,z) = 0$, $F(z) < F(z)$ which leads to a contradiction. Hence P is a H-KKM mapping. This completes the proof. □

Theorem 2.15. *Let $(X, \{\Gamma_A\}\,; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map. Let the multivalued map $P : K \to 2^X$ be defined by*

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0,\ x \in K_h(\eta)\}\,.$$

Assume that

- (a) *for each $x \in K_h(\eta)$, $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$,*
- (b) *T satisfies $\langle T(y), h\,(xy, \eta(x,y))\rangle - \langle T(x), h\,(xy, \eta(y,x))\rangle \leq 0$ for all $x, y \in K_h(\eta)$,*
- (c) *for each $y \in K$, the set*

  $$P^{-1}(y) = \{x \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) < 0\}$$

  is either H-convex or empty.

Then the multivalued map $N : K \to 2^X$ *defined by*

$$N(x) = \{y \in K_h(\eta) : \langle T(x), h\,(xy, \eta(y,x))\rangle + F(x) - F(y) \leq 0,\ x \in K_h(\eta)\}$$

is a H*-KKM mapping.*

Proof. By (a), we have $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$, for each $x \in K_h(\eta)$, that is, $x \in P(x) \cap N(x)$. Therefore $P(x)$ and $N(x)$ are nonempty for each $x \in K_h(\eta)$.

Our aim is to prove that N is a H-KKM mapping, that is, to show that there exists a finite set A in $K_h(\eta)$ such that $\Gamma_A \nsubseteq \bigcup_{x \in A} N(x)$. On the contrary, if N is not a H-KKM mapping, then there exists a $z \in \Gamma_A$ such that $z \notin \bigcup_{x \in A} N(x)$, implying $z \notin N(x)$ for all $x \in A$, that is, $\langle T(x), h\,(xz, \eta(z,x))\rangle + F(x) - F(z) > 0$ for all $x \in A$. Since T is h-η-monotone on $K_h(\eta)$, we have

$$\langle T(y), h\,(xy, \eta(x,y))\rangle - \langle T(x), h\,(xy, \eta(y,x))\rangle \leq 0$$

for all $x, y \in A$. At $y = z$, we have

$$\langle T(x), h\,(xz, \eta(z,x))\rangle - \langle T(z), h\,(xz, \eta(x,z))\rangle \leq 0$$

for all $x \in A$, that is,

$$\begin{aligned}\langle T(z), h\,(xz, \eta(x,z))\rangle + F(x) - F(z) &\leq \langle T(x), h\,(xz, \eta(z,x))\rangle + F(x) - F(z)\\ &< 0\end{aligned}$$

for all $x \in A$, that is, $x \in X - P^{-1}(z)$. Thus $A \subset X - P^{-1}(z)$. Since $X - P^{-1}(z)$ is a weakly H-convex set for each $z \in K_h(\eta)$, we have $\Gamma_A \subset X - P^{-1}(z)$. Thus $z \in X - P^{-1}(z)$, that is,

$$\langle T(z), h\,(zz, \eta(z,z))\rangle + F(z) - F(z) < 0.$$

It implies that $\langle T(z), h\,(zz, \eta(z,z))\rangle < 0$, which contradicts (a). Hence N is a H-KKM mapping. □

3. Existence theorems of solution

The following theorem establishes the existence of solution of the problem h-GVIP without using the weak H-convexity.

Theorem 3.1. *Let* $(X, \{\Gamma_A\}; K_h(\eta))$ *be a* H*-pseudospace. Let* $T : K \to X^*$ *be any continuous map. Assume that*

(a) *for each* $x \in K_h(\eta)$, $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$,
(b) *for each* $y \in K_h(\eta)$, *the mapping* $y \mapsto \langle T(y), h\,(xy, \eta(x,y))\rangle$ *is continuous,*
(c) *for each* $y \in K$, *the set* $B(y) = \{x \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle < 0\}$ *is either* H*-convex or empty,*

(d) *there is a compact set $L \subset X$ and a H-compact set $C \subset X$ such that for each weakly H-convex set $D \subset K_h(\eta)$ with $C \subset D$, we have*

$$\bigcap_{x\in D} \{P(x) \cap D\} \subset L.$$

Then there exists a $\overline{y} \in K_h(\eta)$ such that $\overline{y} \in K_h(\eta)$ solves the problem h-GVIP, that is,

$$\langle T(\overline{y}), h\,(x\overline{y}, \eta(x, \overline{y}))\rangle \geq 0 \text{ for all } x \in K_h(\eta).$$

Proof. By Theorem 2.10, the multivalued mapping $P : K \to 2^X$ defined by $P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x, y))\rangle \geq 0\}$ is a H-KKM mapping for each $x \in K_h(\eta)$. To show the existence of the solution of the problem h-GVIP, we only show that P is closed, that is, if $\{y_n\} \subset P(x)$ with $y_n \to \overline{y}$, then $\overline{y} \in P(x)$. Since for each $y \in K_h(\eta)$, the mapping $y \mapsto \langle T(y), h\,(xy, \eta(x, y))\rangle$ is continuous, we have

$$\langle T(y_n), h\,(xy_n, \eta(x, y_n))\rangle \to \langle T(\overline{y}), h\,(x\overline{y}, \eta(x, \overline{y}))\rangle$$

for each $x \in K_h(\eta)$. Since $\langle T(y_n), h\,(xy_n, \eta(x, y_n))\rangle \geq 0$ for each $x \in K_h(\eta)$ and $\mathbb{R}_+$ is closed, we have $\langle T(\overline{y}), h\,(x\overline{y}, \eta(x, \overline{y}))\rangle \geq 0$ for each $x \in K_h(\eta)$. Thus $\overline{y} \in P(x)$. Hence P is closed in $K_h(\eta) \subset X$. Since the multivalued map P satisfied the followings:

(i) P is a H-KKM mapping,
(ii) for each $x \in X$, $P(x)$ is compactly closed, that is, $B \cap F(x)$ is closed in B, for every compact $B \subset X$,
(iii) there is a compact set $L \subset X$ and a H-compact set $K \subset X$, such that for each weakly H-convex set D with $K \subset D \subset X$, we have

$$\bigcap_{x\in D} \{P(x) \cap D\} \subset L$$

which are all the conditions of Theorem 1.4. Therefore by Theorem 1.4 we have

$$\bigcap_{x\in X} P(x) \neq \varnothing,$$

that is, there exists a $\overline{y} \in K_h(\eta)$ such that $\langle T(\overline{y}), h\,(x\overline{y}, \eta(x, \overline{y}))\rangle \geq 0$ for all $x \in K_h(\eta)$. Hence $\overline{y} \in K_h(\eta)$ solves the problem h-GVIP. This completes the proof. □

Theorem 3.2. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map and $F : K \to \mathbb{R} - \{0\}$ be any differentiable map. Let for each $x \in K_h(\eta)$, the multivalued mapping $P : K \to 2^X$ defined by*

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x, y))\rangle + F(x) - F(y) \geq 0\}$$

be any mapping. Assume that

(a) *for all* $x, y \in K_h(\eta)$, $\langle T(y), h\,(xy, \eta(x,y))\rangle \geq 0$,
(b) *for each* $x \in K_h(\eta)$, $\langle \nabla F(x), h\,(xx, \eta(x,x))\rangle = 0$,
(c) *for each* $y \in K$, *the set*

$$B(y) = \{x \in K_h(\eta) : \langle \nabla F(y), h\,(xy, \eta(y,x))\rangle < 0\}$$

is either H*-convex or empty,*
(d) *for each* $y \in K_h(\eta)$, *the mapping*

$$y \mapsto \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y)$$

is continuous.

Then there exists a $\overline{y} \in K_h(\eta)$ *such that* $\overline{y} \in K_h(\eta)$ *solves the problem h-GVIP*$_F$*, that is,*

$$\langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(\overline{y}) \geq 0 \text{ for all } x \in K_h(\eta).$$

Proof. By Theorem 2.12, the multivalued mapping $P : K \to 2^X$ defined by

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0\}$$

is a H-KKM mapping for each $x \in K_h(\eta)$. To show the existence of the solution of the problem h-GVIP$_F$, we only show that P is closed, that is, if $\{y_n\} \subset P(x)$ with $y_n \to \overline{y}$, then $\overline{y} \in P(x)$. Since for each $y \in K_h(\eta)$, the mapping

$$y \mapsto \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y)$$

is continuous, we have

$$\langle T(y_n), h\,(xy_n, \eta(x,y_n))\rangle + F(x) - F(y_n) \to \langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(y)$$

for each $x \in K_h(\eta)$. Since $\langle T(y_n), h\,(xy_n, \eta(x,y_n))\rangle + F(x) - F(y_n) \geq 0$ for each $x \in K_h(\eta)$ and $\mathbb{R}_+$ is closed, we have

$$\langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(\overline{y}) \geq 0$$

for each $x \in K_h(\eta)$. Thus $\overline{y} \in P(x)$. Hence P is closed in $K_h(\eta) \subset X$. Since the multivalued map P satisfied the followings:

(i) P is a H-KKM mapping,
(ii) for each $x \in X$, $P(x)$ is compactly closed, that is, $B \cap F(x)$ is closed in B, for every compact $B \subset X$,
(iii) there is a compact set $L \subset X$ and a H-compact set $K \subset X$, such that for each weakly H-convex set D with $K \subset D \subset X$, we have

$$\bigcap_{x \in D} \{P(x) \cap D\} \subset L$$

which are all the conditions of Theorem 1.4. Therefore by Theorem 1.4 we have $\bigcap_{x\in X} P(x) \neq \varnothing$, that is, there exists a $\overline{y} \in K_h(\eta)$ such that

$$\langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(\overline{y}) \geq 0$$

for all $x \in K_h(\eta)$. Hence $\overline{y} \in K_h(\eta)$ solves the problem h-GVIP$_F$. This completes the proof. □

Theorem 3.3. *Let $(X, \{\Gamma_A\}\,; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map and $F : K \to \mathbb{R} - \{0\}$ be any differentiable map. Let for each $x \in K_h(\eta)$, the multivalued mapping $P : K \to 2^X$ defined by*

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0\}$$

be any mapping. Assume that

- *(a) for each $y = x \in K_h(\eta)$, $\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$,*
- *(b) for each $y \in K$ and $\lambda \in (0,1)$, the set*

  $$B(y) = \{x \in K_h(\eta) : (1-\lambda)F(x) + \lambda F(y) < F(h(xy, x + \lambda\eta(y,x)))\}$$

  is either H-convex or empty,
- *(c) for each $y \in K_h(\eta)$, the mapping $y \mapsto \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y)$ is continuous,*
- *(d) there is a compact set $L \subset X$ and an H-compact set $C \subset X$ such that for each weakly H-convex set $D \subset K_h(\eta)$ with $C \subset D$, we have*

  $$\bigcap_{x\in D} \{P(x) \cap D\} \subset L.$$

Then there exists a $\overline{y} \in K_h(\eta)$ such that $\overline{y} \in K_h(\eta)$ solves the problem h-GVIP$_F$, that is,

$$\langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(\overline{y}) \geq 0 \text{ for all } x \in K_h(\eta).$$

Proof. By Theorem 2.15, the multivalued mapping $P : K \to 2^X$ defined by

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0\}$$

is a H-KKM mapping for each $x \in K_h(\eta)$. To show the existence of the solution of the problem h-GVIP$_F$, we only show that P is closed, that is, if $\{y_n\} \subset P(x)$ with $y_n \to \overline{y}$, then $\overline{y} \in P(x)$. Since for each $y \in K_h(\eta)$, the mapping

$$y \mapsto \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y)$$

is continuous, we have

$$\langle T(y_n), h\,(xy_n, \eta(x,y_n))\rangle + F(x) - F(y_n) \to \langle T(\overline{y}), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(y)$$

for each $x \in K_h(\eta)$. Since

$$\langle T(y_n), h\,(xy_n, \eta(x,y_n))\rangle + F(x) - F(y_n) \geq 0$$

for each $x \in K_h(\eta)$ and $\mathbb{R}_+$ is closed, we have

$$\langle T(\overline{y}), h\left(x\overline{y}, \eta(x, \overline{y})\right)\rangle + F(x) - F(\overline{y}) \geq 0$$

for each $x \in K_h(\eta)$. Thus $\overline{y} \in P(x)$. Hence P is closed in $K_h(\eta) \subset X$. Since the multivalued map P satisfies

(i) P is a H-KKM mapping,
(ii) for each $x \in X$, $P(x)$ is compactly closed, that is, $B \cap F(x)$ is closed in B, for every compact $B \subset X$,
(iii) there is a compact set $L \subset X$ and an H-compact set $K \subset X$, such that for each weakly H-convex set D with $K \subset D \subset X$, we have

$$\bigcap_{x \in D} \{P(x) \cap D\} \subset L$$

which are all the conditions of Theorem 1.4. Therefore by Theorem 1.4 we have

$$\bigcap_{x \in X} P(x) \neq \varnothing,$$

that is, there exists a $\overline{y} \in K_h(\eta)$ such that

$$\langle T(\overline{y}), h\left(x\overline{y}, \eta(x, \overline{y})\right)\rangle + F(x) - F(\overline{y}) \geq 0$$

for all $x \in K_h(\eta)$. Hence $\overline{y} \in K_h(\eta)$ solves the problem h-GVIP$_F$. This completes the proof. □

In the following theorem we apply Theorem 1.5 to show the existence of solution of the problem h-GDVIP in a h-η-invex set.

Theorem 3.4. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map. Let $P, N : K \to 2^X$ be the multivalued closed maps defined by*

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\left(xy, \eta(x, y)\right)\rangle \geq 0,\ x \in K_h(\eta)\}$$

and

$$N(x) = \{y \in K_h(\eta) : \langle T(x), h\left(xy, \eta(y, x)\right)\rangle \leq 0\}$$

for each $x \in K_h(\eta)$. Assume that

(a) *for each $x \in K_h(\eta)$, $P(x)$ is compactly closed in X,*
(b) *for each $x \in K_h(\eta)$, $\langle T(x), h\left(xx, \eta(x, x)\right)\rangle = 0$,*
(c) *T is h-η-monotone on $K_h(\eta)$,*
(d) *for each $y \in K$, the set $P^{-1}(y) = \{x \in K_h(\eta) : \langle T(y), h\left(xy, \eta(x, y)\right)\rangle < 0\}$ is either H-convex or empty,*
(e) *the map $y \mapsto \langle T(x), h\left(xy, \eta(y, x)\right)\rangle$ is continuous,*

(*f*) *there is a compact set* $L \subset X$ *and an* H*-compact set* $C \subset X$ *such that for each weakly* H*-convex set* $D \subset K_h(\eta)$ *with* $C \subset D$*, we have*

$$\bigcap_{x \in D} \{N(x) \cap D\} \subset L.$$

Then there exists a $\overline{y} \in K_h(\eta)$ *such that* $\overline{y} \in K_h(\eta)$ *solves the problem h-GDVIP, that is,* $\langle T(x), h(x\overline{y}, \eta(\overline{y}, x))\rangle \leq 0$ *for all* $x \in K_h(\eta)$.

Proof. As T is h-η-monotone on $K_h(\eta)$, we have

$$\langle T(x), h(xy, \eta(y,x))\rangle + \langle T(y), h(xy, \eta(x,y))\rangle \leq 0$$

for all $x, y \in K_h(\eta)$. If $y \in P(x)$, then $\langle T(y), h(xy, \eta(x,y))\rangle \geq 0$ for all $x, y \in K_h(\eta)$, implying

$$\langle T(x), h(xy, \eta(y,x))\rangle \leq -\langle T(y), h(xy, \eta(x,y))\rangle \leq 0$$

for all $x, y \in K_h(\eta)$, that is, $y \in N(x)$. Thus $P(x) \subset N(x)$ for each $x \in K_h(\eta)$ which is a condition of Theorem 1.5. Since for each $x \in K_h(\eta)$, $\langle T(x), h(xx, \eta(x,x))\rangle = 0$, we have $x \in P(x)$ which is a condition of Theorem 1.5. Since for each $x \in K_h(\eta)$, the set

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h(xy, \eta(x,y))\rangle \geq 0,\ x \in K_h(\eta)\}$$

is compactly closed in X, we have

$$\begin{aligned} P^{-1}(y) &= \{x \in K_h(\eta) : y \in P(x)\} \\ &= \{x \in K_h(\eta) : \langle T(y), h(xy, \eta(x,y))\rangle \geq 0\} \end{aligned}$$

is closed in X. By Theorem 2.11, N is a H-KKM mapping. Now to apply Theorem 1.5, we need to show that N is closed, that is, if there exists a sequence $\{y_n\} \in N(x)$ such that $y_n \to \overline{y}$ in $K_h(\eta)$, then $\overline{y} \in N(x)$. For $\{y_n\} \in N(x)$, we have $\langle T(x), h(xy_n, \eta(y_n, x))\rangle \leq 0$ for all $x \in K_h(\eta)$. Since the map which takes y to $\langle T(x), h(xy, \eta(y,x))\rangle$ is continuous, we have

$$\langle T(x), h(xy_n, \eta(y_n, x))\rangle \to \langle T(x), h(x\overline{y}, \eta(\overline{y}, x))\rangle$$

for all $x \in K_h(\eta)$ as $y_n \to \overline{y}$ in $K_h(\eta)$. Since $P(x)$ is compactly closed in X, we have

$$\langle T(y_n), h(xy_n, \eta(x, y_n))\rangle \to \langle T(y), h(x\overline{y}, \eta(x, \overline{y}))\rangle \geq 0$$

for all $x \in K_h(\eta)$ as $y_n \to \overline{y}$. Since

$$\begin{aligned} &\langle T(x), h(xy_n, \eta(y_n, x))\rangle + \langle T(y_n), h(xy_n, \eta(x, y_n))\rangle \\ &\quad \to \langle T(x), h(x\overline{y}, \eta(\overline{y}, x))\rangle + \langle T(y), h(x\overline{y}, \eta(x, \overline{y}))\rangle \leq 0 \end{aligned}$$

for all $x \in K_h(\eta)$ as $y_n \to \overline{y}$ in $K_h(\eta)$ and $P(x) \subset N(x)$, we have

$$\langle T(x), h(xy, \eta(y,x))\rangle \leq 0$$

for all $x \in K_h(\eta)$. Therefore $\overline{y} \in N(x)$, that is, N is closed in $K_h(\eta)$. Since all the conditions of Theorem 1.5 are satisfied, we obtain $\bigcap_{x\in D} N(x) \neq \varnothing$, that is, there exists a $\overline{y} \in K_h(\eta)$ such that $\langle T(x), h\left(x\overline{y}, \eta(\overline{y}, x)\right)\rangle \leq 0$ for all $x \in D \subset K_h(\eta)$. Hence $\overline{y} \in K_h(\eta)$ solves the problem h-GDVIP. This completes the proof. □

Theorem 3.5. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Let $T : K \to X^*$ be any continuous map. Let $P : K \to 2^X$ and $N : K \to 2^X$ be two multivalued maps defined by*

$$P(x) = \left\{y \in K_h(\eta) : \langle T(y), h\left(xy, \eta(x, y)\right)\rangle + F(x) - F(y) \geq 0,\ x \in K_h(\eta)\right\},$$

and

$$N(x) = \left\{y \in K_h(\eta) : \langle T(x), h\left(xy, \eta(y, x)\right)\rangle + F(x) - F(y) \leq 0,\ x \in K_h(\eta)\right\}.$$

Assume that

(a) *for each $x \in K_h(\eta)$, $\langle T(x), h\left(xx, \eta(x, x)\right)\rangle = 0$,*
(b) *T satisfies $\langle T(y), h\left(xy, \eta(x, y)\right)\rangle - \langle T(x), h\left(xy, \eta(y, x)\right)\rangle \leq 0$ for all $x, y \in K_h(\eta)$,*
(c) *for each $y \in K$, the set*

$$P^{-1}(y) = \{x \in K_h(\eta) : \langle T(y), h\left(xy, \eta(x, y)\right)\rangle + F(x) - F(y) < 0\}$$

is either H-convex or empty,
(d) *for each $y \in K_h(\eta)$, the mapping $y \mapsto \langle T(y), h\left(xy, \eta(x, y)\right)\rangle + F(x) - F(y)$ is continuous,*
(e) *there is a compact set $L \subset X$ and a H-compact set $C \subset X$ such that for each weakly H-convex set $D \subset K_h(\eta)$ with $C \subset D$, we have*

$$\bigcap_{x\in D} \{N(x) \cap D\} \subset L.$$

Then there exists a $\overline{y} \in K_h(\eta)$ such that $\overline{y} \in K_h(\eta)$ solves the problem h-GDVIP$_F$, that is,

$$\langle T(x), h\left(x\overline{y}, \eta(\overline{y}, x)\right)\rangle + F(x) - F(\overline{y}) \leq 0 \ for\ all\ x \in K_h(\eta).$$

Proof. As T satisfies $\langle T(y), h\left(xy, \eta(x, y)\right)\rangle - \langle T(x), h\left(xy, \eta(y, x)\right)\rangle \leq 0$ for all $x, y \in K_h(\eta)$. If $y \in P(x)$, then $\langle T(y), h\left(xy, \eta(x, y)\right)\rangle + F(x) - F(y) \geq 0$ for all $x, y \in K_h(\eta)$ implying

$$\langle T(y), h\left(xy, \eta(x, y)\right)\rangle + F(x) - F(y) \leq \langle T(x), h\left(xy, \eta(y, x)\right)\rangle + F(x) - F(y)$$

for all $x, y \in K_h(\eta)$, that is, $y \in N(x)$. Thus $P(x) \subset N(x)$ for each $x \in K_h(\eta)$ which is a condition of Theorem 1.5. Since for each $x \in K_h(\eta)$,

$\langle T(x), h\,(xx, \eta(x,x))\rangle = 0$, we have $x \in P(x)$, which is a condition of Theorem 1.5. Since for each $x \in K_h(\eta)$, the set

$$P(x) = \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0,\ x \in K_h(\eta)\}$$

is compactly closed in X, we have

$$\begin{aligned} P^{-1}(y) &= \{x \in K_h(\eta) : y \in P(x)\} \\ &= \{x \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x,y))\rangle + F(x) - F(y) \geq 0\} \end{aligned}$$

is closed in X. By Theorem 2.15, N is a H-KKM mapping. Now to apply Theorem 1.5, we need to show that N is closed, that is, if there exists a sequence $\{y_n\} \in N(x)$ such that $y_n \to \overline{y}$ in $K_h(\eta)$, then $\overline{y} \in N(x)$. For $\{y_n\} \in N(x)$, we have

$$\langle T(x), h\,(xy_n, \eta(y_n,x))\rangle + F(x) - F(y) \leq 0$$

for all $x \in K_h(\eta)$. Since the map which takes y to

$$\langle T(x), h\,(xy, \eta(y,x))\rangle + F(x) - F(y)$$

is continuous, we have

$$\langle T(x), h\,(xy_n, \eta(y_n,x))\rangle + F(x) - F(y_n) \to \langle T(x), h\,(x\overline{y}, \eta(\overline{y},x))\rangle + F(x) - F(y_n)$$

for all $x \in K_h(\eta)$ as $y_n \to \overline{y}$ in $K_h(\eta)$. Since $P(x)$ is compactly closed in X, we have

$$\begin{aligned} &\langle T(y_n), h\,(xy_n, \eta(x,y_n))\rangle + F(x) - F(y_n) \\ &\quad \to \langle T(y), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(\overline{y}) \geq 0 \end{aligned}$$

for all $x \in K_h(\eta)$ as $y_n \to \overline{y}$. Since

$$\begin{aligned} &\langle T(y_n), h\,(xy_n, \eta(x,y_n))\rangle - \langle T(x), h\,(xy_n, \eta(y_n,x))\rangle \\ &\quad \to \langle T(y), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle - \langle T(x), h\,(x\overline{y}, \eta(\overline{y},x))\rangle \leq 0 \end{aligned}$$

for all $x \in K_h(\eta)$ as $y_n \to \overline{y}$ in $K_h(\eta)$ and $P(x) \subset N(x)$, we have

$$\langle T(x), h\,(x\overline{y}, \eta(\overline{y},x))\rangle + F(x) - F(\overline{y}) \leq 0.$$

Hence,

$$\langle T(y), h\,(x\overline{y}, \eta(x,\overline{y}))\rangle + F(x) - F(\overline{y}) \geq 0$$

for all $x \in K_h(\eta)$. Therefore $\overline{y} \in N(x)$, that is, N is closed in $K_h(\eta)$. Since all the conditions of Theorem 1.5 are satisfied, we obtain $\bigcap_{x\in D} N(x) \neq \varnothing$, that is, there exists a $\overline{y} \in K_h(\eta)$ such that

$$\langle T(x), h\,(x\overline{y}, \eta(\overline{y},x))\rangle + F(x) - F(\overline{y}) \leq 0$$

for all $x \in D \subset K_h(\eta)$. Hence $\overline{y} \in K_h(\eta)$ solves the problem h-GDVIP$_F$. This completes the proof. □

In the following theorem, we modify the Lemma 1.7 to show the existence of solution of the problems in a H-space.

Theorem 3.6. *Let $(X, \{\Gamma_A\}; K_h(\eta))$ be a H-pseudospace. Assume that the multivalued maps $P : X \to 2^X$ and $N : X \to 2^X$ satisfy the following conditions:*

(i) *$N(x)$ is compactly closed for all x in X with $P(x) \subset N(x)$,*
(ii) *P is an H-KKM mapping,*
(iii) *there exists a compact subset $L(x)$ of X and a H-compact subset $C(x)$ of X such that for each weakly H-convex subset $D(x)$ of X with $C(x) \subset D(x)$ we have $\bigcap_{x\in D(x)} (P(x) \cap D(x)) \subset L(x)$,*
(iv) *for every H-convex subset $D(x)$ of X with $C(x) \subset D(x)$, we have $\bigcap_{x\in D(x)} (N(x) \cap D(x)) \neq \varnothing$ if and only if $\bigcap_{x\in D(x)} (P(x) \cap D(x)) \neq \varnothing$.*

Then $\bigcap_{x\in D(x)} P(x) \neq \varnothing$.

Proof. Since generalized H-space is a H-space, the proof of this theorem is directly followed from the Lemma 1.7. If we can consider X is a H-space, $L = L(x)$, $D = D(x)$ and $C = C(x)$, then according to the result studied by Verma (see [15], Theorem 2.1, p.134), we have

$$\bigcap_{x\in D(x)} P(x) \neq \varnothing.$$

This completes the proof. □

Theorem 3.7. *Let $K_h(\eta)$ be a h-η-invex set where η satisfies condition $HC_\star$. Let $T : K \to X^*$ be any continuous map and $F : K \to \mathbb{R} - \{0\}$ be any differentiable map. Assume that for $x \in K_h(\eta)$, the following sets are nonempty:*

$$\begin{aligned}
P(x) &= \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x, y))\rangle + F(x) - F(y) \geq 0\},\\
N(x) &= \{y \in K_h(\eta) : \langle T(x), h\,(xy, \eta(y, x))\rangle + F(x) - F(y) \leq 0\},\\
K_1(x) &= \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x, y))\rangle + \langle T(x), h\,(xy, \eta(y, x))\rangle \geq 0\},\\
K_2(x) &= \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x, y))\rangle = \langle T(x), h\,(xy, \eta(y, x))\rangle\},\\
K_3(x) &= \{y \in K_h(\eta) : \langle T(y), h\,(xy, \eta(x, y))\rangle + \langle T(x), h\,(xy, \eta(y, x))\rangle\\
&\qquad + 2\,(F(x) - F(y)) \leq 0\},\\
K_4(x) &= \{y \in K_h(\eta) : \langle \nabla F(y), h\,(xy, \eta(y, x))\rangle \leq 0\},\\
L(x) &= \{y \in K_h(\eta) : F(x) \geq F(y)\}.
\end{aligned}$$

Let P be a H-KKM mapping, $N(x)$ be a compactly closed set, $L(x)$ be a compact set for each $x \in K$, $K_3(x)$ be a H-compact in K, $N(x) \cap K_2(x) \cap K_4(x)$

be a weakly H-convex set. Then there exists a $\overline{y} \in K_h(\eta)$ such that $\overline{y}$ solves the problem h-GVIP$_F$ on $D(x) = N(x) \cap K_2(x) \cap K_4(x)$.

Proof. We consider $C(x) = K_3(x)$ and $D(x) = N(x) \cap K_2(x) \cap K_4(x)$. Now $C(x) \subset D(x)$ for all $x \in K_1(x)$. For $x \in D(x)$, we have $F(x) - F(y) \leq 0$ and for $x \in P(x) \cap K_4(x)$, we have $F(x) - F(y) \geq 0$, therefore we obtain $\bigcap_{x\in D(x)} (N(x) \cap D(x)) \neq \varnothing$ if and only if $\bigcap_{x\in D(x)} (P(x) \cap D(x)) \neq \varnothing$. Thus

(i) $N(x)$ is compactly closed in $K_h(\eta)$ and $P(x) \subset N(x)$ for all $x \in C(x)$,

(ii) P is an H-KKM mapping in $K_h(\eta)$,

(iii) there exists a compact subset $L(x)$ of X and a H-compact subset $C(x)$ of X such that for each weakly H-convex subset $D(x)$ of X with $C(x) \subset D(x)$ we have

$$\bigcap_{x\in D(x)} (P(x) \cap D(x)) \subset L(x),$$

(iv) for every H-convex subset $D(x)$ of X with $C(x) \subset D(x)$, we have

$$\bigcap_{x\in D(x)} (N(x) \cap D(x)) \neq \varnothing \quad \text{if and only if} \quad \bigcap_{x\in D(x)} (P(x) \cap D(x)) \neq \varnothing,$$

which are the conditions of Theorem 3.6. Therefore by Theorem 3.6,

$$\bigcap_{x\in D(x)} P(x) \neq \varnothing,$$

that is, there exists a $\overline{y} \in K_h(\eta)$ such that

$$\langle T(\overline{y}), h\left(x\overline{y}, \eta(x, \overline{y})\right)\rangle + F(x) - F(\overline{y}) \geq 0$$

for all $x \in D(x)$. Hence $\overline{y} \in K_h(\eta)$ solves the problem h-GVIP$_F$ on $D(x)$. This completes the proof. □

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