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NEW SUBCLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS ASSOCIATED WITH HYPERGEOMETRIC FUNCTION

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Abstract. As hypergeometric meromorphic multivalent functions of the form

$$L^{t,\rho}_{\varpi,\sigma} f(\zeta) = \frac{1}{\zeta^{\rho}} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \ . \ \frac{(\rho-(\kappa+2\rho)t)}{\rho} a_{\kappa+\rho}\zeta^{\kappa+\rho}$$

contains a new subclass in the punctured unit disk $\Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$ for $-1 \le D < S \le 1$, this paper aims to determine sufficient conditions, distortion properties and radii of starlikeness and convexity for functions in the subclass $L^{t,\rho}_{\varpi,\sigma} f(\zeta)$.

1. Introduction

Let Σ_ρ denote the class of meromorphic multivalent functions $f(\zeta)$ normalized by

$$f(\zeta) = \tfrac{1}{\zeta^{\rho}} + \textstyle\sum_{\kappa=0}^{\infty} a_{\kappa+\rho}\zeta^{\kappa+\rho} \ \ ; \rho \in \mathbb{N}\backslash\{0\} \tag{1.1}$$

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which are analytic in the punctured unit disk

$$\mathbb{U}^* = \{\zeta : \zeta \in \mathbb{C} \text{ and } 0 < |\zeta| < 1\} = \mathbb{U}\backslash \{0\},$$

where $\mathbb{C}$ is the set of complex numbers.

The starlike and convex functions are most important subclass of meromorphic functions, as they have very useful characteristics. See for example (Aouf *et al.* [4], Ghanim and Darus [8], Srivastava [12], Kulkarni *et al.* [13], Morga [17], Owa *et al.* [18], Srivastava and Owa [19], Uralegaddi and Somantha [20], and Yang [21]).

The functions $f_v(\zeta), (v = 1, 2)$ are defined by:

$$f_v(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho,v}\zeta^{\kappa+\rho}, (v = 1, 2).$$

The convolution (or Hadamard product) of $f_1(\zeta)$ and $f_2(\zeta)$ are defined by :

$$(f_1 * f_2)(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho,1}a_{\kappa+\rho,2}\zeta^{\kappa+\rho} = (f_2 * f_1)(\zeta) .$$

Let the function$\widetilde{}_{\rho}(\varpi, \sigma; \zeta)$ be defined by:

$$\widetilde{}_{\rho}(\varpi, \sigma; \zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \zeta^{\kappa+\rho}, (\sigma \in \mathbb{C}/(\mathbb{Z}^- \bigcup \{0\}) ; \varpi \in \mathbb{C}) , \quad (1.2)$$

where $(\tau)_\kappa$ is defined by:

$$(\tau)_\kappa := \frac{\Gamma(\tau + \kappa)}{\Gamma(\tau)} = \begin{cases} \tau(\tau + 1)...(\tau + \kappa - 1), & (\kappa = n \in \mathbb{N}; \tau \in \mathbb{C}) \\ 1, & (\kappa = 0; \tau \in \mathbb{C}\backslash \{0\}), \end{cases}$$

$(\tau)_\kappa$ is called the Pochhammer symbol and Γ is Gamma function.

The meromorphic functions with the generalized hypergeometric functions were considered recently by Al-Janaby and Ghanim [2], Al-Janaby *et. al.* [3], Cho and Kim [5], Dziok and Srivastava [6], El-Ashwah [7], Ghanim and Darus [9-11], Liu [14], and Liu and Srivastava [15].

Let the function f which is defined in (1.1) belong to the subclass Σ_ρ, we start by reintroducing $c^t f(\zeta)$ which was studied by [16]:

$$\begin{aligned} c^t f(\zeta) &= (1 - t)f(\zeta) + \tfrac{t\ \zeta(-f(\zeta))'}{\rho} \\ &= \tfrac{1}{\zeta^\rho} + \textstyle\sum_{\kappa=0}^{\infty} \tfrac{(\rho-(\kappa+2\rho)t)}{\rho} a_{\kappa+\rho}\zeta^{\kappa+\rho}, (t \geq 0). \end{aligned} \quad (1.3)$$

Using the convolution between (1.2) and (1.3), we will introduce a new function $L^{t,\rho}_{\varpi,\sigma}$ defined on Σ_ρ by:

$$\begin{aligned} L^{t,\rho}_{\varpi,\sigma} f(\zeta) &= \widetilde{}_{\rho}(\varpi,\sigma;\zeta) \ * c^t f(\zeta) \\ &= \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho-(\kappa+2\rho)t)}{\rho} \ a_{\kappa+\rho}\zeta^{\kappa+\rho}, (\zeta \in \ \mathbb{U}^*) . \end{aligned} \tag{1.4}$$

For all $\zeta \in \mathbb{U}^*$ and $-1 \leq D < S \leq 1$, the function $f \in \Sigma_\rho$ is said to be a member of the subclass $\Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$ if it satisfies:

$$\left| \frac{\zeta \ (L^{t,\rho}_{\varpi,\sigma} f(\zeta))' + \rho \ L^{t,\rho}_{\varpi,\sigma} f(\zeta)}{D\zeta \ (\ L^{t,\rho}_{\varpi,\sigma} f(\zeta))' \ + \ S\rho \ (\ L^{t,\rho}_{\varpi,\sigma} \ f(\zeta))} \right| < 1. \tag{1.5}$$

See for example [1] and [8].

2. Coefficient estimates and distortion theorems

In this section, our first result will be concerning the coefficient estimates and distortion theorem for the subclass $\Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$.

Theorem 2.1. *Let the function f be defined by* (1.4) *and satisfies* (1.5). *Then*

$$\begin{aligned} &\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho) \ t)}{\rho} \right| [(\kappa+2\rho) - (\ (\kappa+\rho)D + S\rho)] \ |a_{\kappa+\rho}| \\ &\leq \ \rho(\ S-D), \end{aligned} \tag{2.1}$$

where $-1 \leq D < S \leq 1$.

Proof. Suppose that (1.5) holds. Let

$$M = \zeta \left(\frac{-\rho}{\zeta^{\rho+1}} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} . \frac{(\rho-(\kappa+2\rho)t)}{\rho} (\kappa+\rho) a_{k+\rho} \zeta^{k+\rho-1} \right),$$

$$N = \frac{\rho}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} . \frac{(\rho-(k+2\rho)t)}{\rho} \rho a_{\kappa+\rho} \zeta^{\kappa+\rho},$$

$$P = D\zeta \left(\frac{-\rho}{\zeta^{\rho+1}} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} . \frac{(\rho-(\kappa+2\rho)t)}{\rho} (\kappa+\rho) \quad a_{\kappa+\rho} \zeta^{\kappa+\rho-1} \right)$$

and

$$Q = S\rho \left(\frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} a_{\kappa+\rho} \zeta^{\kappa+\rho} \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right).$$

Then

$$\left| \frac{M+N}{P+Q} \right| \leq 1,$$

it implies that

$$\left|\frac{\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\cdot\frac{(\rho-(\kappa+2\rho)t)}{\rho}\{\ \kappa+2\rho\}\ \ a_{\kappa+\rho}\zeta^{\kappa+\rho}}{\frac{S\rho-\rho D}{\zeta^{\rho}}+\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\cdot\frac{(\rho-(\kappa+2\rho)t)}{\rho}\{(\kappa+\rho)D+S\rho\}\ \ a_{\kappa+\rho}\zeta^{\kappa+\rho}}\right|\ \leq 1.$$

Thus we have

$$\left|\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\cdot\ \frac{(\rho-(\kappa+2\rho)t)}{\rho}\ \{\kappa+2\rho\}\,a_{\kappa+\rho}\zeta^{\kappa+\rho}\right|$$
$$-\left|\frac{S\rho-\rho D}{\zeta^{\rho}}+\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\ \frac{(\rho-(\kappa+2\rho)t)}{\rho}\{(\kappa+\rho)D+S\rho\}\,a_{\kappa+\rho}\zeta^{\kappa+\rho}\right|\leq 0.$$

Therefore,

$$\left|\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\frac{(\rho-(\kappa+2\rho)t)}{\rho}\{\kappa+2\rho\}\ \ a_{\kappa+\rho}\zeta^{k+2\rho}\right|$$
$$-\left|S\rho-\rho D+\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\frac{(\rho-(k+2\rho)t)}{\rho}\ \ \{(k+\rho)D+S\rho\}\,a_{\kappa+\rho}\zeta^{\kappa+2\rho}\right|\leq 0.$$

Hence we have

$$\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{k+2}}\left|\frac{\rho-(\kappa+2\rho)\ t}{\rho}\ \right|\ \ (\kappa+2\rho)\left|a_{\kappa+\rho}\right|\left|\zeta^{\kappa+2\rho}\right|$$
$$-S\rho+\rho D-\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\left|\frac{\rho-(\kappa+2\rho)t}{\rho}\right|((\kappa+\rho)D+S\rho)\left|a_{\kappa+\rho}\right|\left|\zeta^{\kappa+2\rho}\right|\ \leq 0.$$

If $|\zeta|=r\ \rightarrow 1$, then we have

$$\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{k+2}}{(\sigma)_{k+2}}\left|\frac{(\rho-(\kappa+2\rho)\ t)}{\rho}\right|[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]\left|a_{\kappa+\rho}\right|\leq\rho(S-D).$$

Hence it follows from (1.5) that $f\in\Sigma_{\varpi,\sigma}^{S,D}(t,\kappa,\rho)$ and the intended result is achieved. □

Corollary 2.2. *Let the function f be defined by (1.4). If $f\in\Sigma_{\varpi,\sigma}^{S,D}(t,\kappa,\rho)$, then*

$$|a_{\kappa+\rho}|\leq\frac{\rho^{2}\ \ (\sigma)_{\kappa+2}\ (S-D)}{(\varpi)_{\kappa+2}\left|(\rho-(\kappa+2\rho)t)\right|[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]},(\kappa\geq 0).\tag{2.2}$$

Our assertion in Theorem 2.1 is sharp for functions of the form:

$$f_\kappa(\zeta) = \frac{1}{\zeta^\rho} + \frac{\rho^2(\sigma)_{k+2}(S-D)}{(\varpi)_{\kappa+2}\,|(\rho-(\kappa+2\rho)t)|\,[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}\zeta^{\kappa+\rho}. \tag{2.3}$$

Corollary 2.3. *Let $S = 1$ and $D = -1$ in Theorem 2.1. Then we have*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| (\kappa+\rho)\,|a_{\kappa+\rho}| \le \rho$$

and therefore the function $L^{t;\rho}_{\varpi,\sigma}f(\zeta)$ is starlike in $\mathbb{U}^*$.

Corollary 2.4. *If $t = 0, S = 1$ and $D = -1$ in Theorem 2.1, then $f \in \Sigma_\rho$ satisfying the following condition:*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} (\kappa+\rho)\,|a_{\kappa+\rho}| \le \rho$$

and it is starlike in $\mathbb{U}^*$.

Corollary 2.5. *For $S = 1$ and $D - 1$ in Theorem 2.1, we have:*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa+2\rho)t)}{\rho} \right| (\kappa+\rho)^2\,|a_{\kappa+\rho}| \le \rho$$

and therefore the function $L^{t;\rho}_{\varpi,\sigma}f(\zeta)$ is convex in $\mathbb{U}^*$.

Corollary 2.6. *If $t = 0, S = 1$ and $D = -1$ in Theorem 2.1, then $f \in \Sigma_\rho$ satisfies the following condition:*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} (\kappa+\rho)^2\,|a_{\kappa+\rho}| \le \rho$$

and it is convex in $\mathbb{U}^*$.

A distortion property for functions in the subclass $\Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$ is given in the following result:

Theorem 2.7. *If the function f is defined by (1.4) in the subclass $\Sigma^{E,S}_{\varpi,\sigma}(t,\kappa,\rho)$, then for $0 < |\zeta| = r < 1$, we have*

$$\frac{1}{r^\rho} - r^\rho\frac{(S-D)}{[2-(S+D)]} \le |f(\zeta)| \le \frac{1}{r^\rho} + r^\rho\frac{(S-D)}{[2-(S+D)]} \tag{2.4}$$

and

$$\frac{\rho}{r^{\rho+1}} - \frac{\rho(S-D)}{[2-(S+D)]}\,r^{\rho-1} \le |f'(\zeta)| \le \frac{\rho}{r^{\rho+1}} + \frac{\rho(S-D)}{[2-(S+D)]}\,r^{\rho-1} \tag{2.5}$$

with equality for

$$f(\zeta) = \frac{1}{\zeta^\rho} + \frac{(S-D)}{[2-(D+S)]} \zeta^\rho .$$

Proof. Let $f \in \Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$. Then

$$|f(\zeta)| \leq \frac{1}{|\zeta^\rho|} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+\rho}$$

$$\leq \frac{1}{r^\rho} + r^\rho \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (k+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|.$$

Theorem 2.1 readily yields the inequality:

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{k+2}}{(\sigma)_{k+2}} \left| \frac{(\rho - (\kappa+2\rho)\ t)}{\rho} \right| |a_{\kappa+\rho}| \leq \frac{\rho(S-D)}{[(\kappa+2\rho) - ((\kappa+\rho)\ D + S\rho)]} \quad (2.6)$$

thus, for $0 < |\zeta| = r < 1$, we have

$$|f(\zeta)| \leq \frac{1}{r^\rho} + r^\rho \frac{\rho(S-D)}{[(\kappa+2\rho) - ((\kappa+\rho)D + S\rho)]}, (\kappa \geq 0).$$

Hence, we have

$$|f(\zeta)| \leq \frac{1}{r^\rho} + r^\rho \frac{(S-D)}{[2-(S+D)]} \quad (2.7)$$

and the other side of the inequality in (2.4) can be obtained using similar procedure. For (2.5),

$$|f'(\zeta)| \leq \frac{\rho}{|\zeta^{\rho+1}|} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa+2\rho)t)}{\rho} \right| (\kappa+\rho) |a_{\kappa+\rho}| |\zeta|^{\kappa+\rho-1}.$$

It follows from Theorem 2.1 that

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{k+2}}{(\sigma)_{k+2}} \left| \frac{(\rho - (\kappa+2\rho)\ t)}{\rho} \right| (\kappa+\rho) |a_{\kappa+\rho}| \leq \frac{(\kappa+\rho)\rho(SD)}{[(\kappa+2\rho) - ((\kappa+\rho)D + S\rho)]} \quad (2.8)$$

thus, for $0 < |\zeta| = r < 1$, we have

$$|f'(\zeta)| \leq \frac{\rho}{r^{\rho+1}} + \frac{(\kappa+\rho)\rho(S-D)}{[(\kappa+2\rho) - ((\kappa+\rho)D + S\rho)]} r^{\kappa+\rho-1}, (\kappa \geq 0).$$

Hence, we have

$$|f'(\zeta)| \leq \frac{\rho}{r^{\rho+1}} + \frac{\rho(S-D)}{[2-(S+D)]} r^{\rho-1} \quad (2.9)$$

By similarity, the other side of the inequality follows and the proof is complete. □

3. Starlikeness and Radii of convexity

Theorem 3.1. *If the function f is in the subclass $\Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$, then f is meromorphically starlike of order $\chi(0 \leq \chi < 1)$ in $|\zeta| < r_1$,*

$$r_1 = r_1(S,D,\kappa,\rho) = \inf_{\kappa \geq 0} \left\{ \frac{(\rho-\chi)[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}{(\kappa+3\rho-\chi)\rho(S-D)} \right\}^{\frac{1}{\kappa+2\rho}}, \quad (3.1)$$

where the result is sharp for the function f_κ given by (2.3).

Proof. It suffices to prove that

$$\left| \frac{\zeta(\ L^{t,\rho}_{\varpi,\sigma} f(\zeta))'}{L^{t,\rho}_{\varpi,\sigma} f(\zeta)} + \rho \right| \leq \rho - \chi, \quad (3.2)$$

for $|\zeta| < r_1$. We note that

$$\left| \frac{\zeta(\ L^{t,\rho}_{\varpi,\sigma} f(\zeta))'}{L^{t,\rho}_{\varpi,\sigma} f(\zeta)}\ +\ \rho \right|$$
$$= \left| \frac{\frac{-\rho}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho-(\kappa+2\rho)t)}{\rho} (\kappa+\rho)\ a_{\kappa+\rho}\zeta^{\kappa+\rho}}{\frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho-(\kappa+2\rho)t)}{\rho} a_{\kappa+\rho}\zeta^{\kappa+\rho}} + \rho \right|$$
$$= \left| \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho-(\kappa+2\rho)t)}{\rho} a_{\kappa+\rho}\zeta^{\kappa+\rho}(\kappa+2\rho)}{\frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho-(\kappa+2\rho)t)}{\rho} a_{\kappa+\rho}\zeta^{\kappa+\rho}} \right|$$
$$\leq \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|(\kappa+2\rho)\,|\zeta|^{\kappa+\rho}}{\frac{1}{|\zeta|^\rho} - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|\,|\zeta|^{\kappa+\rho}}$$
$$\leq \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|(\kappa+2\rho)\,|\zeta|^{\kappa+2\rho}}{1 - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|\,|\zeta|^{\kappa+2\rho}}. \quad (3.3)$$

Hence, if

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|(\kappa+2\rho)\,|\zeta|^{\kappa+2\rho}$$
$$\leq (\rho-\chi)\left[1 - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|\,|\zeta|^{\kappa+2\rho}\right] \quad (3.4)$$

or

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho-(\kappa+2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|\ |\zeta|^{\kappa+2\rho} \left(\frac{\kappa+3\rho-\chi}{\rho-\chi} \right) \leq 1,$$

that is,

$$\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\left|\frac{(\rho-(\kappa+2\rho)\ t\ )}{\rho}\right| |a_{\kappa+\rho}|\ |\zeta|^{\kappa+2\rho}\left(\frac{\kappa+3\rho-\chi}{\rho-\chi}\right)\leq 1,\ for\,\kappa\geq 0 \tag{3.5}$$

with the aid of (2.2) and (3.5), for all $\kappa \geq 0$,

$$\begin{aligned}&\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}}\left|\frac{(\ \rho\ -\ (\ \kappa+2\rho)\ t\ )}{\rho}\right||\zeta|^{\kappa+2\rho}\left(\frac{\kappa+3\rho-\chi}{\rho-\chi}\right)\\ &\leq \frac{(\varpi)_{\kappa+2}\,|(\rho-(\kappa+2\rho)t)|\,[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}{\rho^2(\sigma)_{\kappa+2}\ (S-D)}.\end{aligned} \tag{3.6}$$

Solving (3.6) for $|\zeta|$, we obtain

$$|\zeta|\leq\left\{\frac{(\rho-\chi)[(\kappa+2\rho)-((\kappa+\rho)\ D\ +S\rho\ )]}{(\ \kappa\ +3\rho\ -\ \chi\ )\rho(\ S\ -\ D\ )}\right\}^{\frac{1}{\kappa+2\rho}},\ \forall\kappa\geq 0. \tag{3.7}$$

□

Theorem 3.2. *If the function f is defined by (1.4) in the $\Sigma^{S,D}_{\varpi,\sigma}(t,\kappa,\rho)$, then f is meromorphically convex of order $\chi(0\leq\chi<1)$ in $|\zeta|<r_2$, where*

$$r_2=r_2(S,D,\kappa,\rho)=\inf_{\kappa\geq 0}\left\{\frac{(\rho-\chi)[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}{(k+\rho)(k+3\rho-\chi)(S-D)}\right\}^{\frac{1}{\kappa+2\rho}}, \tag{3.8}$$

the result is sharp for the function f_κ given by (2.3).

Proof. By using the technique employed in the proof of Theorem 3.2, we can show that

$$\left|\frac{\zeta\ (\ L^{t,\rho}_{\varpi,\sigma}f(\zeta))''}{(L^{t,\rho}_{\varpi,\sigma}f(\zeta))'}\ +\ \rho+1\right|\leq(\rho-\chi), \tag{3.9}$$

for $|\zeta| < r_2$, with the aid of Theorem 2.1. Thus, we have the assertion of Theorem 3.2. □

4. Convex linear combinations

Our next results involve a linear combination of several functions of the type (2.3).

Theorem 4.1. *Let*

$$f_{\rho-1}(\zeta)=\frac{1}{\zeta^\rho} \tag{4.1}$$

and for $\kappa \geq 0$,

$$f_{\kappa+\rho}(\zeta)=\frac{1}{\zeta^\rho}+\frac{\rho^2(\sigma)_{k+2}(S-D)}{(\varpi)_{\kappa+2}\,|(\rho-(\kappa+2\rho)t)|\,[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}\zeta^{\kappa+\rho}. \tag{4.2}$$

Then $f \in \Sigma_{\varpi,\sigma}^{S,D}(t,\kappa,\rho)$ *if and only if it can be expressed in the form*

$$f(\zeta)=\sum_{\kappa=0}^{\infty}\eta_{\kappa+\rho-1}f_{\kappa+\rho-1}(\zeta) \tag{4.3}$$

where $\eta_{\kappa+\rho-1}\geq 0$ *and* $\sum_{\kappa=0}^{\infty}\eta_{\kappa+\rho-1}=1$.

Proof. From (4.1), (4.2) and (4.3), it is easily seen that

$$\begin{aligned} f(\zeta)&=\sum_{\kappa=0}^{\infty}\eta_{\kappa+\rho-1}f_{\kappa+\rho-1}(\zeta)\\ &=\frac{1}{\zeta^{\rho}}+\sum_{\kappa=0}^{\infty}\frac{\rho^2\ (\sigma)_{\kappa+2}\ (S-D)\eta_{\kappa+\rho}}{(\varpi)_{\kappa+2}\left|(\rho-(\kappa+2\rho)t)\right|\ [(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}\zeta^{\kappa+\rho}\\ &=:\frac{1}{\zeta^{\rho}}+\sum_{\kappa=0}^{\infty}d_{\kappa+\rho}\zeta^{\kappa+\rho}. \end{aligned} \tag{4.4}$$

Since

$$\begin{aligned} &\sum_{\kappa=0}^{\infty}\frac{(\varpi)_{\kappa+2}\left|(\rho-(\kappa+2\rho)t)\right|[(\kappa+2\rho)-((\ \kappa+\rho)\ D\ +\ S\rho)]}{\varpi\rho^2(\sigma)_{\kappa+2}(S-D)}\\ &\times\frac{\rho^2(\sigma)_{\kappa+2}(S-D)\eta_{\kappa+\rho}}{(\varpi)_{\kappa+2}\left|(\rho-(\kappa+2\rho)t)\right|[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}\\ &=\sum_{\kappa=0}^{\infty}\eta_{\kappa+\rho}\ =1-\eta_{\rho-1}\leq 1, \end{aligned}$$

it follows from Theorem 2.1 that the function $f \in \Sigma_{\varpi,\sigma}^{S,D}(t,k,\rho)$.

Conversely, suppose that $f\in\Sigma_{\varpi,\sigma}^{S,D}(t,k,\rho)$. Du to

$$|a_{\kappa+\rho}|\leq\frac{\rho^2(\sigma)_{\kappa+2}\ (S-D)}{(\varpi)_{\kappa+2}\left|(\rho-(\kappa+2\rho)t)\right|[(k+2\rho)-((k+\rho)D+S\rho)]},(\kappa\ \geq\ 0\ ).$$

Setting

$$\eta_{\kappa+\rho}\ =\ \frac{(\varpi)_{\kappa+2}\left|(\rho-(\kappa+2\rho)t)\right|[(\kappa+2\rho)-((\kappa+\rho)D+S\rho)]}{\rho^2(\sigma)_{\kappa+2}\ (S-D)}\ \left|a_{\kappa+\rho}\right|,(\kappa\geq 0)$$

and

$$\eta_{\rho-1}=1-\sum_{\kappa=0}^{\infty}\eta_{\kappa+\rho},$$

it follows that $f\ (\zeta)=\sum_{\kappa=0}^{\infty}\eta_{\kappa+\rho-1}f_{\kappa+\rho-1}(\zeta)$. □

Theorem 4.2. *The subclass* $\Sigma_{\varpi,\sigma}^{S,D}(t,\kappa,\rho)$ *is closed under convex linear combinations.*

Proof. Suppose that the functions $f_1(\zeta)$ and $f_2(\zeta)$ defined by

$$f_v(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho,v}\zeta^{\kappa+\rho}, (v = 1, 2; \zeta \in \mathbb{U}^*) \tag{4.5}$$

are in the subclass $\Sigma_{\varpi,\sigma}^{S,D}(t, \kappa, \rho)$. Let

$$f(\zeta) = \mu f_1(\zeta) + (1 - \mu)f_2(\zeta), (0 \leq \mu \leq 1). \tag{4.6}$$

Then, from (4.5), we find that

$$f(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left(\frac{\rho - (\kappa + 2\rho)t}{\rho}\right) \{\mu a_{\kappa+\rho,1} + (1 - \mu)a_{\kappa+\rho,2}\}\zeta^{\kappa+\rho}, \quad (0 \leq \mu \leq 1; \zeta \in \mathbb{U}^*). \tag{4.7}$$

In view of Theorem 2.1, we have

$$\begin{aligned}
&\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left(\frac{(\rho - (\kappa + 2\rho)t}{\rho}\right) [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] \\
&\times |(\mu a_{\kappa+\rho,1} + (1 - \mu)a_{\kappa+\rho,2})| \\
&\leq \mu \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left(\frac{(\rho - (\kappa + 2\rho)t}{\rho}\right) [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] |a_{\kappa+\rho,1}| \\
&\quad + (1 - \mu) \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left(\frac{(\rho - (\kappa + 2\rho)t}{\rho}\right) \\
&\quad \times [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] |a_{\kappa+\rho,2}| \\
&\leq \mu\rho(S - D) + (1 - \mu)\rho(S - D) \\
&= \rho(S - D).
\end{aligned}$$

This means that $f \in \Sigma_{\varpi,\sigma}^{S,D}(t, \kappa, \rho)$. □

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