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INEQUALITIES FOR THE DERIVATIVE OF A POLYNOMIAL WITH RESTRICTED ZEROS

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Abstract. In this paper we establish several sharp results concerning the maximum modulus of the polar derivative of a polynomial $P(z)$ which does not vanish outside a disk. Our results generalize and refine some known polynomial inequalities including some results of Turán, Malik, Govil and others.

1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. If $P(z)$ has all its zeros in $|z| \leq 1$, then it was shown by Turán [14] that

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{2}\underset{|z|=1}{Max}\left|P(z)\right|. \tag{1.1}$$

Inequality (1.1) was generalized and refined by Aziz and Dawood[4] who under the same hypothesis proved that

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{2}\left\{\underset{|z|=1}{Max}\left|P(z)\right| + n\underset{|z|=1}{Min}\left|P(z)\right|\right\}. \tag{1.2}$$

Both the inequalities (1.1) and (1.2) are sharp and equality in (1.1) and (1.2) holds for $P(z) = az^n + b$ where $|a| = |b|$. As an extension of (1.1), Malik [9] proved that if $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k}\underset{|z|=1}{Max}\left|P(z)\right|, \tag{1.3}$$

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whereas Govil [7] showed that if $P(z) = 0$ for $|z| \leq k$ where $k \geq 1$, then

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k^n}\underset{|z|=1}{Max}\left|P(z)\right|. \tag{1.4}$$

Recently Govil [8] refined both the inequalities (1.3) and (1.4) and proved that if $P(z)$ has all its zeros in $|z| \leq k$, then for $k \leq 1$,

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k}\left\{\underset{|z|=1}{Max}\left|P(z)\right| + \frac{1}{k^{n-1}}\underset{|z|=k}{Min}\left|P(z)\right|\right\} \tag{1.5}$$

and for $k \geq 1$,

$$\underset{|z|=1}{Max}\left|P'(z)\right| \geq \frac{n}{1+k^n}\left\{\underset{|z|=1}{Max}\left|P(z)\right| + \underset{|z|=k}{Min}\left|P(z)\right|\right\}. \tag{1.6}$$

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to a point α, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P'(z). \tag{1.7}$$

A. Aziz [2] proved several sharp results concerning the maximum modulus of the polar derivative of a polynomial $P(z)$ with restrocted zeros. Recently Shah [13] extended (1.1) to the polar derivative of a polynomial $P(z)$ and proved that if $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| \geq \frac{n}{2}(|\alpha|-1)\underset{|z|=1}{Max}\left|P(z)\right|. \tag{1.8}$$

More recently, the authors [5] refined inequality (1.8) by showing that

$$\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| \geq \frac{n}{2}\left\{|\alpha|-1)\underset{|z|=1}{Max}\left|P(z)\right| + (|\alpha|+1)\underset{|z|=1}{Min}\left|P(z)\right|\right\}. \tag{1.9}$$

The result is sharp and equality in (1.9) holds for $P(z) = (z-1)^n$ where $\alpha \geq 1$.

The authors [5] also considered the class of polynomials $P(z)$ of degree n having all its zeros in $|z| \leq k$ and extended the inequalities (1.3) and (1.4) to the polar derivative of a polynomial $P(z)$ and obtained some generalizations of the inequality (1.8). More precisely they have shown that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for every real or complex number α, $|\alpha| \geq k$,

$$\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| \geq n\left\{\frac{|\alpha|-k}{1+k}\right\}\underset{|z|=1}{Max}\left|P(z)\right| \tag{1.10}$$

whereas if $P(z) = 0$ for $|z| \leq k$ where $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,

$$\underset{|z|=1}{Max}\,|D_\alpha P(z)| \geq n\left\{\frac{|\alpha| - k}{1 + k^n}\right\}\underset{|z|=1}{Max}\,|P(z)|\,. \tag{1.11}$$

The estimate (1.10) is sharp and equality in (1.10) holds for $P(z) = (z - k)^n$ with $\alpha \geq k$.

In the present paper, we first extend the inequality (1.5) to the polar derivative of a polynomial $P(z)$ by presenting the following sharp result which is a refinement of the inequality (1.3) as well as a generalization of the inequality (1.9).

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

$$\underset{|z|=1}{Max}\,|D_\alpha P(z)| \geq \frac{n}{1 + k}\left\{(|\alpha| - k)\underset{|z|=1}{Max}\,|P(z)| + \frac{|\alpha| + 1}{k^{n-1}}\underset{|z|=k}{Min}\,|P(z)|\right\}. \tag{1.12}$$

The result is best possible and equality in (1.12) holds for $P(z) = (z - k)^n$ with $\alpha \geq k$.

Remark 1.1. For $k = 1$, (1.12) reduces to the inequality (1.9).

Remark 1.2. Dividing the two sides of (1.12) by $|\alpha|$ and letting $\alpha \to \infty$ and noting (1.7), we get the inequality (1.5).

Next we present the following generalization of the inequality (1.9) which extends the inequality (1.6) to the polar derivative of a polynomial $P(z)$.

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \underset{|z|=k}{Min}\,|P(z)|$, then for every real or complex numbers α, β with $|\alpha| \geq k$ and $|\beta| \leq 1$,*

$$\underset{|z|=1}{Max}\,|D_\alpha P(z) + \beta nm| \geq n\frac{|\alpha| - k}{1 + k^n}\left\{\underset{|z|=1}{Max}\,|P(z)| + |\beta|m\right\} \tag{1.13}$$

and

$$\underset{|z|=1}{Max}\,|D_\alpha P(z)| \geq \frac{n}{1 + k^n}\left\{(|\alpha| - k)\underset{|z|=1}{Max}\,|P(z)| + |\beta|(|\alpha| - 1 - k - k^n)m\right\}. \tag{1.14}$$

Remark 1.3. For $\beta = 0$, Theorem 1.2 reduces to the inequality (1.11).

Remark 1.4. Dividing the two sides of the inequality (1.14) by $|\alpha|$ and letting $\alpha \to \infty$ with $\beta = 1$, we get the inequality (1.6).

As a refinement of the inequality (1.11), we establish the following result.

Theorem 1.3. *If all the zeros of polynomial $P(z) = \prod_{\nu=1}^{n}(z - z_\nu)$ of degree n lie in $|z| \leq k$ where $k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

$$\underset{|z|=1}{Max}|D_\alpha P(z)| \geq 2\left\{\frac{|\alpha| - k}{1 + k^n}\right\} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \underset{|z|=1}{Max}|P(z)|\,. \qquad (1.15)$$

Remark 1.5. Dividing the two sides of the inequality (1.15) by $|\alpha|$ and letting $\alpha \to \infty$, we obtain a result due to A. Aziz [3].

The following result which is a refinement of the inequality (1.8) immediately follows from Theorem 1.3.

Corollary 1.1. *If all the zeros of polynomial $P(z) = \prod_{\nu=1}^{n}(z - z_\nu)$ of degree n lie in $|z| \leq 1$, then for every real or complex number α with $|\alpha| \geq 1$,*

$$\underset{|z|=1}{Max}|D_\alpha P(z)| \geq (|\alpha| - 1) \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \underset{|z|=1}{Max}|P(z)|\,. \qquad (1.16)$$

The result is sharp and equality in (1.16) holds for $P(z) = (z - 1)^n$ with $\alpha \geq 1$.

We also prove the following result which is a refinement of the inequality (1.10) and a generalization of (1.9).

Theorem 1.4. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \geq 1$, then for every real or complex numbers α with $|\alpha| \geq k$,*

$$\underset{|z|=1}{Max}|D_\alpha P(z)| \geq n\left\{\frac{|\alpha| - k}{1 + k^n} \underset{|z|=1}{Max}|P(z)| + \frac{|\alpha| + k}{2k^n} \underset{|z|=k}{Min}|P(z)|\right\}. \qquad (1.17)$$

Remark 1.6. For $k = 1$, Theorem 1.4 reduces to the inequality (1.9).

Finally we present the following result which is a generalization of a result Aziz and Dawood [4, Theorem 1].

Theorem 1.5. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq t$ where $t > 0$, then*

$$\underset{|z|=R\geq t}{Min}|P(z)| \geq \frac{R^n}{t^n} \underset{|z|=t}{Min}|P(z)| \qquad (1.18)$$

and

$$\underset{|z|=R\geq t}{Min}\left|P'(z)\right| \geq n\frac{R^{n-1}}{t^n} \underset{|z|=t}{Min}|P(z)|. \qquad (1.19)$$

Both the estimates are sharp and equality in (1.18) and (1.19) holds for $P(z) = \lambda z^n$, $\lambda \neq 0$.

The following corollary immediately follows from Theorem 1.5 by taking $R = t$ in (1.19).

Corollary 1.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le t$ where $t > 0$, then*

$$\underset{|z|=t}{Min}\left|P'(z)\right| \ge \frac{n}{t}\underset{|z|=t}{Min}|P(z)|.$$

2. Lemma

For the proof of these theorems we need the following lemma due to Aziz [3].

Lemma. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$ where $k \ge 1$, then*

$$\underset{|z|=k}{Max}|P(z)| \ge \frac{2k^n}{1+k^n}\underset{|z|=1}{Max}|P(z)|.$$

3. Proofs of the theorems

Proof of Theorem 1.1. Let $m = \underset{|z|=k}{Min}|P(z)|$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from the inequality (1.10). Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < k$ where $k \le 1$, therefore, $m > 0$ and $m \le |P(z)|$ for $|z| = k$. Now if β is any real or complex number with $|\beta| < 1$,then for $|z| = k$

$$|m\beta z^n/k^n| < |P(z)|.$$

By Rouche's theorem, it follows that the polynomial $G(z) = P(z) - m\beta z^n/k^n$ has all its zeros in $|z| < k$. Using now the same argument as in the proof of Theorem 1.1 of [5] with $P(z)$ replaced by $G(z) = P(z) - m\beta z^n/k^n$, we get for every α with $|\alpha| \ge k$ and $|z| = 1$,

$$\begin{aligned} |D_\alpha(P(z) - m\beta z^n/k^n)| &\ge (|\alpha| - k)\sum_{\nu=1}^{n}\frac{1}{1+|z_\nu|}|P(z) - m\beta z^n/k^n| \\ &\ge n\left\{\frac{|\alpha|-k}{1+k}\right\}|P(z) - m\beta z^n/k^n|. \end{aligned}$$

Equivalently for $|z| = 1$,

$$|D_\alpha(P(z) - mn\beta z^{n-1}/k^n)| \ge n\left\{\frac{|\alpha|-k}{1+k}\right\}|P(z) - m\beta z^n/k^n|. \tag{3.1}$$

A simple application of Laguerre theorem (see [1] or [10, p.52]) on the polar derivative of a polynomial shows that for every real or complex number α with

$|\alpha| \geq k$, the polynomial $D_\alpha G(z) = D_\alpha P(z) - mn\alpha\beta z^n/k^n$ has all its zeros in $|z| < k \leq 1$ where $|\beta| < 1$. This implies that

$$|D_\alpha P(z)| \geq mn|\alpha||z|^{n-1}/k^n \quad \text{for} \quad |z| \geq k. \tag{3.2}$$

Now choosing the argument of β in the left hand side of (3.1) such that

$$|D_\alpha P(z) - mn\alpha\beta z^{n-1}/k^n| = |D_\alpha P(z)| - mn|\alpha||\beta|/k^n \quad \text{for} \quad |z| = 1,$$

which is possible by (3.2), we get for $|z| = 1$,

$$|D_\alpha P(z)| - mn|\alpha||\beta|/k^n \geq n\left\{\frac{|\alpha| - k}{1 + k}\right\}\{|P(z)| - m|\beta|/k^n|\}. \tag{3.3}$$

Letting $|\beta| \to 1$, we obtain

$$\underset{|z|=1}{Max}|D_\alpha P(z)| \geq \frac{n}{1+k}\left\{(|\alpha| - k)\underset{|z|=1}{Max}|P(z)| + \frac{|\alpha| + 1}{k^{n-1}}\underset{|z|=k}{Min}|P(z)|\right\},$$

which is inequality (1.12) and this completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By hypothesis, all the zeros of $P(z)$ lie in $|z| < k$ where $k \geq 1$ and $m = \underset{|z|=k}{Min}|P(z)|$. If $P(z)$ has a zero on $|z| = k$, then result follows from the inequality (1.11). Henceforth, we assume that all the zeros of $P(z)$ lie in $|z| < k$ where $k \geq 1$ so that $m > 0$ and $m \leq |P(z)|$ for $|z| = k$. By the maximum modulus theorem for every β with $|\beta| \leq 1$, it follows that

$$m|\beta| < |P(z)| \quad \text{for} \quad |z| > k.$$

A direct application of Rouche's theorem shows that the polynomial $G(z) = P(z) + \beta m$ has all its zeros in $|z| \leq k$. Applying inequality (1.11) to the polynomial $G(z)$, we get for every α with $|\alpha| \geq k$,

$$\underset{|z|=1}{Max}|D_\alpha G(z)| \geq n\left\{\frac{|\alpha| - k}{1 + k^n}\right\}\underset{|z|=1}{Max}|G(z)|.$$

That is,

$$\begin{aligned}\underset{|z|=1}{Max}|D_\alpha P(z) + \beta mn| &\geq n\left\{\frac{|\alpha| - k}{1 + k^n}\right\}\underset{|z|=1}{Max}|P(z) + \beta m| \\ &\geq n\left\{\frac{|\alpha| - k}{1 + k^n}\right\}|P(z) + \beta m| \quad \text{for } |z| = 1. \end{aligned} \tag{3.4}$$

Now choosing the argument of β in the right hand side of inequality (3.4) suitably, we obtain

$$\underset{|z|=1}{Max}|D_\alpha P(z) + \beta mn| \geq n\left\{\frac{|\alpha| - k}{1 + k^n}\right\}\underset{|z|=1}{Max}|P(z)| + |\beta|m|,$$

which proves the inequality (1.13).

To prove the inequality (1.14), we use the fact that

$$|D_\alpha P(z)| + |\beta| mn \geq |D_\alpha P(z) + \beta mn|$$

and it follows by inequality (1.13) that

$$\begin{aligned} \underset{|z|=1}{Max}\, |D_\alpha P(z)| + |\beta| mn &\geq \underset{|z|=1}{Max}\, |D_\alpha P(z) + \beta mn| \\ &\geq n\left\{\frac{|\alpha| - k}{1+k^n}\right\}\left\{\underset{|z|=1}{Max}\, |D_\alpha P(z)| + |\beta| m|\right\} \end{aligned}$$

for every β with $|\beta| \leq 1$. This gives

$$\underset{|z|=1}{Max}\, |D_\alpha P(z)| \geq \frac{n}{1+k^n}\left\{(|\alpha| - k)\underset{|z|=1}{Max}\, |P(z)| + |\beta|(|\alpha| - 1 - k - k^n)m\right\}$$

which is inequality (1.14) and this completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Since all the zeros of polynomial

$$P(z) = \prod_{\nu=1}^{n}(z - z_\nu)$$

lie in $|z| \leq k$, where $k \geq 1$, the polynomial

$$G(z) = P(kz) = \prod_{\nu=1}^{n}(kz - z_\nu)$$

lie in $|z| \leq 1$ and we have

$$\frac{G'(z)}{G(z)} = \sum_{\nu=1}^{n} \frac{1}{z - z_\nu/k},$$

so that for $0 \leq \theta < 2\pi$,

$$\left|\frac{G'(e^{i\theta})}{G(e^{i\theta})}\right| = Re\left\{\frac{e^{i\theta}G'(e^{i\theta})}{G(e^{i\theta})}\right\} = \sum_{\nu=1}^{n} Re\left\{\frac{e^{i\theta}}{e^{i\theta} - z_\nu/k}\right\} \geq \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|}.$$

This implies

$$|G'(z)| \geq \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} |G(z)| \quad \text{for} \quad |z| = 1. \tag{3.5}$$

If $H(z) = z^n\, \overline{G(1/\overline{z})}$, then the polynomial $H(z)$ has all its zeros in $|z| \geq 1$ and

$$|H(z)| = |G(z)| \quad \text{for} \quad |z| = 1.$$

Therefore, it follows by a result of De Bruijn (see [6, Theorem 1, p.1265]) that

$$|H'(z)| \leq |G'(z)| \quad \text{for} \quad |z| = 1. \tag{3.6}$$

Since

$$H'(z) = nz^{n-1}\overline{G(1/\overline{z})} - z^{n-2}\overline{G'(1/\overline{z})},$$

therefore, for $0 \leq \theta < 2\pi$, we have

$$|H'(e^{i\theta})| = |nG(e^{i\theta}) - e^{i\theta}G'(e^{i\theta})|$$

so that

$$|H'(z)| = |nG(z) - zG'(z)| \quad \text{for} \quad |z| = 1.$$

Using this in (3.6), we get

$$|nG(z) - zG'(z)| \leq |G'(z)| \quad \text{for} \quad |z| = 1. \tag{3.7}$$

Now for every α with $|\alpha| \geq k$, we have

$$\begin{aligned} |D_{\alpha/k}G(z)| &= |nG(z) - zG'(z) + \frac{\alpha}{k}G'(z)| \\ &\geq \frac{|\alpha|}{k}|G'(z)| - |nG(z) - zG'(z)|. \end{aligned}$$

This gives with the help of (3.7) that

$$|D_{\alpha/k}G(z)| \geq \left\{\frac{|\alpha| - k}{k}\right\}|G'(z)| \quad \text{for} \quad |z| = 1. \tag{3.8}$$

Combining (3.5)and (3.8), we obtain for every α with $|\alpha| \geq k$,

$$|D_{\alpha/k}G(z)| \geq (|\alpha| - k)\sum_{\nu=1}^{n}\frac{1}{k + |z_\nu|}|G(z)| \quad \text{for} \quad |z| = 1.$$

Replacing $G(z)$ by $P(kz)$, we get

$$|D_{\alpha/k}P(kz)| \geq (|\alpha| - k)\sum_{\nu=1}^{n}\frac{1}{k + |z_\nu|}|P(kz)| \quad \text{for} \quad |z| = 1,$$

that is,

$$|nP(kz) + (\frac{\alpha}{k} - z)kP'(kz)| \geq (|\alpha| - k)\sum_{\nu=1}^{n}\frac{1}{k + |z_\nu|}|P(kz)| \quad \text{for} \quad |z| = 1,$$

which implies

$$\underset{|z|=1}{Max}|nP(kz) + (\alpha - kz)P'(kz)| \geq (|\alpha| - k)\sum_{\nu=1}^{n}\frac{1}{k + |z_\nu|}\underset{|z|=1}{Max}|P(kz)|.$$

Equivalently,

$$\underset{|z|=k}{Max}|nP(z)+(\alpha-z)P'(z)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\underset{|z|=k}{Max}|P(z)|.$$

Using the Lemma, we obtain

$$\begin{aligned}\underset{|z|=1}{Max}|D_\alpha P(z)| &= \underset{|z|=k}{Max}|nP(z)+(\alpha-z)P'(z)| \\ &\geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\left(\frac{2k^n}{1+k^n}\right)\underset{|z|=1}{Max}|P(z)|. \qquad (3.9)\end{aligned}$$

Now, if $F(z)$ is a polynomial of degree n, then ([12, p. 346] or [11, vol. I, p. 137])

$$\underset{|z|=R>1}{Max}|F(z)| \leq R^n \underset{|z|=1}{Max}|F(z)|.$$

Applying this result to the polynomial $nP(z)+(\alpha-z)P'(z)=D_\alpha P(z)$ which is of degree at most $n-1$ with $R=k$, we get

$$\underset{|z|=k}{Max}|D_\alpha P(z)| \leq k^{n-1}\underset{|z|=1}{Max}|D_\alpha P(z)|. \qquad (3.10)$$

Together (3.9) and (3.10) leads to

$$\underset{|z|=1}{Max}|D_\alpha P(z)| \geq 2\left\{\frac{|\alpha|-k}{1+k^n}\right\}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\underset{|z|=1}{Max}|P(z)|\,.$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. By hypothesis all the zeros of $P(z)$ lie in $|z|\leq k$ where $k\geq 1$, therefore, all the zeros of $G(z)=P(kz)$ lie in $|z|\leq 1$. Applying inequality (1.9) to the polynomial $G(z)$ and noting that $\frac{|\alpha|}{k}\geq 1$, we get

$$\underset{|z|=1}{Max}|D_{\frac{\alpha}{k}}G(z)| \geq \frac{n}{2}\left\{\frac{|\alpha|-k}{k}\underset{|z|=1}{Max}|G(z)|+\frac{|\alpha|+k}{k}\underset{|z|=1}{Min}|G(z)|\right\}.$$

Replacing $G(z)$ by $P(kz)$, we obtain

$$\underset{|z|=1}{Max}|D_{\frac{\alpha}{k}}P(kz)| \geq \frac{n}{2}\left\{\frac{|\alpha|-k}{k}\underset{|z|=1}{Max}|P(kz)|+\frac{|\alpha|+k}{k}\underset{|z|=1}{Min}|P(kz)|\right\}.$$

This implies with the help of the Lemma that

$$\begin{aligned}\underset{|z|=1}{Max}|nP(kz) &+ (\frac{\alpha}{k}-z)kP'(kz)| \\ &\geq \frac{n}{2}\left\{\frac{|\alpha|-k}{k}\underset{|z|=k}{Max}|P(z)|+\frac{|\alpha|+k}{k}\underset{|z|=k}{Min}|P(z)|\right\} \\ &\geq \frac{n}{2}\left\{\frac{|\alpha|-k}{k}\frac{2k^n}{1+k^n}\underset{|z|=1}{Max}|P(z)|+\frac{|\alpha|+k}{k}\underset{|z|=k}{Min}|P(z)|\right\},\end{aligned}$$

which gives

$$\begin{aligned}\underset{|z|=k}{Max}|D_\alpha P(z)| &= \underset{|z|=k}{Max}|nP(z)+(\alpha-z)P'(z)| \\ &= \underset{|z|=1}{Max}|nP(kz)+(\frac{\alpha}{k}-z)kP'(kz)| \\ &\geq n\left\{k^{n-1}(\frac{|\alpha|-k}{1+k^n})\underset{|z|=1}{Max}|P(z)|+(\frac{|\alpha|+k}{2k})\underset{|z|=k}{Min}|P(z)|\right\}.\end{aligned}$$

Combining this with the inequality (3.10), it follows that

$$\begin{aligned}k^{n-1}\underset{|z|=1}{Max}|D_\alpha P(z)| &\geq n\left\{k^{n-1}(\frac{|\alpha|-k}{1+k^n})\underset{|z|=1}{Max}|P(z)|\right. \\ &+ \left.(\frac{|\alpha|+k}{2k})\underset{|z|=k}{Min}|P(z)|\right\},\end{aligned}$$

which immediately leads to the inequality (1.17) and this completes the proof of Theorem 1.4. □

Proof of Theorem 1.5. If $P(z)$ has a zero on $|z|=t$, then $m=\underset{|z|=t}{Min}|P(z)|=0$, and the result is trivial in this case. We now assume that all the zeros of $P(z)$ lie in $|z|<t$ where $t>0$, therefore, $m>0$ and

$$m \leq |P(z)| \quad \text{for} \quad |z|=t.$$

If $F(z)=P(tz)$, then all the zeros of polynomial $F(z)$ lie in $|z|<1$ and

$$m|z|^n \leq |F(z)| \quad \text{for} \quad |z|=1. \tag{3.11}$$

By the maximum modulus theorem, it follows that

$$m|z|^n \leq |F(z)| \quad \text{for} \quad |z|\geq 1.$$

Equivalently,

$$m|z|^n \leq |P(tz)| \quad \text{for} \quad |z|\geq 1.$$

Taking $z=\frac{R}{t}e^{i\theta}$, $0\leq\theta<2\pi$ and noting that $|z|=\frac{R}{t}\geq 1$, we obtain0

$$|P(Re^{i\theta}| \geq m\frac{R^n}{t^n}$$

for every θ, $0 \leq \theta < 2\pi$ and $R \geq t$. This implies

$$\underset{|z|=R\geq t}{Min}|P(z)| \geq \frac{R^n}{t^n}\underset{|z|=t}{Min}|P(z)|,$$

which is the inequality (1.18).

To prove the inequality (1.19), we have by (3.11),

$$m|z|^n \leq |F(z)| \quad \text{for} \quad |z| = 1.$$

A direct application of Rouche's theorem shows that for every real or complex number β with $|\beta| < 1$, all the zeros of polynomial $F(z) - m\beta z^n$ lie in $|z| < 1$ and therefore, by the Gauss- Lucas theorem, all the zeros of derived polynomial

$$G(z) = F'(z) - nm\beta z^{n-1} \tag{3.12}$$

also lie in $|z| < 1$. This implies

$$|F'(z)| \geq nm|z|^{n-1} \quad \text{for} \quad |z| \geq 1. \tag{3.13}$$

For, if inequality (3.13) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|F'(z_0)| < nm|z_0|^{n-1}.$$

We take

$$\beta = \frac{F'(z_0)}{nmz_0}^{n-1},$$

then $|\beta| < 1$ and with this choice of β, we have $G(z_0) = 0$ with $|z_0| \geq 1$, which is clearly a contradiction to (3.12). This establishes inequality (3.13). Equivalently,

$$t|P'(tz)| \geq nm|z|^{n-1} \quad \text{for} \quad |z| \geq 1.$$

Taking $z = \frac{R}{t}e^{i\theta}$, $0 \leq \theta < 2\pi$ and noting that $|z| = \frac{R}{t} \geq 1$, we obtain

$$t|P'(Re^{i\theta})| \geq nm\frac{R^{n-1}}{t^{n-1}} \quad \text{for} \quad |z| \geq 1.$$

This implies

$$\underset{|z|=R\geq t}{Min}|P'(z)| \geq n\frac{R^{n-1}}{t^n}\underset{|z|=t}{Min}|P(z)|,$$

which is inequality (1.19) and this completes the proof of Theorem 1.5. □

References

[1] A. Aziz, *A new proof of Laguerre's theorem about the zeros of polynomials*, Bull. Austral. Math. Soc. **33** (1986), 131–138.

[2] A. Aziz, *Inequalities for the polar derivative of a polynomial*, J. Approx. Theory, **55** (1988), 181–193.

[3] A. Aziz, *Inequalities for the derivative of a polynomial*, Proc. Amer. Soc. **89** (1983), 259–266.

[4] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **54** (1988), 306–313.

[5] A. Aziz and N. A. Rather, *A refinement of a theorem of Paul Turán concerning the polynomials*, J. Math. Ineq. Appl. **1** (1998), 231–238.

[6] N. G. De Bruijn, *Inequalities concerning the polynomials in the complex donmain*, Nederl. Akad. Wetenreh. Proc. **50** (1947), 67–98.

[7] N. K. Govil, *On the derivative of a polynomial*, Proc. Amer. Math. Soc. **41** (1973), 543–546.

[8] N. K. Govil, *Some inequalities for derivative of polynomials*, J. Approx. Theory, **66** (1991), 29–335.

[9] M. A. Malik, *On the derivative of a polynomial*, J. London Math. Soc. **1** (1969), 57–60.

[10] M. Marden, *Geometry of polynomials*, 2nd ed. Mathematical Surveys, No. 3, Amer. Math. Soc. Providence, RI, 1966.

[11] G. Pólya and G. Sgezö, *Aufgaben und Lahrsatze aus der Analysis*, Springer- Verlag, Berlin, 1925.

[12] M. Reisz, *Über einen satz des Herrn Serge Bernstein*, Acta Math. **40** (1916), 337–347.

[13] W. M. Shah, *A generalization of a Theorem of Paul Turán*, J. Ramanujan Math. Soc. **11** (1996), 67–72.

[14] Paul Turán, *Über die ableitung von polynomen*, Compositio Math. **7** (1939), 89–95.