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COMMON FIXED POINT THEOREMS FOR GENERALIZED $\psi_{\int\varphi}$-WEAKLY CONTRACTIVE MAPPINGS IN G-METRIC SPACES

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Abstract. In this paper, first of all we prove a fixed point theorem for $\psi_{\int\varphi}$-weakly contractive mapping. Next, we prove some common fixed point theorems for a pair of weakly compatible self maps along with E.A. property and (CLR) property. An example is also given to support our results.

1. Introduction

Dhage [4, 5] introduced a new class of generalized metric spaces named D-metric spaces. Mustafa and Sims [7, 8] proved that most of claims concerning the fundamental topological structures are incorrect and introduced

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appropriate notion of generalized metric spaces, named G-metric spaces. In fact, Mustafa, Sims and other authors proved many fixed point results for self mapping under certain conditions in [7, 8, 9] and in other papers [2, 10, 13, 14].

2. Preliminaries

We give some definitions and their properties for our main results.

Definition 2.1. Let X be a nonempty set and $G : X^3 \to R_+$ be a function satisfying the following properties:

(i) $G(x, y, z) = 0$ if $x = y = z$,
(ii) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(iii) $G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(iv) $G(x, y, z) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (triangle inequality).

The function G is called a G-metric on X and (X, G) is called a G-metric space.

Remark 2.2. Let (X, G) be a G-metric space. If $y = z$, then $G(x, y, y)$ is a quasi-metric on X. Hence (X, Q) is a G-metric space, where $Q(x, y) = G(x, y, y)$ is a quasi-metric and since every metric space is a particular case of quasi-metric space, it follow that the notion of G-metric space is a generalization of a metric space.

Lemma 2.3. ([7]) *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 2.4. Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X is G-convergent if for $\epsilon > 0$, there exists $x \in X$ and $k \in N$ such that for all $m, n \geq k,\ G(x, x_n, x_m) < \epsilon$.

Lemma 2.5. ([7]) *Let (X, G) be a G-metric space. Then the following conditions are equivalent.*

(i) $\{x_n\}$ *is G-convergent to x,*
(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,
(iv) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Jungck [6] introduced the new notion of weakly compatible maps as follows:

Definition 2.6. Let f and g be two self-mappings of a metric space (X, d). Then a pair (f, g) is said to be weakly compatible if they commute at coincidence points.

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 2.7. Let f and g be two self-mappings of a metric space (X, d). Then a pair (f, g) is said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 2011, Sintunavarat and Kumam [12] introduced the notion of (CLR) property as follows:

Definition 2.8. Let f and g be two self- mappings of a metric space (X, d). Then a pair (f, g) is said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some $x \in X$.

3. Main result

In this section, we give a new notion of $\psi_{\int\varphi}$-weakly contractive mapping and prove a fixed point theorem for a single map in G-metric spaces. Also, common fixed point theorems for a pair of weakly compatible maps along with E. A. property and (CLR) property are proved.

Definition 3.1. Let (X, G) be a G-metric space and $\varphi : [0, \infty) \to [0, \infty)$ be a Lebesgue integrable mapping. A mapping $T : X \to X$ is said to be $\psi_{\int\varphi}$-weakly contractive if for all x, y, z in X,

$$\psi\left(\int_0^{G(Tx,Ty,Tz)} \varphi(t)dt\right) \le \psi\left(\int_0^{G(x,y,z)} \varphi(t)dt\right) - \phi\left(\int_0^{G(x,y,z)} \varphi(t)dt\right), \quad (3.1)$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous and non-decreasing function such that $\phi(t) = 0 = \psi(t)$ if and only if $t = 0$.

Theorem 3.2. *Let (X, G) be a complete G-metric space and $T : X \to X$ is $\psi_{\int\varphi}$-weakly contractive mapping, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that*

$$\int_0^{\epsilon} \varphi(t)dt > 0, \quad (3.2)$$

for each $\epsilon > 0$ and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous and non-decreasing function such that $\phi(t) = 0 = \psi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and choose a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$ for all $n > 0$. From (3.1), we have

$$\begin{aligned}\psi\Big(\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{G(Tx_n,Tx_{n-1},Tx_{n-1})}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt\Big)\\ &\quad - \phi\Big(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt\Big).\end{aligned}$$

Using monotone property of ψ -function, we have

$$\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt \le \int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt. \tag{3.3}$$

Let $y_n = \int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt$. Then $0 \le y_n \le y_{n-1}$ for all $n > 0$. It follows that the sequence $\{y_n\}$ is monotone decreasing and lower bounded. So, there exists $r \ge 0$, such that

$$\lim_{n\to\infty}\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt = \lim_{n\to\infty} y_n = r.$$

Then, by the lower semi-continuity of ϕ, we get

$$\phi(r) \le \liminf_{n\to\infty}\phi\Big(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt\Big).$$

Let $r > 0$. Taking upper limit as $n \to \infty$ on either side of (3.3), we get

$$\begin{aligned}\psi(r) &\le \psi(r) - \liminf_{n\to\infty}\phi\Big(\int_0^{G(x_n,x_{n-1},x_{n-1})}\varphi(t)dt\Big)\\ &\le \psi(r) - \phi(r),\end{aligned}$$

which is a contradiction. Thus, $r = 0$, that is,

$$\lim_{n\to\infty}\Big(\int_0^{G(x_{n+1},x_n,x_n)}\varphi(t)dt\Big) = \lim_{n\to\infty} y_n = 0.$$

Therefore, we have

$$\lim_{n\to\infty} G(x_{n+1},x_n,x_n) = 0. \tag{3.4}$$

Now, we prove that $\{x_n\}$ is a G-Cauchy sequence. Suppose that $\{x_n\}$ is not a G-Cauchy sequence, there exists an $\epsilon > 0$ and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \geq \epsilon. \tag{3.5}$$

Let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (3.5) such that

$$G(x_{n(k)-1}, x_{m(k)}, x_{m(k)}) < \epsilon, \tag{3.6}$$

for every integer k. Then, we have

$$\begin{aligned}\epsilon &\leq G(x_{n(k)}, x_{m(k)}, x_{m(k)})\\ &\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)}, x_{m(k)})\\ &< \epsilon + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}).\end{aligned}$$

Now

$$\begin{aligned}0 < \delta = \int_0^{\epsilon} \varphi(t)dt &\leq \int_0^{G(x_{n(k)}, x_{m(k)}, x_{m(k)})} \varphi(t)dt\\ &\leq \int_0^{\epsilon + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1})} \varphi(t)dt.\end{aligned}$$

Letting $k \to \infty$ and using (3.4), we get

$$\lim_{k\to\infty} \int_0^{G(x_n(k), x_m(k), x_m(k))} \varphi(t)dt = \delta. \tag{3.7}$$

By the triangular inequality,

$$\begin{aligned}G(x_{n(k)}, x_{m(k)}, x_{m(k)}) \leq\ & G(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)-1})\\ &+ G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})\\ &+ G(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)})\end{aligned}$$

and

$$\begin{aligned}G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) \leq\ & G(x_{n(k)-1}, x_{n(k)}, x_{n(k)})\\ &+ G(x_{n(k)}, x_{m(k)}, x_{m(k)})\\ &+ G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_0^{G(x_{n(k)}, x_{m(k)}, x_{m(k)})} \varphi(t)dt\\ &\leq \int_0^{G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)-1}, x_{m(k)})} \varphi(t)dt\end{aligned}$$

and

$$\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})}\varphi(t)dt$$
$$\leq \int_0^{G(x_{n(k)-1},x_{n(k)},x_{n(k)})+G(x_{n(k)},x_{m(k)},x_{m(k)})+G(x_{m(k)},x_{m(k)-1},x_{m(k)-1})}\varphi(t)dt.$$

Letting $\lim k \to \infty$ in the above two inequalities and using (3.4) and (3.7),we get

$$\lim_{k\to\infty}\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})}\varphi(t)dt = \delta. \tag{3.8}$$

Taking $x = x_{n(k)-1}, y = x_{m(k)-1}, z = x_{m(k)-1}$ in (3.1), we get

$$\begin{aligned}\psi\bigg(\int_0^{G(Tx_{n(k)-1},Tx_{m(k)-1},Tx_{m(k)-1})}\varphi(t)dt\bigg) &= \psi\bigg(\int_0^{G(x_{n(k)},x_{m(k)},x_{m(k)})}\varphi(t)dt\bigg)\\ &\leq \psi\bigg(\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})}\varphi(t)dt\bigg)\\ &\quad - \phi\bigg(\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})}\varphi(t)dt\bigg).\end{aligned}$$

Letting $k \to \infty$, using (3.7), (3.8) and properties of ψ and ϕ, we get

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta),$$

which is a contradiction from $\delta > 0$. Hence $\{x_n\}$ is a G-Cauchy sequence.

Since X is a complete metric space, there exists u in X such that

$$\lim_{n\to\infty} x_n = u. \tag{3.9}$$

Taking $x = x_{n-1}$, $y = u$, $z = u$ in (3.1), we get

$$\begin{aligned}\psi\bigg(\int_0^{G(Tx_{n-1},Tu,Tu)}\varphi(t)dt\bigg) &= \psi\bigg(\int_0^{G(x_n,Tu,Tu)}\varphi(t)dt\bigg)\\ &\leq \psi\bigg(\int_0^{G(x_{n-1},u,u)}\varphi(t)dt\bigg) - \phi\bigg(\int_0^{G(x_{n-1},u,u)}\varphi(t)dt\bigg).\end{aligned}$$

Letting $n \to \infty$, using (3.9) and properties of ψ and ϕ, we get

$$\psi(\int_0^{G(u,Tu,Tu)}\varphi(t)dt) \leq \psi(0) - \phi(0) = 0,$$

which implies that $\int_0^{G(u,Tu,Tu)}\varphi(t)dt = 0$. Thus, $G(u, Tu, Tu) = 0$, this means that, $u = Tu$.

Now, we prove that u is the unique fixed point of T. Let v be an another common fixed point of T, that is, $Tv = v$.

Putting $x = u$, $y = v$, $z = v$ in (3.1), we get

$$\begin{aligned}\psi\Big(\int_0^{G(Tu,Tv,Tv)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big).\end{aligned}$$

Hence we have

$$\phi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big) = 0,$$

which implies that, $G(u, v, v) = 0$, that is, $u = v$. This completes the proof. □

Theorem 3.3. *Let (X, G) be a G-metric space and let f and g be self-mappings on X satisfying the following:*

$$gX \subset fX, \tag{3.10}$$

$$fX \ \text{or} \ gX \ \text{is complete} \tag{3.11}$$

and

$$\psi\Big(\int_0^{G(gx,gy,gz)}\varphi(t)dt\Big) \le \psi\Big(\int_0^{G(fx,fy,fz)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(fx,fy,fz)}\varphi(t)dt\Big), \tag{3.12}$$

for all x, y, z in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\epsilon}\varphi(t)dt > 0, \quad \text{for each } \epsilon > 0 \tag{3.13}$$

and $\psi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous and non-decreasing function such that $\phi(t) = 0 = \psi(t)$ if and only if $t = 0$. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. From (3.10), we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_{n+1} = gx_n$, for each $n = 0, 1, 2, \ldots$. Then, from (3.12), we have

$$\begin{aligned}
\psi\bigg(\int_0^{G(y_{n+1},y_n,y_n)} \varphi(t)dt\bigg) &= \psi\bigg(\int_0^{G(gx_{n+1},gx_n,gx_n)} \varphi(t)dt\bigg) \\
&\le \psi\bigg(\int_0^{G(fx_{n+1},fx_n,fx_n)} \varphi(t)dt\bigg) \\
&\quad - \phi\bigg(\int_0^{G(fx_{n+1},fx_n,fx_n)} \varphi(t)dt\bigg) \\
&\le \psi\bigg(\int_0^{G(y_n,y_{n-1},y_{n-1})} \varphi(t)dt\bigg) \\
&\quad - \phi\bigg(\int_0^{G(y_n,y_{n-1},y_{n-1})} \varphi(t)dt\bigg) \\
&\le \psi\bigg(\int_0^{G(y_n,y_{n-1},y_{n-1})} \varphi(t)dt\bigg). \qquad (3.14)
\end{aligned}$$

Using monotone property of function ψ, we have

$$\int_0^{G(y_{n+1},y_n,y_n)} \varphi(t)dt \le \int_0^{G(y_n,y_{n-1},y_{n-1})} \varphi(t)dt.$$

Let $u_n = \int_0^{G(y_{n+1},y_n,y_n)} \varphi(t)dt$. Then $0 \le u_n \le u_{n-1}$ for all $n > 0$. It follows that the sequence $\{u_n\}$ is monotone decreasing and lower bounded. So, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \int_0^{G(y_{n+1},y_n,y_n)} \varphi(t)dt = \lim_{n\to\infty} u_n = r.$$

Then, from the lower semi-continuity of ϕ, we have

$$\phi(r) \le \liminf_{n\to\infty} \phi\bigg(\int_0^{G(y_n,y_{n-1},y_{n-1})} \varphi(t)dt\bigg).$$

Let $r > 0$ and taking upper limit as $n \to \infty$ on either side of (3.14), we get

$$\begin{aligned}
\psi(r) &\le \psi(r) - \liminf_{n\to\infty} \phi\bigg(\int_0^{G(y_n,y_{n-1},y_{n-1})} \varphi(t)dt\bigg) \\
&\le \psi(r) - \phi(r)
\end{aligned}$$

which is a contradiction. Then, $r = 0$, that is,

$$\lim_{n\to\infty} \int_0^{G(y_{n+1},y_n,y_n)} \varphi(t)dt = \lim_{n\to\infty} u_n = 0.$$

Therefore, we have

$$\lim_{n\to\infty} G(y_{n+1}, y_n, y_n) = 0. \tag{3.15}$$

Now, we prove that $\{y_n\}$ is a G-Cauchy sequence. Suppose that $\{y_n\}$ is not a G-Cauchy sequence. Then, there exists, an $\epsilon > 0$ and subsequences $\{y_m(k)\}$ and $\{y_n(k)\}$ of $\{y_n\}$ with $n(k) > m(k)$ such that

$$G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \geq \epsilon. \tag{3.16}$$

Let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (3.16) such that

$$G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) < \epsilon, \text{ for every integer } k. \tag{3.17}$$

Then, we have

$$\begin{aligned}\epsilon &\leq G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\ &\leq G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{m(k)}, y_{m(k)}) \\ &< \epsilon + G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}).\end{aligned}$$

Hence, we have

$$\begin{aligned}0 < \delta &= \int_0^{\epsilon} \varphi(t)dt \\ &\leq \int_0^{G(y_{n(k)}, y_{m(k)}, y_{m(k)})} \varphi(t)dt \leq \int_0^{\epsilon + G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1})} \varphi(t)dt.\end{aligned}$$

Letting $k \to \infty$ and using (3.15), we get

$$\lim_{k\to\infty} \int_0^{G(y_{n(k)}, y_{m(k)}, y_{m(k)})} \varphi(t)dt = \delta. \tag{3.18}$$

By the triangular inequality, we have

$$\begin{aligned}G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \leq\ & G(y_{n(k)}, y_{n(k)-1}, y_{n(k)-1}) \\ &+ G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \\ &+ G(y_{m(k)-1}, y_{m(k)-1}, y_{m(k)})\end{aligned}$$

and

$$\begin{aligned}G(y_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}) \leq\ & G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \\ &+ G(y_{n(k)}, y_{m(k)}, y_{m(k)}) \\ &+ G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\int_0^{G(y_{n(k)},y_{m(k)},y_{m(k)})}\varphi(t)dt\\&\leq\int_0^{G(y_{n(k)},y_{n(k-1)},y_{n(k-1)})+G(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1})+G(y_{m(k)-1},y_{m(k)-1},y_{m(k)})}\varphi(t)dt\end{aligned}$$

and

$$\begin{aligned}&\int_0^{G(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1})}\varphi(t)dt\\&\leq\int_0^{G(y_{n(k)-1},y_{n(k)},y_{n(k)})+G(y_{n(k)},y_{m(k)},y_{m(k)})+G(y_{m(k)},y_{m(k)-1},y_{m(k)-1})}\varphi(t)dt.\end{aligned}$$

Letting $k\to\infty$ in the above two inequalities and using (3.15) and (3.18), we get

$$\lim_{k\to\infty}\int_0^{G(y_{n(k)},y_{m(k)},y_{m(k)})}\varphi(t)dt=\delta. \tag{3.19}$$

Taking $x=x_{n(k)}$, $y=x_{m(k)}$, $z=x_{m(k)}$ in (3.1), we get

$$\begin{aligned}\psi\Big(\int_0^{G(gx_{n(k)},gx_{m(k)},gx_{m(k)})}\varphi(t)dt\Big)&=\psi\Big(\int_0^{G(y_{n(k)},y_{m(k)},y_{m(k)})}\varphi(t)dt\Big)\\&\leq\psi\Big(\int_0^{G(fx_{n(k)},fx_{m(k)},fx_{m(k)})}\varphi(t)dt\Big)\\&\quad-\phi\Big(\int_0^{G(x_{n(k)-1},x_{m(k)-1},x_{m(k)-1})}\varphi(t)dt\Big)\\&=\psi\Big(\int_0^{G(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1})}\varphi(t)dt\Big)\\&\quad-\phi\Big(\int_0^{G(y_{n(k)-1},y_{m(k)-1},y_{m(k)-1})}\varphi(t)dt\Big).\end{aligned}$$

Letting $k\to\infty$, using (3.18), (3.19) and properties of ψ and ϕ, we get

$$\psi(\delta)\leq\psi(\delta)-\phi(\delta),$$

which is a contradiction from $\delta>0$. Thus $\{y_n\}$ is a G-Cauchy sequence.

Now, since fX is complete, there exists a point $u\in fX$ such that

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}fx_{n+1}=u. \tag{3.20}$$

Now, we prove that u is the common fixed point of f and g. Since $u \in fX$, there exists a point $p \in X$ such that $fp = u$. From (3.12), we have

$$\begin{aligned}\psi\left(\int_0^{G(fp,gp,gp)} \varphi(t)dt\right) &= \lim_{n\to\infty} \psi\left(\int_0^{G(gx_n,gp,gp)} \varphi(t)dt\right)\\ &\leq \lim_{n\to\infty} \psi\left(\int_0^{G(fx_n,fp,fp)} \varphi(t)dt\right)\\ &\quad - \lim_{n\to\infty} \phi\left(\int_0^{G(fx_n,fp,fp)} \varphi(t)dt\right).\end{aligned}$$

From (3.20) and using properties of ψ and ϕ, we get

$$\psi\left(\int_0^{G(fp,gp,gp)} \varphi(t)dt\right) \leq \psi(0) - \phi(0) = 0,$$

implies that,

$$\psi\left(\int_0^{G(fp,gp,gp)} \varphi(t)dt\right) = 0.$$

Thus, $G(fp, gp, gp) = 0$, that is, $fp = gp = u$. Hence u is the coincidence point of f and g.

Now, we show that u is the common fixed point of f and g.
Since, $fp = gp$ and f, g are weakly compatible maps, we have $fu = fgp = gfp = gu$.

We claim that $fu = gu = u$. Suppose that $gu \neq u$. From (3.12), we have

$$\begin{aligned}\psi\left(\int_0^{G(gu,u,u)} \varphi(t)dt\right) &= \psi\left(\int_0^{G(gu,gp,gp)} \varphi(t)dt\right)\\ &\leq \psi\left(\int_0^{G(fu,fp,fp)} \varphi(t)dt\right) - \phi\left(\int_0^{G(fu,fp,fp)} \varphi(t)dt\right)\\ &= \psi\left(\int_0^{G(gu,u,u)} \varphi(t)dt\right) - \phi\left(\int_0^{G(gu,u,u)} \varphi(t)dt\right)\\ &< \psi\left(\int_0^{G(gu,u,u)} \varphi(t)dt\right).\end{aligned}$$

This is a contradiction. Thus, we get, $gu = u = fu$. Hence u is the common fixed point of f and g.

For the uniqueness, let v be an another common fixed point of f and g, We claim that $u = v$. Suppose that $u \neq v$. From (3.2), we have

$$\begin{aligned}\psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{G(gv,gv,gv)}\varphi(t)dt\Big)\\ &\leq \psi\Big(\int_0^{G(fu,fv,fv)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(fv,fv,fv)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big).\end{aligned}$$

This is a contraction. Thus, we get, $u = v$. Hence u is the unique common fixed point of f and g. This completes the proof. □

Theorem 3.4. *Let (X, G) be a G-metric space and let f and g be weakly compatible self-maps of X satisfying* (3.12), (3.13) *and the following conditions:*

$$f \text{ and } g \text{ satisfy the E.A. property}, \tag{3.21}$$

$$fX \text{ is closed subset of } X. \tag{3.22}$$

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = x_0$$

for some $x_0 \in X$. Since fX is closed subset of X, using (3.21), we have

$$\lim_{n\to\infty} fx_n = fz \quad \text{for some } z \in X. \tag{3.23}$$

Now, we claim that $fz = gz$. From (3.12), we have

$$\psi\Big(\int_0^{G(gx_n,gz,gz)}\varphi(t)dt\Big) \leq \psi\Big(\int_0^{G(gx_n,fz,fz)}\varphi(t)dt\Big)-\phi\Big(\int_0^{G(fx_n,fz,fz)}\varphi(t)dt\Big).$$

From (3.23) and properties of ψ and ϕ, we have

$$\psi\Big(\int_0^{G(fx_n,gz,gz)}\varphi(t)dt\Big) \leq \psi(0) - \phi(0) = 0,$$

it implies that

$$\int_0^{G(fz,gz,gz)}\varphi(t)dt = 0.$$

Thus, we have, $G(fz, gz, gz) = 0$, and so $fz = gz$.

Now, we show that gz is common fixed point of f and g. Suppose that, $gz \neq fz$. Since f and g are weakly compatible, $gfz = fgz$ and therefore, $ffz = ggz$. From (3.12), we have

$$\begin{aligned}\psi\Big(\int_0^{G(gx_n,ggz,ggz)}\varphi(t)dt\Big) &\leq \psi\Big(\int_0^{G(fz,fgz,fgz)}\varphi(t)dt\Big)-\phi\Big(\int_0^{G(fz,fgz,fgz)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{G(gz,ggz,ggz)}\varphi(t)dt\Big)-\phi\Big(\int_0^{G(gz,ggz,ggz)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{G(gz,ggz,ggz)}\varphi(t)dt\Big),\end{aligned}$$

which is a contradiction. Thus, $ggz = gz$. Hence gz is the common fixed point of f and g.

Finally, we show that the common fixed point is unique. Let u and v be two common fixed points of f and g such that $u \neq v$. From (3.12), we have

$$\begin{aligned}\psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{G(gu,gv,gv)}\varphi(t)dt\Big)\\ &\leq \psi\Big(\int_0^{G(fu,fv,fv)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(fu,fv,fv)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{G(u,v,v)}\varphi(t)dt\Big),\end{aligned}$$

which is a contradiction. Therefore $u = v$. This completes the proof. □

Theorem 3.5. *Let (X, G) be a G-metric space and let f and g be weakly compatible self-maps of X satisfying* (3.12)*,* (3.13) *and the following:*

$$f \text{ and } g \text{ satisfy } (CLR_f) \text{ property.} \tag{3.24}$$

Then f and g have a unique fixed point.

Proof. Since f and g satisfy the (CLR_f) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$$

for some $x \in X$. From (3.12), we have

$$\psi\Big(\int_0^{G(gx_n,gx,gx)}\varphi(t)dt\Big) \leq \psi\Big(\int_0^{G(fx_n,fx,fx)}\varphi(t)dt\Big)-\phi\Big(\int_0^{G(fx_n,fx,fx)}\varphi(t)dt\Big).$$

Letting $n \to \infty$ and using the properties of ψ and ϕ, we get

$$\begin{aligned}\psi\Big(\int_0^{G(fx,gx,gx)}\varphi(t)dt\Big) &\le \psi\Big(\int_0^{G(fx,fx,fx)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(fx,fx,fx)}\varphi(t)dt.\Big)\\ &= \psi(0) - \phi(0) = 0.\end{aligned}$$

Hence $\int_0^{G(fx,gx,gx)}\varphi(t)dt = 0$. Thus, $G(fx,gx,gx) = 0$, that is, $fx = gx$. Let $w = fx = gx$. Since f and g are weakly compatible, $fgx = gfx$, implies that, $fw = fgx = gfx = gw$.

Now, we claim that $Tw = w$. Suppose that $Tw \neq w$. Then. from (3.12), we have

$$\begin{aligned}\psi\Big(\int_0^{G(gw,w,w)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{G(gw,gx,gx)}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{G(fw,fx,fx)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(fw,fx,fx)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{G(gw,w,w)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(gw,w,w)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{G(gw,w,w)}\varphi(t)dt\Big),\end{aligned}$$

which is a contradiction. Hence $fw = w = gw$. Hence, w is the common fixed point of f and g.

Finally, we show that the common fixed point is unique. Let v be an another common fixed point of f and g such that $fv = v = gv$ and $w \neq v$. From (3.12), we have

$$\begin{aligned}\psi\Big(\int_0^{G(w,v,v)}\varphi(t)dt\Big) &= \psi\Big(\int_0^{G(gw,gv,gv)}\varphi(t)dt\Big)\\ &\le \psi\Big(\int_0^{G(fw,fv,fv)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(fw,fv,fv)}\varphi(t)dt\Big)\\ &= \psi\Big(\int_0^{G(w,v,v)}\varphi(t)dt\Big) - \phi\Big(\int_0^{G(w,v,v)}\varphi(t)dt\Big)\\ &< \psi\Big(\int_0^{G(w,v,v)}\varphi(t)dt\Big),\end{aligned}$$

which is a contradiction. Therefore $w = v$. This completes the proof. □

Example 3.6. Let $X = [1, \infty)$ and let $G : X^3 \to R_+$ be the G-metric defined as follows:

$$G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\} \text{ for all } x,y,z \in X.$$

Clearly (X, G) is a G-metric space. Define $f, g : X \to X$ by $f(x) = x$ and $g(x) = \frac{x+1}{2}$. Let $\{x_n\} = \{1 + \frac{1}{n}\}$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1 = f(1) \in X.$$

Hence, the pair (f, g) satisfy $(CLR_f)-$property. Let us define $\psi(t) = 2t$, $\varphi(t) = t$ and $\phi(t) = \frac{t}{2}$. Without loss of generality, we assume that for $x > y > z$

$$\begin{aligned} G(gx, gy, gz) &= G\left(\frac{x+1}{2}, \frac{y+1}{2}, \frac{z+1}{2}\right) \\ &= \max\left(\frac{|x-y|}{2}, \frac{|y-z|}{2}, \frac{|x-z|}{2}\right) = \frac{|x-z|}{2}. \end{aligned}$$

Clearly, $G(fx, fy, fz) = |x - z|$. Also, we have

$$\psi \int_0^{\frac{|x-z|}{2}} tdt = \psi\left(\frac{t^2}{2}\right) = \psi\left(\frac{|x-z|^2}{8}\right) = 2\frac{|x-z|^2}{8} = \frac{|x-z|^2}{4},$$

$$\psi \int_0^{|x-z|} tdt = \psi\left(\frac{|x-z|^2}{2}\right) = |x-z|^2,$$

and

$$\phi\left(\frac{|x-z|^2}{2}\right) = \frac{|x-z|^2}{4} = |x-z|^2 - \frac{|x-z|^2}{4} = \frac{3}{4}|x-z|^2.$$

By applying all these, we see that equation (3.12) is satisfied. Hence all the conditions of Theorem 3.5 are satisfied and f and g have a unique common fixed point $x = 1$.

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References

[1] M. Aamri and D. El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002), 181-188.

[2] M. Asadi and P. Salimi, *Some fixed point and common fixed point theorems on G-metric spaces*, Nonlinear Funct. Anal. Appl., **21**(3) (2016), 5213-530.

[3] A. Branciari, *A Fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29**(9) (2002), 531-536.

[4] B.C. Dhage, *Generalized metric spaces and mappings with fixed point*, Bull. Calcutta Math. , Soc., **84** (1982), 329-336.

[5] B.C. Dhage, *Generalized metric spaces and topological structures*, Anal. St. Univ. Al. I. Cuza, Iasi Ser. Mat., **46**(1) (2000), 3-24.

[6] G. Jungck, *Common fixed points for non-continuous non-self mappings on non-metric spaces*, Far East J. Math. Sci., **4** (1996), 199-212.

[7] Z. Mustafa and B. Sims, *Some remarks concerning D-metric spaces*, Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia (Spain), **2003** (2003), 189-198.

[8] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7**(2) (2006), 289-297.

[9] Z. Mustafa, H. Obiedat and F. Awawdeh, *Some fixed point theorems for mappings on complete G-metric spaces*, Fixed Point Theory Appl., Article ID 189870, **2008** (2008), 10 pages.

[10] B. Nurwahyu, *Some properties of common fixed point for two self-mappings on some contraction mappings in quasi αb-metric space,* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 175-188

[11] W. Shatanawi, *Fixed point theory for contractive mappings satisfying φ-maps in G-metric spaces*, Fixed Point Theory Appl., Article ID 181670, **2010** (2010), 9 pages.

[12] W. Sintunavarat and P. Kumam, *Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space*, J. Appl. Math. , Article ID 637958, **2011** (2011), 14 pages.

[13] D. Singh, V. Chauhan and V. Joshi, *Results on* n*-tupled coincidence and fixed points in partially ordered* G*-metric spaces via symmetric (ϕ,ψ)-contractions* Nonlinear Funct. Anal. Appl., **20**(3) (2015), 491-516.

[14] D. Singh, V. Joshi and J.K. Kim, *Existence of solution to Bessel-type boundary value problem via G-cyclic F-contractive mapping with graphical veriication,* Nonlinear Funct. Anal. Appl., **23**(2) (2018), 205-224.