



## FIXED POINT THEOREMS IN ORDERED $b$ -METRIC SPACES WITH ALTERNATING DISTANCE FUNCTIONS

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**Abstract.** In this paper we obtain a unique common fixed point theorem for four self-maps which are involved in  $(\phi, \psi)$ -weak contraction of a partially ordered  $b$ -metric space. The necessary condition has been given to a space for the existence of an unique common fixed of the maps. And our work changed conditions and nonlinear contraction, and search for the unique common fixed point of the maps.

### 1. INTRODUCTION

The Banach contraction principle is one of the basic results in fixed point theory which asserts that every contraction function in a complete metric space has a unique fixed point. Many authors extended this crucial theorem to many directions, see ([1],[2],[6],[8],[9],[10],[11]).

Bakhtin in [7] extended the notion of metric space to the notion of  $b$ -metric space. For some work in  $b$ -metric spaces, see ([3],[4],[13],[14],[17],[20],[21],[22],[30],[31],[45]).

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Many authors studied many fixed point theorems, for example see ([9], [15], [16], [18], [19], [22]-[29], [32]-[44]). The benefit of fixed point theorems is to prove the existence and uniqueness of such equations in partial differential equations, integral equations, and ordinary differential equations.

**Definition 1.1.** A metric space is a pair  $(X, d)$ , where  $X$  is a nonempty set and  $d : X \times X \rightarrow \mathbb{R}$  is a function such that for all  $x, y, z \in X$ , the following conditions hold:

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ,
- (ii)  $d(x, y) = d(y, x)$ ,
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$ .

**Definition 1.2.** ([5]) Let  $X$  be a nonempty set and let  $s \geq 1$  be a given real number. A function  $d : X \times X \rightarrow [0, \infty)$  is called  $b$ -metric if for all  $x, y, z \in X$  the following conditions are satisfied:

- (i)  $d(x, y) = 0$  if and only if  $x = y$ ,
- (ii)  $d(x, y) = d(y, x)$ ,
- (iii)  $d(x, z) \leq s[d(x, y) + d(y, z)]$ .

$(X, d, s)$  is said to be  $b$ -metric space, with coefficient  $s \geq 1$ .

**Remark 1.3.** Every  $b$ -metric space is a metric space with coefficient  $s = 1$ , but the converse is not true.

**Example 1.4.** Let  $X = \mathbb{R}$  and define  $d : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  by  $d(x, y) = (x - y)^2$ . Then  $d$  is a  $b$ -metric space which is not a metric space.

**Definition 1.5.** A metric  $d$  on  $X$  together with a partially ordered relation  $\leq$  is called a partially ordered metric space. It is denoted by  $(X, d, \leq)$ .

**Definition 1.6.** If the  $b$ -metric  $d$  is complete, then  $(X, d, \leq)$  is called a complete partially ordered  $b$ -metric space.

**Proposition 1.7.** ([12]) *In a  $b$ -metric space  $(X, d)$ , the following assertions hold:*

- (1) *A  $b$ -convergent sequence has a unique limit.*
- (2) *Each  $b$ -convergent sequence is  $b$ -Cauchy.*
- (3) *In general, a  $b$ -metric is continuous.*

2. PREVIOUS RESULTS

In the sequel, we have to recall previous notations and results. Let  $f$  and  $g$  be self-mappings on a set  $X$ . If  $w = fx = gx$  for some  $x \in X$ , then  $x$  is called a coincidence point of  $f$  and  $g$ , and  $w$  is called the point of coincidence of  $f$  and  $g$ .

Two self mappings  $f$  and  $g$  are said to be weakly compatible if they commute at their coincidence point, that is, if  $fx = gx$ , then  $fgx = gfx$ . Now, consider  $(X, \leq)$  to be partially ordered set. Two self-mappings  $f$  and  $g$  are said to be compatible, if for any sequence  $\{x_n\}$  with  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = \mu$ , then  $\lim_{n \rightarrow \infty} d(gfx_n, fgx_n) = 0$ . Also  $g$  is called monotone  $f$  nondecreasing, if  $fx \leq fy$  then  $gx \leq gy$  for any  $x, y \in X$ .

**Definition 2.1.** Let  $f, g, S$  and  $T$  be self-maps on a partial  $b$ -metric space  $(X, p, \leq)$  with  $(s > 1)$ . Then  $f$  and  $g$  are said to satisfy almost generalized  $(S, T)$ -contractive condition if there  $\delta \in [0, 1)$  such that

$$s^2 p(fx, gy) \leq \delta \max \left\{ p(Sx, Ty), p(fx, Sx), p(gy, Ty), \frac{p(Sx, gy) + p(fx, Ty)}{2s} \right\}, \tag{2.1}$$

for all  $x, y \in X$ .

3. MAIN RESULT

Let  $\Psi$  is the family of all functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  such that

- (1)  $\psi$  is continuous and nondecreasing,
- (2)  $\psi(t) = 0$  if and only if  $t = 0$ .

Also, let  $\Phi$  denote all functions  $\phi : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  such that

- (1)  $\phi$  is continuous,
- (2)  $\phi(t, s, u) = 0$  if and only if  $u = s = t = 0$ .

If  $\psi \in \Psi$ , then  $\Psi$  is called an altering distance function (see [18]).

Now, we introduce our definition.

**Definition 3.1.** Let  $f, g, S$ , and  $T$  be self-mappings on a  $b$ -metric space  $(X, d)$ . Then  $f$  and  $g$  are said to satisfy the almost nonlinear  $(S, T, \psi, \phi)$ -contractive condition if there exist  $\psi \in \Psi, \phi \in \Phi$  such that

$$\begin{aligned} & \psi(s^2 d(fx, gy)) \\ & \leq \psi \left( \max \left\{ d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2s} \right\} \right) \\ & \quad - \phi(d(Sx, Ty), d(Sx, gy), d(fx, Ty)), \end{aligned} \tag{3.1}$$

for all  $x, y \in X$ , where  $\phi(x, y, z) = \psi(x, x, x, x)$ , for all  $x \in [0, +\infty)$ .

**Theorem 3.2.** Let  $(X, d, \leq)$  be a complete ordered  $b$ -metric space. Suppose  $f, g, T, S : X \rightarrow X$  are continuous mappings such that  $f$  and  $g$  satisfy the almost nonlinear  $(S, T, \psi, \phi)$ -contractive condition for any two comparable element  $x, y \in X$ . Suppose that  $f, g, S$  and  $T$  satisfy the following conditions:

- (1)  $fX \subseteq TX$ ,
- (2)  $gX \subseteq SX$ ,
- (3) one of four mappings  $f, g, S, T$  is continuous,
- (4)  $\{f, S\}$  and  $\{g, T\}$  are compatible.

Then  $f, g, S$ , and  $T$  have a common fixed point.

*Proof.* Let  $x_0 \in X$  be an arbitrary. From  $fX \subseteq TX$ ,  $gX \subseteq SX$ , construct the sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that  $fx_{2n} = Tx_{2n+1} = y_{2n}$ ,  $gx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ . Putting  $y = x_{2n+1}$ ,  $x = x_{2n+2}$ .

Suppose that  $y_{2n} = y_{2n-1}$ , we have

$$\begin{aligned} & \psi(s^2 d(y_{2n}, y_{2n+1})) \\ &= \psi(s^2 d(fx_{2n}, gx_{2n+1})) \\ &\leq \psi \left( \max \left\{ d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), \right. \right. \\ &\quad \left. \left. \frac{d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})}{2s} \right\} \right) \\ &\quad - \phi(d(Sx_{2n}, Tx_{2n+1}), d(gx_{2n+1}, Sx_{2n}), d(fx_{2n}, Tx_{2n+1})). \\ &= \psi \left( \max \left\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{d(y_{2n-1}, y_{2n+1})}{2s} \right\} \right) \\ &\quad - \phi(d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n})) \\ &= \psi \left( \max \left\{ 0, 0, d(y_{2n+1}, y_{2n}), \frac{d(y_{2n+1}, y_{2n})}{2} \right\} \right) \\ &\quad - \phi(0, d(y_{2n+1}, y_{2n-1}), 0) \\ &\leq \psi(d(y_{2n+1}, y_{2n})). \end{aligned}$$

Therefore  $\phi(0, d(y_{2n+1}, y_{2n-1}), 0) = 0$  and hence  $y_{2n-1} = y_{2n+1} = y_{2n}$ . Similarly, we may show that  $y_{2n+2} = y_{2n+1}$ . Thus  $\{y_n\}$  is a constant sequence in  $X$ , hence it is a Cauchy sequence in  $(X, d)$ .

Suppose  $y_n \neq y_{n+1}$  for all  $n \in \mathbb{N}$ . If  $n$  is even, then  $n = 2t$  for some  $t \in \mathbb{N}$ . Since  $x_{2t}$  and  $x_{2t+1}$  are comparable, we have

$$\begin{aligned} & \psi(s^2 d(y_n, y_{n+1})) \\ &= \psi(s^2 d(fx_{2t}, gx_{2t+1})) \end{aligned}$$

$$\begin{aligned}
 &\leq \psi \left( \max \left\{ d(Sx_{2t}, Tx_{2t+1}), d(fx_{2t}, Sx_{2t}), d(gx_{2t+1}, Tx_{2t+1}), \right. \right. \\
 &\quad \left. \left. \frac{d(Sx_{2t}, gx_{2t+1}) + d(fx_{2t}, Tx_{2t+1})}{2s} \right\} \right) \\
 &\quad - \phi(d(Sx_{2t}, Tx_{2t+1}), d(gx_{2t+1}, Sx_{2t}), d(fx_{2t}, Tx_{2t+1})). \\
 &= \psi \left( \max \left\{ d(y_{2t-1}, y_{2t}), d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), \frac{d(y_{2t-1}, y_{2t+1})}{2s} \right\} \right) \\
 &\quad - \phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), d(y_{2t}, y_{2t})) \\
 &= \psi \left( \max \left\{ d(y_{2t-1}, y_{2t}), d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), \right. \right. \\
 &\quad \left. \left. \frac{d(y_{2t-1}, y_{2t}) + d(y_{2t}, y_{2t+1})}{2} \right\} \right) - \phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0) \\
 &\leq \psi(\max\{d(y_{2t-1}, y_{2t}), d(y_{2t+1}, y_{2t})\}).
 \end{aligned}$$

If  $\max\{d(y_{2t-1}, y_{2t}), d(y_{2t+1}, y_{2t})\} = d(y_{2t+1}, y_{2t})$ , then

$$\phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0) = 0$$

and hence

$$d(y_{2t-1}, y_{2t}) = d(y_{2t-1}, y_{2t+1}) = 0.$$

Thus  $y_{2t} = y_{2t-1}$ . That is,  $y_n = y_{n-1}$  which is a contradiction. Thus,

$$\max\{d(y_{2t-1}, y_{2t}), d(y_{2t+1}, y_{2t})\} = d(y_{2t-1}, y_{2t}). \tag{3.2}$$

Therefore,

$$\psi(s^2 d(y_{2t}, y_{2t+1})) \leq \psi(d(y_{2t-1}, y_{2t})) - \phi(d(y_{2t+1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0). \tag{3.3}$$

If  $n$  is odd, then  $n = 2t + 1$  for some  $t \in \mathbb{N}$ . Since  $x_{2t+2}$  and  $x_{2t+1}$  are comparable, we have

$$\begin{aligned}
 &\psi(s^2 d(y_n, y_{n+1})) \\
 &= \psi(s^2 d(y_{2t+2}, y_{2t+1})) \\
 &= \psi(s^2 d(fx_{2t+2}, gx_{2t+1})) \\
 &\leq \psi \left( \max \left\{ d(Sx_{2t+2}, Tx_{2t+1}), d(fx_{2t+2}, Sx_{2t+2}), d(gx_{2t+1}, Tx_{2t+1}), \right. \right. \\
 &\quad \left. \left. \frac{d(Sx_{2t+2}, gx_{2t+1}) + d(fx_{2t+2}, Tx_{2t+1})}{2s} \right\} \right) \\
 &\quad - \phi(d(Sx_{2t+2}, Tx_{2t+1}), d(gx_{2t+1}, Sx_{2t+2}), d(fx_{2t+2}, Tx_{2t+1})).
 \end{aligned}$$

$$\begin{aligned}
&= \psi \left( \max \left\{ d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1}), \frac{d(y_{2t+2}, y_{2t})}{2s} \right\} \right) \\
&\quad - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})). \\
&\leq \psi \left( \max \left\{ d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1}), \frac{d(y_{2t+2}, y_{2t+1}) + d(y_{2t+1}, y_{2t})}{2} \right\} \right) \\
&\quad - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})). \\
&= \psi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\}) \\
&\quad - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})). \\
&\leq \psi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\}).
\end{aligned}$$

If  $\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+2}, y_{2t+1})$ , then

$$\phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})) = 0,$$

and hence  $d(y_{2t+1}, y_{2t}) = d(y_{2t+2}, y_{2t})$ . Thus  $y_{2t+1} = y_{2t}$  which is a contradiction. So we have

$$\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+1}, y_{2t}). \quad (3.4)$$

Therefore,

$$\psi(s^2 d(y_{2t+2}, y_{2t+1})) \leq \psi(d(y_{2t+1}, y_{2t}) - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t}))).$$

From (3.2) and (3.4), we have

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n).$$

Therefore  $\{d(y_{n+1}, y_n) : n \in \mathbb{N}\}$  is a nonincreasing sequence. Thus there exists  $r \geq 0$  such that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = r.$$

On taking limsup in (3.3) and (3.4), we have

$$\psi(s^2 r) \leq \psi(r) - \liminf_{t \rightarrow \infty} \phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0)$$

and

$$\begin{aligned}
\psi(s^2 r) &\leq \psi(r) - \liminf_{t \rightarrow \infty} \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t}, y_{2t+2})) \\
&= \psi(r).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\liminf_{t \rightarrow \infty} d(y_{2t-1}, y_{2t}) &= \liminf_{t \rightarrow \infty} d(y_{2t-1}, y_{2t+1}) \\
&= \liminf_{t \rightarrow \infty} d(y_{2t}, y_{2t+2}) \\
&= \liminf_{t \rightarrow \infty} d(y_{2t}, y_{2t+1}) \\
&= 0.
\end{aligned}$$

Therefore,  $r = 0$  and hence

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (3.5)$$

We show that  $\{y_n\}$  is a Cauchy sequence in the metric space  $(X, d)$ . Suppose to the contrary, that is,  $\{y_{2n}\}$  is not a Cauchy sequence in  $(X, d)$ . Then there exists  $\epsilon \geq 0$  and two subsequences  $\{y_{2m(i)}\}$  and  $\{y_{2n(i)}\}$  of  $\{y_{2n}\}$  such that  $n(i)$  is the smallest index for which,  $n(i) > m(i) > i$ ,

$$d(y_{2m(i)}, y_{2n(i)}) \geq \epsilon$$

and

$$d(y_{2m(i)}, y_{2n(i)-1}) < \epsilon. \quad (3.6)$$

From (3.5), (3.6) and the triangular inequality, we get that

$$\begin{aligned} \epsilon &\leq d(y_{2m(i)}, y_{2n(i)}) \\ &\leq sd(y_{2m(i)}, y_{2n(i)-1}) \\ &\quad + sd(y_{2n(i)-1}, y_{2n(i)}) \\ &\leq s\epsilon + sd(y_{2n(i)-1}, y_{2n(i)}). \end{aligned}$$

Furthermore,

$$\begin{aligned} d(y_{2n(i)}, y_{2m(i)+1}) &\leq sd(y_{2n(i)}, y_{2n(i)-1}) \\ &\quad + sd(y_{2n(i)-1}, y_{2m(i)+1}) \\ &\leq sd(y_{2n(i)}, y_{2n(i)-1}) \\ &\quad + s^2 d(y_{2n(i)-1}, y_{2m(i)}) \\ &\quad + s^2 d(y_{2m(i)}, y_{2m(i)+1}) \\ &\leq sd(y_{2n(i)}, y_{2n(i)-1}) + s^2 \epsilon \\ &\quad + s^2 d(y_{2m(i)}, y_{2m(i)+1}). \end{aligned}$$

Also, from triangular inequality, we get

$$\begin{aligned} \epsilon &\leq d(y_{2m(i)}, y_{2n(i)}) \\ &\leq sd(y_{2m(i)}, y_{2m(i)-1}) + sd(y_{2m(i)-1}, y_{2n(i)}) \\ &\leq sd(y_{2m(i)}, y_{2m(i)-1}) + s^2 d(y_{2m(i)-1}, y_{2m(i)+1}) + s^2 d(y_{2m(i)+1}, y_{2n(i)}) \\ &\leq sd(y_{2m(i)}, y_{2m(i)-1}) + s^3 d(y_{2m(i)-1}, y_{2m(i)}) + s^3 d(y_{2m(i)}, y_{2m(i)+1}) \\ &\quad + s^2 d(y_{2m(i)+1}, y_{2n(i)}). \end{aligned}$$

Letting  $i \rightarrow +\infty$  in above inequalities in (3.5) and (3.6), we get

$$\limsup_{i \rightarrow \infty} d(y_{2m(i)}, y_{2n(i)}) \leq s\epsilon,$$

and

$$\frac{\epsilon}{s^2} \leq \limsup_{i \rightarrow \infty} d(y_{2n(i)}, y_{2m(i)+1}) \leq s^2 \epsilon.$$

Similarly, we get

$$\frac{\epsilon}{s^2} \leq \liminf_{i \rightarrow \infty} d(y_{2n(i)}, y_{2m(i)+1}) \leq s^2 \epsilon.$$

Also,

$$\begin{aligned} d(y_{2n(i)-1}, y_{2m(i)+1}) &\leq sd(y_{2n(i)-1}, y_{2m(i)}) + sd(y_{2m(i)}, y_{2m(i)+1}) \\ &\leq s\epsilon + sd(y_{2m(i)}, y_{2m(i)+1}). \end{aligned}$$

Letting  $i \rightarrow +\infty$  in above inequalities in (3.5) and (3.6), we get

$$\limsup_{i \rightarrow \infty} d(y_{2n(i)-1}, y_{2m(i)+1}) \leq s\epsilon.$$

Since  $x_{2n(i)}$  and  $x_{2m(i)+1}$  are comparable, we have

$$\begin{aligned} &\psi(s^2 d(y_{2n(i)}, y_{2m(i)+1})) \\ &= \psi(s^2 d(fx_{2n(i)}, gx_{2m(i)+1})) \\ &\leq \psi\left(\max\left\{d(Sx_{2n(i)}, Tx_{2m(i)+1}), d(fx_{2n(i)}, Sx_{2n(i)}), d(gx_{2m(i)+1}, Tx_{2m(i)+1}), \right. \right. \\ &\quad \left. \left. \frac{d(Sx_{2n(i)}, gx_{2m(i)+1}) + d(fx_{2n(i)}, Tx_{2m(i)+1})}{2s}\right\}\right) \\ &\quad - \phi(d(Sx_{2n(i)}, Tx_{2m(i)+1}), d(Sx_{2n(i)}, gx_{2m(i)+1}), d(fx_{2n(i)}, Tx_{2m(i)+1})). \\ &= \psi\left(\max\left\{d(y_{2n(i)-1}, y_{2m(i)}), d(y_{2n(i)-1}, y_{2m(i)}), d(y_{2m(i)+1}, y_{2m(i)}), \right. \right. \\ &\quad \left. \left. \frac{d(y_{2n(i)-1}, y_{2m(i)+1}) + d(y_{2n(i)}, y_{2m(i)})}{2s}\right\}\right) \\ &\quad - \phi(d(y_{2n(i)-1}, y_{2m(i)}), d(y_{2n(i)-1}, y_{2m(i)+1}), d(y_{2n(i)}, y_{2m(i)})). \end{aligned}$$

Letting  $i \rightarrow +\infty$ . And using the continuity of  $\psi$ , we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon, s\epsilon, s\epsilon) < \psi(\epsilon).$$

So  $\phi(\epsilon, s\epsilon, s\epsilon) = 0$ , hence  $\epsilon = 0$ . Thus  $\{y_n\}$  is a Cauchy sequence in  $(X, d)$ . Since  $(X, d)$  is complete, there is  $z \in X$  such that  $y_n \rightarrow z$  in the metric space  $(X, d)$ . Thus  $d(y_n, z) = 0$ , for all  $n \rightarrow +\infty$ . Hence, by the compatibility of  $S$  and  $f$ , we obtain

$$\lim_{n \rightarrow \infty} d(f(Sx_{2n}), S(fx_{2n})) = 0,$$



further by using triangular inequality,

$$\begin{aligned} d(Sz, fz) &\leq sd(Sz, S(fx_{2n})) \\ &\quad + s^2d(S(fx_{2n}), f(Sx_{2n})) \\ &\quad + s^2d(fz, f(Sx_{2n})). \end{aligned} \tag{3.7}$$

Therefore, we arrive at  $d(Sz, fz) = 0$  as  $n \rightarrow +\infty$  in (3.7). Hence,  $z$  is a coincidence point for  $S$  and  $f$  in  $X$ .

Putting  $x = z, y = x_{2n+1}$ , we get

$$\psi(s^2d(Sz, z)) \leq \psi(d(Sz, z)),$$

hence  $Sz = z$  and  $fz = z$ . Since  $z = fz \in fX \subseteq TX$ , there exists  $u \in X$  such that  $z = Tu$ . Putting  $x = x_{2n}, y = u$ , and letting  $n \rightarrow \infty$ , we get  $z = gu$ , so  $gu = Tu$  since  $(g, T)$  is compatible, we have  $gz = Tz$ . Putting  $x = x_{2n}, y = z$ , we get  $gz = z$  so that  $Tz = z$ . This completes the proof.  $\square$

**Example 3.3.** Let  $X = \{0, 1, 2, 3, 4\}$ .

$$\begin{cases} d(x, y) = 0, & \text{if } x = y, \\ d(x, y) = (x + y)^2, & \text{if } x \neq y. \end{cases}$$

Then  $(X, d)$  is a  $b$ -metric space with constant  $s > 1$ . Let  $f, g, S, T : X \rightarrow X$  be defined by

$$f(x) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix}, S(x) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, g(x) = 1, T(x) = x.$$

Then, we know that  $f(X) \subseteq T(X), g(X) \subseteq S(X)$ , and the pairs  $\{f, S\}$  and  $\{g, T\}$  are compatible. Let the control functions  $\psi : [0, \infty) \rightarrow [0, \infty), \phi : [0, \infty) \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  be defined by  $\psi(t) = t$  and  $\phi(t_1, t_2, t_3) = t_1 + t_2 + t_3$ . Then all the conditions of Theorem 3.2 are satisfied except the contractive condition of this theorem. Hence  $f, g, S, T$  have no common fixed point in  $X$ .

**Remark 3.4.** If inequality of Theorem 3.2 is replaced by

$$\begin{aligned} &\psi(s^2d(fx, gy)) \tag{3.8} \\ &\leq k\psi \left( \max \left\{ d(Sx, Ty), d(fx, Sy), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2s} \right\} \right) \end{aligned}$$

for all  $x, y \in X$ , where  $0 < k < 1$ , then the result of the Theorem 3.2 holds.

**Example 3.5.** Let  $X = \{1, 2, 3, 4\}$ , define  $d : X \times X \rightarrow \mathbb{R}$  by

$$\begin{cases} d(x, y) = 0, & \text{if } x = y \\ d(x, y) = 1, & \text{if } 2 \neq x \neq y \neq 3. \\ d(x, y) = \frac{1}{2}, & \text{if } x = 2, y = 3. \end{cases}$$

Also define  $f, g : X \rightarrow X$ ,  $f1 = f4 = 2$ ,  $f2 = f3 = 2$ ,  $g2 = 2$ ,  $g4 = 3$ ,  $g1 = g3 = 4$ . Moreover take  $S, T = I$  (Identity map) in Theorem 3.2. Then  $f, g, S, T$  have a common fixed point.

**Theorem 3.6.** Let  $(X, d, s, \leq)$  be a complete partially ordered  $b$ -metric space with parameter  $s \geq 1$  and  $f, g, S, T : X \rightarrow X$  be self-mappings such that

(1)

$$\phi(sd(fx, gy)) \leq \phi(M(x, y)) - \psi(M(x, y)),$$

where

$$M(x, y) = \max \left\{ d(gy, Ty) \frac{[1 + d(fx, Sx)]}{1 + d(fx, gy)}, \frac{d(fx, Ty) + d(gy, Sx)}{2s}, \right. \\ \left. d(fx, Sx), d(gy, Ty), d(fx, gy) \right\}$$

for  $x, y \in X$ , and for some  $\phi \in \Phi$  and  $\psi \in \Psi$ .

- (2) One of four mappings  $f, g, S, T$  is continuous nondecreasing map with regards to  $\leq$  such that there exists  $x_0$  with  $x_0 \leq Sx_0$  (if  $S$  is continuous and nondecreasing map).
- (3)  $fX \subseteq TX$ .
- (4)  $gX \subseteq SX$ .
- (5)  $f$  is monotone  $T$ -nondecreasing map.
- (6)  $g$  is monotone  $S$ -nondecreasing map.
- (7)  $\{f, s\}$  and  $\{g, T\}$  are weakly compatible.

Then  $f, g, S$  and  $T$  have a common fixed point in  $X$ .

*Proof.* Let  $x_0 \in X$  be an arbitrary. From  $fX \subseteq TX$ ,  $gX \subseteq SX$ , we can construct the sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that

$$fx_{2n} = Tx_{2n+1} = y_{2n}, \quad gx_{2n+1} = Sx_{2n+2} = y_{2n+1}.$$

Putting  $y = x_{2n+1}$ ,  $x = x_{2n+2}$ . Suppose  $y_{2n} = y_{2n-1}$ , we have

$$\begin{aligned}
& \phi(sd(y_{2n}, y_{2n+1})) \\
&= \phi(sd(fx_{2n}, gx_{2n+1})) \\
&\leq \phi\left(\max\left\{d(gx_{2n+1}, Tx_{2n+1})\frac{[1+d(fx_{2n}, Sx_{2n})]}{1+d(fx_{2n}, gx_{2n})},\right. \right. \\
&\quad \left.\left.\frac{d(fx_{2n}, Tx_{2n+1})+d(gx_{2n+1}, Sx_{2n})}{2s},\right. \right. \\
&\quad \left.\left. d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1})\right\}\right) \\
&\quad - \psi\left(\max\left\{d(gx_{2n+1}, Tx_{2n+1})\frac{[1+d(fx_{2n}, Sx_{2n})]}{1+d(fx_{2n}, gx_{2n})},\right. \right. \\
&\quad \left.\left.\frac{d(fx_{2n}, Tx_{2n+1})+d(gx_{2n+1}, Sx_{2n})}{2s}, d(fx_{2n}, Sx_{2n}),\right. \right. \\
&\quad \left.\left. d(gx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1})\right\}\right) \\
&= \phi\left(\max\left\{d(y_{2n+1}, y_{2n})\frac{[1+d(y_{2n}, y_{2n-1})]}{1+d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n}, y_{2n})+d(y_{2n+1}, y_{2n-1})}{2s},\right. \right. \\
&\quad \left.\left. d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}\right) \\
&\quad - \psi\left(\max\left\{d(y_{2n+1}, y_{2n})\frac{[1+d(y_{2n}, y_{2n-1})]}{1+d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n}, y_{2n})+d(y_{2n+1}, y_{2n-1})}{2s},\right. \right. \\
&\quad \left.\left. d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}\right) \\
&\leq \phi\left(\max\left\{\frac{d(y_{2n+1}, y_{2n})}{1+d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n+1}, y_{2n})+d(y_{2n}, y_{2n-1})}{2},\right. \right. \\
&\quad \left.\left. d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}\right) \\
&\quad - \psi\left(\max\left\{d\left(\frac{d(y_{2n+1}, y_{2n})}{1+d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n+1}, y_{2n})+d(y_{2n}, y_{2n-1})}{2},\right. \right. \right. \\
&\quad \left.\left. d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}\right) \\
&= \phi(d(y_{2n+1}, y_{2n})) - \psi(d(y_{2n+1}, y_{2n})) \\
&\leq \phi(d(y_{2n+1}, y_{2n})).
\end{aligned}$$

Therefore  $\psi(d(y_{2n+1}, y_{2n})) = 0$  and hence  $y_{2n-1} = y_{2n+1} = y_{2n}$ .

Similarly, we may show that  $y_{2n+2} = y_{2n+1}$ . Thus  $\{y_n\}$  is a constant sequence in  $X$ , hence it is a Cauchy sequence in  $(X, d)$ .

Suppose  $y_n \neq y_{n+1}$  and  $y_{n-1} = y_{n+1}$  for all  $n \in \mathbb{N}$ . If  $n$  is even, then  $n = 2t$  for some  $t \in \mathbb{N}$ . Since  $x_{2t}$  and  $x_{2t+1}$  are comparable, we have

$$\begin{aligned}
& \phi(sd(y_n, y_{n+1})) \\
&= \phi(sd(fx_{2t}, gx_{2t+1})) \\
&\leq \phi\left(\max\left\{d(gx_{2t+1}, Tx_{2t+1})\frac{[1 + d(fx_{2t}, Sx_{2t})]}{1 + d(fx_{2t}, gx_{2t})}, \right. \right. \\
&\quad \left. \left. \frac{d(fx_{2t}, Tx_{2t+1}) + d(gx_{2t+1}, Sx_{2t})}{2s}, \right. \right. \\
&\quad \left. \left. d(fx_{2t}, Sx_{2t}), d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t}, gx_{2t+1})\right\}\right) \\
&\quad - \psi\left(\max\left\{d(gx_{2t+1}, Tx_{2t+1})\frac{[1 + d(fx_{2t}, Sx_{2t})]}{1 + d(fx_{2t}, gx_{2t+1})}, \right. \right. \\
&\quad \left. \left. \frac{d(fx_{2t}, Tx_{2t+1}) + d(gx_{2t+1}, Sx_{2t})}{2s}, d(fx_{2t}, Sx_{2t}), \right. \right. \\
&\quad \left. \left. d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t}, gx_{2t+1})\right\}\right) \\
&= \phi\left(\max\left\{d(y_{2t+1}, y_{2t})\frac{[1 + d(y_{2t}, y_{2t-1})]}{1 + d(y_{2t}, y_{2t+1})}, \right. \right. \\
&\quad \left. \left. \frac{d(y_{2t}, y_{2t}) + d(y_{2t+1}, y_{2t-1})}{2s}, \right. \right. \\
&\quad \left. \left. d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), d(y_{2t}, y_{2t+1})\right\}\right) \\
&\quad - \psi\left(\max\left\{d(y_{2t+1}, y_{2t})\frac{[1 + d(y_{2t}, y_{2t-1})]}{1 + d(y_{2t}, y_{2t+1})}, \right. \right. \\
&\quad \left. \left. \frac{d(y_{2t}, y_{2t}) + d(y_{2t+1}, y_{2t-1})}{2s}, \right. \right. \\
&\quad \left. \left. d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), d(y_{2t}, y_{2t+1})\right\}\right) \\
&= \phi(d(y_{2t-1}, y_{2t})) - \psi(d(y_{2t-1}, y_{2t})).
\end{aligned}$$

Therefore,

$$\phi(sd(y_{2t}, y_{2t+1})) \leq \phi(d(y_{2t-1}, y_{2t})) - \psi(d(y_{2t-1}, y_{2t})). \quad (3.9)$$

If  $n$  is odd, then  $n = 2t + 1$  for some  $t \in \mathbb{N}$ . Since  $x_{2t+2}$  and  $x_{2t+1}$  are comparable, we have

$$\begin{aligned}
 & \phi(sd(y_n, y_{n+1})) \\
 &= \phi(sd(y_{2n+2}, y_{2n+1})) = \phi(sd(fx_{2t+2}, gx_{2t+1})) \\
 &\leq \phi\left(\max\left\{d(gx_{2t+1}, Tx_{2t+1})\frac{[1+d(fx_{2t+2}, Sx_{2t+2})]}{1+d(fx_{2t+2}, gx_{2t+1})},\right.\right. \\
 &\quad \left.\left.\frac{d(fx_{2t+2}, Tx_{2t+1})+d(gx_{2t+1}, Sx_{2t+2})}{2s},\right.\right. \\
 &\quad \left.\left.d(fx_{2t+2}, Sx_{2t+2}), d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t+2}, gx_{2t+1})\right\}\right) \\
 &\quad - \psi\left(\max\left\{d(gx_{2t+1}, Tx_{2t+1})\frac{[1+d(fx_{2t+2}, Sx_{2t+2})]}{1+d(fx_{2t+2}, gx_{2t+1})},\right.\right. \\
 &\quad \left.\left.\frac{d(fx_{2t+2}, Tx_{2t+1})+d(gx_{2t+1}, Sx_{2t+2})}{2s},\right.\right. \\
 &\quad \left.\left.d(fx_{2t+2}, Sx_{2t+2}), d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t+2}, gx_{2t+1})\right\}\right) \\
 &= \phi\left(\max\left\{d(y_{2t+1}, y_{2t})\frac{[1+d(y_{2t+2}, y_{2t+1})]}{1+d(y_{2t+2}, y_{2t+1})},\right.\right. \\
 &\quad \left.\left.\frac{d(y_{2t+2}, y_{2t})+d(y_{2t+1}, y_{2t+1})}{2s}, d(y_{2t+2}, y_{2t+1}),\right.\right. \\
 &\quad \left.\left.d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\right\}\right) \\
 &\quad - \psi\left(\max\left\{d(y_{2t+1}, y_{2t})\frac{[1+d(y_{2t+2}, y_{2t+1})]}{1+d(y_{2t+2}, y_{2t+1})},\right.\right. \\
 &\quad \left.\left.\frac{d(y_{2t+2}, y_{2t})+d(y_{2t+1}, y_{2t+1})}{2s}, d(y_{2t+2}, y_{2t+1}),\right.\right. \\
 &\quad \left.\left.d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\right\}\right) \\
 &= \phi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+1}, y_{2t+2})\}) \\
 &\quad - \psi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+1}, y_{2t+2})\}).
 \end{aligned}$$

If

$$\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+2}, y_{2t+1}),$$

then,  $\psi(d(y_{2t+2}, y_{2t+1})) = 0$ , thus  $y_n = y_{n+1}$  which is a contraction. So,

$$\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+1}, y_{2t}). \quad (3.10)$$

Therefore,

$$\phi(sd(y_{2n+2}, y_{2n+1})) \leq \phi(d(y_{2t+1}, y_{2t})) - \psi(d(y_{2t+1}, y_{2t})).$$

From (3.9) and (3.10), we have

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n).$$

Therefore  $\{d(y_{n+1}, y_n) : n \in \mathbb{N}\}$  is a nonincreasing sequence. Thus there exists  $r \geq 0$  such that

$$\lim_{t \rightarrow \infty} d(y_n, y_{n+1}) = r.$$

On taking limsup in (3.9) and (3.10), we have

$$\begin{aligned} \phi(sr) &\leq \phi(r) - \psi(r) \\ &= \phi(r). \end{aligned}$$

Thus,

$$\begin{aligned} \liminf_{t \rightarrow \infty} d(y_{2t-1}, y_{2t}) &= \liminf_{t \rightarrow \infty} d(y_{2t-1}, y_{2t+1}) \\ &= \liminf_{t \rightarrow \infty} d(y_{2t}, y_{2t+2}) \\ &= \liminf_{t \rightarrow \infty} d(y_{2t}, y_{2t+1}) \\ &= 0. \end{aligned}$$

Therefore,  $r = 0$  and hence

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (3.11)$$

To show that  $\{y_{2n}\}$  is a Cauchy sequence. If  $\{y_{2n}\}$  is not Cauchy, there exists an  $\epsilon > 0$ , and monotone increasing sequence of natural numbers  $\{2m(k)\}$  and  $\{2n(k)\}$  such that  $n(k) > m(k)$ ,

$$d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$$

and

$$d(y_{2m(k)}, y_{2n(k)-1}) < \epsilon. \quad (3.12)$$

From (3.11), (3.12) and the triangular inequality, we obtain that

$$\begin{aligned} \epsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq sd(y_{2m(k)}, y_{2n(k)-1}) + sd(y_{2n(k)-1}, y_{2n(k)}) \\ &\leq s\epsilon + sd(y_{2n(k)-1}, y_{2n(k)}). \end{aligned}$$

Furthermore,

$$\begin{aligned} &d(y_{2n(k)}, y_{2m(k)+1}) \\ &\leq sd(y_{2n(k)}, y_{2n(k)-1}) + sd(y_{2n(k)-1}, y_{2m(k)+1}) \\ &\leq sd(y_{2n(k)}, y_{2n(k)-1}) + s^2d(y_{2n(k)-1}, y_{2m(k)}) \\ &\quad + s^2d(y_{2m(k)}, y_{2m(k)+1}) \\ &\leq s^2\epsilon + s^2d(y_{2m(k)}, y_{2m(k)+1}). \end{aligned}$$

Also, from triangular inequality, we get

$$\begin{aligned} \epsilon \leq d(y_{2m(k)}, y_{2n(k)}) &\leq sd(y_{2m(k)}, y_{2m(k)-1}) + sd(y_{2m(k)-1}, y_{2n(k)}) \\ &\leq sd(y_{2m(k)}, y_{2m(k)-1}) + s^2 d(y_{2m(k)-1}, y_{2m(k)+1}) \\ &\quad + s^2 d(y_{2m(k)+1}, y_{2n(k)}) \\ &\leq sd(y_{2m(k)}, y_{2m(k)-1}) + s^3 d(y_{2m(k)-1}, y_{2m(k)}) \\ &\quad + s^3 d(y_{2m(k)}, y_{2m(k)+1}) \\ &\quad + s^2 d(y_{2m(k)+1}, y_{2n(k)}). \end{aligned}$$

Letting  $k \rightarrow +\infty$  in above inequalities and using (3.11) and (3.12), we get

$$\limsup_{k \rightarrow \infty} d(y_{2m(k)}, y_{2n(k)}) \leq s\epsilon$$

and

$$\frac{\epsilon}{s^2} \leq \limsup_{k \rightarrow \infty} d(y_{2n(k)}, y_{2m(k)+1}) \leq s^2\epsilon.$$

Similarly, we get

$$\frac{\epsilon}{s^2} \leq \liminf_{k \rightarrow \infty} d(y_{2n(k)}, y_{2m(k)+1}) \leq s^2\epsilon.$$

Since  $x_{2n(k)}$  and  $x_{2m(k)+1}$  are comparable, we have

$$\begin{aligned} &\phi(sd(y_{2n(k)}, y_{2m(k)+1})) \\ &\leq \phi\left(\max\left\{d(y_{2m(k)+1}, y_{2m(k)}) \frac{[1 + d(y_{2n(k)}, y_{2n(k)-1})]}{1 + d(y_{2n(k)}, y_{2m(k)+1})}, \right. \right. \\ &\quad \left. \frac{d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)+1}, y_{2m(k)-1})}{2s}, \right. \\ &\quad \left. d(y_{2n(k)}, y_{2n(k)-1}), d(y_{2m(k)+1}, y_{2m(k)}), d(y_{2n(k)}, y_{2m(k)+1})\right\}) \\ &- \psi\left(\max\left\{d(y_{2m(k)+1}, y_{2m(k)}) \frac{[1 + d(y_{2n(k)}, y_{2n(k)-1})]}{1 + d(y_{2n(k)}, y_{2m(k)+1})}, \right. \right. \\ &\quad \left. \frac{d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)+1}, y_{2m(k)-1})}{2s}, \right. \\ &\quad \left. d(y_{2n(k)}, y_{2n(k)-1}), d(y_{2m(k)+1}, y_{2m(k)}), d(y_{2n(k)}, y_{2m(k)+1})\right\}). \end{aligned}$$

Letting  $k \rightarrow \infty$  and using (3.11) and (3.12), we have

$$\begin{aligned} \phi(s^2\epsilon) &\leq \phi\left(\max\left(0, \frac{\epsilon}{2}, 0, 0, s^2\epsilon\right)\right) - \psi\left(\max\left(0, \frac{\epsilon}{2}, 0, 0, s^2\epsilon\right)\right) \\ &< \phi(s^2\epsilon). \end{aligned}$$

So  $\psi(s^2\epsilon) = 0$ , and hence  $\epsilon = 0$ . Thus  $\{y_{2n}\}$  is a Cauchy sequence in  $(X, d)$ . Since  $X$  is complete, there exists  $z \in X$  such that  $y_n \rightarrow z$  as  $n \rightarrow \infty$ .

Now, we show that  $z$  is the fixed point of  $g$  and  $T$ . Assume that  $TX$  is closed, since  $\{y_{2n} = Tx_{2n+1}\}$  is a sequence in  $TX$  converging to  $z$ , we have  $z \in TX$ . So, there exists  $u \in X$  such that  $z = Tu$ . Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} fx_{2n} &= \lim_{n \rightarrow \infty} gx_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+2} \\ &= z = Tu. \end{aligned}$$

Now, we show that  $gu = z$ . Since  $x_{2n} \leq fx_{2n}$  and  $y_{2n} = fx_{2n} \rightarrow z$ , we have  $x_{2n} \leq z$ . Since the mapping  $f$  is monotone  $T$  nondecreasing, we obtain  $x_{2n} \leq z = Tu \leq fTu \leq u$  and

$$\begin{aligned} &\phi(sd(y_{2n}, gu)) \\ &= \phi(sd(fx_{2n}, gu)) \\ &\leq \phi\left(\max\left\{d(gu, Tu) \frac{[1 + d(fx_{2n}, Sx_{2n})]}{1 + d(fx_{2n}, gu)}, \frac{d(fx_{2n}, Tu) + d(gu, S_{2n})}{2S}, \right. \right. \\ &\quad \left. \left. d(fx_{2n}, Sx_{2n}), d(gu, Tu), d(fx_{2n}, gu)\right\}\right) \\ &= \phi\left(\max\left\{d(gu, y) \frac{[1 + d(y_{2n}, y_{2n-1})]}{1 + d(y_{2n}, gu)}, \frac{d(y_{2n}, y) + d(gu, y_{2n-1})}{2S}, \right. \right. \\ &\quad \left. \left. d(y_{2n}, y_{2n-1}), d(gu, y), d(y_{2n}, gu)\right\}\right). \end{aligned}$$

Letting  $n \rightarrow +\infty$  in the above inequalities and using (3.11) we get

$$\phi(sd(y, gu)) \leq \phi(d(y, gu)) - \psi(d(y, gu)).$$

Therefore  $\psi(d(y, gu)) = 0$ , and hence  $d(y, gu) = 0$ . Thus  $y = gu$ . Since  $g$  and  $T$  are weakly compatible,  $gu = gTu = Tgu = Ty$ .

Again, since  $x_{2n}$  and  $y$  are comparable, we have

$$\begin{aligned} &\phi(sd(y_{2n}, gy)) = \phi(sd(fx_{2n}, gy)) \\ &\leq \phi\left(\max\left\{d(gy, Ty) \frac{[1 + d(fx_{2n}, Sx_{2n})]}{1 + d(fx_{2n}, gy)}, \frac{d(fx_{2n}, Ty) + d(gy, S_{2n})}{2S}, \right. \right. \\ &\quad \left. \left. d(fx_{2n}, Sx_{2n}), d(gy, Ty), d(fx_{2n}, gy)\right\}\right) \\ &= \phi\left(\max\left\{d(gy, gy) \frac{[1 + d(y_{2n}, y_{2n-1})]}{1 + d(y_{2n}, gy)}, \frac{d(y_{2n}, y) + d(gy, y_{2n-1})}{2S}, \right. \right. \\ &\quad \left. \left. d(y_{2n}, y_{2n-1}), d(gy, gy), d(y_{2n}, gy)\right\}\right). \end{aligned}$$



Letting  $n \rightarrow \infty$  in the above inequalities and using (3.11), we have

$$\phi(sd(y, gy)) \leq \phi(d(y, gy)) - \psi(d(y, gy)).$$

Therefore,  $\psi(d(y, gy)) = 0$  and hence  $d(y, gy) = 0$ . Thus  $y = gy$ .

Finally, we have to show that  $y$  is also a fixed point of  $f$  and  $T$ . Since  $gX \subseteq SX$ , there exists  $a \in X$  such that  $y = gy = Sa$ . Since  $g$  is monotone  $S$ -nondecreasing map, we have  $y \leq gy = Sa \leq gSa \leq a$ . Thus  $y$  and  $a$  are comparable.  $\square$

**Corollary 3.7.** *Let  $(X, d, s, \leq)$  be a complete partially  $b$ -metric space parameter  $s \geq 1$ . Suppose  $S : X \rightarrow X$  is a continuous, nondecreasing map with regards to  $\leq$  such that there exists  $x_0$  with  $x_0 \leq Sx_0$ . Suppose that*

$$\phi(sd(Sx, Sy)) \leq \phi(M(x, y)) - \psi(M(x, y)), \tag{3.13}$$

where  $\phi \in \Phi, \psi \in \Psi$ , for any  $x, y \in X$  with  $x \leq y$  and,

$$M(x, y) = \max \left\{ \frac{d(y, Sy)}{1 + d(x, y)}, \frac{d(x, Sy) + d(y, Sx)}{2s}, d(x, Sx), d(x, y) \right\}. \tag{3.14}$$

Then  $S$  has a fixed point in  $X$ .

**Example 3.8.** Define a metric  $d : X \rightarrow X$  as below and  $\leq$  is an usual order on  $X$ , where  $X = \{1, 2, 3, 4\}$ ,

$$\begin{aligned} d(x, y) &= 0 && \text{if } x, y = 1, 2, 3 \text{ and } x = y, \\ d(x, y) &= 1 && \text{if } x, y = 1, 2, 3 \text{ and } x \neq y, \\ d(x, y) &= 5 && \text{if } x, y = 1, 2 \text{ and } y = 4, \\ d(x, y) &= 20 && \text{if } x = 3 \text{ and } y = 4. \end{aligned}$$

Define a map  $f : X \rightarrow X$  by  $f1 = f2 = f3 = 1, f4 = 2$  and let  $\phi(t) = t, \psi(t) = \frac{t}{2}$  for  $t \in [0, \infty)$ . Then  $f$  has a fixed point in  $X$ . In fact, it is apparent that,  $(X, d, s, \leq)$  is a complete partially ordered  $b$ -metric space for  $s = 2$ . Consider the possible cases for  $x, y$  in  $X$ :

**Case 1.** Suppose  $x, y \in \{1, 2, 3\}$  and  $x < y$ . Then

$$\phi(2d(fx, fy)) = 0 \leq \phi(M(x, y)) - \psi(M(x, y)).$$

**Case 2.** Suppose  $x \in \{1, 2, 3\}$  and  $y = 4$ . Then  $d(fx, fy) = d(1, 2) = 1, M(x, 4) = 5$  if  $x = \{1, 2\}$  and  $M(3, 4) = 20$ . Therefore, we have the following inequality,

$$\phi(2d(fx, fy)) \leq \phi(M(x, y)) - \psi(M(x, y)).$$

Thus, condition (3) of Corollary 3.7 holds. Then,  $f$  has a fixed point in  $X$ .

**Corollary 3.9.** *If the inequality (1) of Theorem 3.6 is replaced by*

$$\begin{aligned} & \phi(sd(fx, gy)) \\ & \leq \phi \left( \max \left\{ d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2s} \right\} \right) \\ & \quad - \phi(d(Sx, Ty), d(Sx, gy), d(fx, Ty)) \\ & \quad + L \min\{d(Sx, Ty), d(Sx, gy), d(fx, Ty)\}, \end{aligned}$$

for all  $x, y \in X, L \in [0, \infty)$ . Then  $f, g, S$  and  $T$  have a common fixed point in  $X$ .

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