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CONTROLLABILITY OF DAMPED SECOND-ORDER INITIAL VALUE PROBLEM FOR A CLASS OF DIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS ON NONCOMPACT INTERVALS

D. N. Chalishajar

Carroll Hall
Virginia Military Institute, Lexington, VA-24450, USA
e-mail: dipu17370@yahoo.com

Abstract. In this short article, we investigate the controllability of damped second-order initial value problems for a class of differential inclusions with nonlocal conditions on unbounded real interval. We shall employ a theorem of Ma, which is an extension to multivalued maps on locally convex topological spaces, of Schaefer's theorem. Example is provided to illustrate the theory. This work is motivated by the papers of Benchohra and Ntouyas [10] and Benchohra, Gatsori and Ntouyas [8].

1. Introduction

The IVP with nonlocal conditions is of significance since they hace applications in many physical problems. Existence of mild, strong and classical solutions for differential and integro-differential equations in abstract spaces with nonlocal conditions have received much attention in recent years. We refer to the papers of Balachandran and Chandrasekharan ([1]-[3]), Balachandran and Ilamaran [4], Byszewski ([12],[13]), Dauer and Balachandran [17], Ntouyas and Tsamatos [26]. For the importance of nonlocal conditions in different fields we refer to [13] and the references cited there in. Several authors have studied controllability properties of different types of first order inclusion systems with nonlocal conditions.

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Also, controllability of the second-order systems with local and nonlocal conditions has received much attention in the recent years. It is advantageous to treat the second-order abstract differential equations directly rather than to convert them to first-order system, (refer, Fitzgibbon [20] and Ball [7]). Fitzgibbon [20] used the second-order abstract system for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful tool for the study of abstract second-order equations is the theory of strongly continuous cosine families of operators ([31],[32]). Quinn and Carmichael [29] have shown that the controllability problem in Banach spaces can be converted into a fixed point problem for a single-valued mapping. Balachandran, Park and Marshal Anthoni [6] discussed the controllability of second-order semilinear Volterra integro-differential systems in Banach spaces. Balachandran and Marshal Anthoni [5] studied the controllability of second-order semilinear neutral functional differential systems in Banach spaces by using Leray-Schauder alternative. Our aim in this paper is to obtain the sufficient conditions for the controllability of second-order initial value problems (IVP) for a class of damped differential inclusions with nonlocal conditions on noncompact intervals, whose existence was proved by Benchohra and Ntouyas [9]. The fundamental tools used in the proof of the above mentioned works are essentially fixed point arguments due to Ma [24], semigroups method [28] and the set-valued analysis ([18],[22]). This work extends the recent work of Benchohra and Ntouyas [10]in which authors considered the controllability of second order nonlinear inclusions with nonlocal conditions and the work of Benchohra, Gatsori and Ntouyas [8] in which authors have studied the nonlocal quasilinear damped differential inclusions.

Consider the inclusion for the damped second-order system of the form

$$\begin{aligned} & y''(t) - Ay(t) \in Gy'(t) + Bu(t) + F(t, y(t), y'(t)), \\ & y(0) + g(y) = \phi, \quad y'(0) = y_0, \end{aligned} \tag{1.1}$$

where, the state $y(t)$ takes values in a real Banach space X with the norm $\|.\|$ and the control $u(.)$ is given in $L^2(J, U)$, a Banach space of admissible control function with U as a Banach space. B is a bounded linear operator from U to X, $g : C(J, X) \to X, \phi : J \to X, y_0 \in X, J$ is an unbounded real interval. A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in a Banach space X. For the sack of simplicity we choose $J = [0, +\infty)$. here G is bounded linear operator on X and $F : J \times X \times X \to 2^X$ is a bounded, closed, convex multivalued map.

The method we are going to use is to reduce the controllability problem of (1.1) to the search for fixed points of a suitable multivalued map on the Frechet space $C(J, X)$. In order to prove the existence of fixed points, we shall rely on a theorem due to Ma [24], which is an extension to multivalued maps between locally convex topological spaces, of Schaefer's theorem [30].

The study of the dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force in a Hilbert space, can be modeled by the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial t^4} - \Big(\alpha + \beta \int_0^L \big|\frac{\partial u}{\partial t}(\xi, t)\big|^2 d\xi\Big) \frac{\partial^2 u}{\partial x^2} + g\Big(\frac{\partial u}{\partial t}\Big), \tag{1.2}$$

where $\alpha, \beta, L > 0 u(t, x)$ is the deflection of the point x of the beam at the time t, g is a nondecreasing numerical function, and L is the length of the beam.

Equation (1.2) has its analogue in R^n and can be included in a general mathematical model

$$u" + A^2 u + M\big(\|A^{\frac{1}{2}} u\|_H^2\big) Au + g(u') = 0, \tag{1.3}$$

where A is a linear operator in a Hilbert space H and M, $g$are real functions. Equation (1.2) was studied by Patcheu [27] and (1.3) was studied by Matos and Pereira [25].These equations are the special cases of the following second order damped nonlinear differential equation in an abstract space

$$u" + Au + Gu' = f(t, u, u'); \; u(0) = u_0, \; u'(0) = u_1,$$

where A, B are linear operators.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which will be used throughout this paper. Let J_b be the compact real interval $[0, b] (b \in N)$. Let $C(J, X)$ be the linear metric Frechet space of continuous functions from J into X with the metric (see [19])

$$d(y, z) = \sum_{b=0}^{\infty} \frac{2^{-b} \|y - z\|_b}{1 + \|y - z\|_b}, \quad \text{for each } \; y, z \in C(J, X).$$

where,

$$\|y\|_b := \sup\Big\{ \|y(t)\| : t \in J_b \Big\}.$$

Let $B(X)$ denote the Banach space of bounded linear operators X into X with standard norm.

A measurable function $y : J \to X$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable, refer [33]. Let $L^1(J, X)$ denotes the Banach space of continuous functions $y : J \to X$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{\infty} \|y(t)\| dt < \infty \qquad \text{for all} \quad y \in L^1(J, X).$$

Up denotes the neighbourhood of 0 in $C(J, X)$ defined by

$$Up := \Big\{ y \in C(J, X) : \|y\|_b < p \Big\}.$$

The convergence in $C(J, X)$ is the uniform convergence in the compact intervals, i.e., $y_j \to y$ in $C(J, X)$ if and only if each $b \in N, \|y_j - y\|_b \to 0$ in $C(J_b, X)$ as $j \to \infty. M \subseteq C(J, X)$ is a bounded set if and only if there exists a positive function $\xi \in C(J, R_+)$ such that

$$\|y(t)\| \leq \xi(t) \qquad \text{for all} \quad t \in J \quad \text{and all} \quad y \in M.$$

The Arzela-Ascoli theorem says that a set $M \subseteq C(J, X)$ is compact if and only if for each $b \in N$, M is a compact set in the Banach space $(C(J_b, X), \|.\|_b)$

We say that one-parameter family $\Big\{ C(t) : t \in R \Big\}$ of bounded linear operators in $B(X)$ is a strongly continuous cosine family if and only if

(1) $C(0) = I, I$ is the identify operator on X;
(2) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in R$;
(3) The map $t \mapsto C(t)y$ is strongly continuous in t on R for each fixed $y \in X$.

The strongly continuous sine family $\{S(t) : t \in R\}$, associated to the strongly continuous cosine family $\{C(t) : t \in R\}$ is defined by

$$S(t)y = \int_0^t C(s)y \, ds, \quad y \in X, \quad t \in R.$$

Assume the following condition on A.

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$ of bounded linear operators X into itself and the adjoint operator A^* is densely defined i.e., $\overline{D(A^*)} = X^*$ (refer, [11]).

The infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$ is the operator $A : X \to X$ defined by

$$Ay = \frac{d^2}{dt^2} C(t)y|_{t=0}, \qquad y \in D(A)$$

where, $D(A) = \{y \in X : C(t)y \text{ is twice continuously differentiable in } t\}$ i.e.,

$$D(A) = \{y \in X : C(.)y \in C^2(R, X)\}.$$

Define $X_1 = \{y \in X : C(t)y \text{ is once continuously differentiable in } t\} = \{y \in X : C(.)y \in C^1(R, X)$.

Lemma 2.1. ([31]) *Let* $(H1)$ *hold. Then*

(1) *there exists constant* $M_1 \geq 1$ *and* $w \geq 0$ *such that*

$$\|C(t)\| \leq M_1 e^{w|t|} \quad and$$

$$\|S(t) - S(t^*)\| \leq M_1 \| \int_0^{t^*} e^{w|s|} ds \|; \quad for \quad t, t^* \in R;$$

(2) $S(t)y \subset X_1$ *and* $S(t)X \subset D(A)$, *for* $t \in R$;

(3) $\frac{d}{dt}C(t)y = AS(t)y$, *for* $y \in X_1$ *and* $t \in R$; $C(t)y \in X_1, S(t)y \in D(A)$.

(4) $\frac{d^2}{dt^2}C(t)y = AC(t)y$, *for* $y \in D(A)$ *and* $t \in R$.

Lemma 2.2. ([31]) *Let (H1) hold and* $v : R \to X$ *such that* v *is continuously differentiable and* $q(t) = \int_0^t S(t-s)v(s)ds$ *then,*

$$q \in C^2(R, X) \;\; for \quad t \in R, q(t) \in D(A),$$

$$q'(t) = \int_0^t C(t-s)v(s)ds \quad and \quad q''(t) = Aq(t) + v(t).$$

For more details on strongly continuous cosine and sine family, WE refer to the book of Goldstein [21] and papers of Travis and Webb ([31], [32]). We now recall some preliminaries about multivalued maps.

Let $(X, \|.\|)$ be a Banach space. A multivalued map $G_1 : X \to 2^X$ is convex (closed) valued if $G_1(x)$ is convex (closed) for all $x \in X$. G_1 is bounded on bounded sets if $G_1(B) = U_{x \in B} G_1(x)$ is bounded in X for any bounded set B of X (i.e., $\sup_{x \in B}\{\sup\{\|y\| : y \in G_1(x)\}\} < \infty$).

The multimap G_1 is called upper semi continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G_1(x_0)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G_1(x_0)$, there exists an open neighbourhood A of x_0 such that $G_1(A) \subseteq B$.

The multimap G_1 is said to be completely continuous if $G_1(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G_1 is completely continuous with nonempty compact values, then G_1 is u.s.c. if and only if G_1 has a closed graph (i.e., $x_n \to x_0, y_n \to y_0, y_n \in G_1(x_n)$ imply $y_0 \in G_1(x_0)$).

G_1 has a fixed point if there is $x \in X$ such that $x \in G_1 x$.
In the following, $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G_1 : J \to BCC(X)$ is said to be measurable, if for each $x \in X$, the distance between x and $G_1(x)$ is a measurable function on J. i.e., for each $x \in X$, the function $Y : J \to R$ defined by

$$Y(t) = d(x, G_1(t)) = \inf\{|x - z| : x \in G_1(t)\} \in L^1(J, R)$$

is measurable. For more details on mutivalued map, see ([18],[22]).
We assume the following hypotheses:

(H2) $C(t), t > 0$ is compact.
(H3) $t \to B(u(t))$ is continuous in t.
(H4) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b S(t-s)Bu(s)ds$$

induces a bounded invertible operator $\widetilde{W}^{-1}$ which takes the values in $L^2(J_b, U)$ $\backslash \ker W$, (for construction of $\widetilde{W}^{-1}$, refer [5]), and there exist positive constants M_2 and M_3 such that $\|B\| \leq M_2$ and $\|\widetilde{W}^{-1}\| \leq M_3$.
(H5) $F : J \times X \times X \to BCC(X); (t, y, y') \mapsto F(t, y, y')$ is measurable with respect to t for each $y \in X$, u.s.c. with respect to y, for each $t \in J$ and for each fixed $y \in C(J, X)$and$z \in C(J_1, X)$ the set

$$S_{F,y,z} = \{v \in L^1(J, X) : v(t) \in F(t, y(t), y'(t)) \text{ for a.e. } t \in J\}$$

is nonempty.
(H6) There exists a constant L such that

$$\|g(y)\| \leq L, \qquad \text{for} \quad y \in X.$$

(H7) $\|F(t, y, z)\| = \sup\{\|v\| : v \in F(t, y, z)\} \leq p(t)\psi(\|y\|+\|y'\|)$ for almost all $t \in J$ and all $y \in X$, where $p \in L^1(J, R_+)$ and $\psi : R_+ \to (0, \infty)$ is continuous and increasing with

$$\int_0^b p(s)ds < \int_c^\infty \frac{ds}{s + \psi(s)} = \infty, \text{ for each } \quad b > 0$$

where,

$$\begin{aligned} c &= (M_1 + M_1^*)[\|\phi(0)\| + L] + \|G\|\|y(t)\| \\ &\quad + (1 + b)\Big[M_1\|y(0)\| + M_1 Nb + M_1\|G\|\|\phi(0)\|\Big], \\ M_1 &= \sup\Big\{\|C(t)\| : t \in J\Big\}, M_1^* = \sup\{\|AS(t)\| : t \in J\}, \\ N &= M_2 M_3\Big[\|y_1\| + (M_1 + M_1 b + \|G\|)\|\phi(0)\| + M_1 L + M_1 b\|y_0\| \\ &\quad + M_1\|G\| \int_0^t \|y(s)\|ds + M_1 b \int_0^b p(s)\psi(\|y(s)\| + \|y'(s)\|)ds\Big]. \end{aligned}$$

(H8) For fixed $u \in L^2(J, X)$, each neighbourhood Up of 0, $y \in Up$ and $t \in J$, the set

$$\begin{aligned} &\{C(t) - S(t)G\}\phi(0) - C(t)g(y) + S(t)y_0 + \int_0^t C(t-s)Gy(s)ds \\ &\quad + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)v(s)ds; v \in S_{F,y,y'}\} \end{aligned}$$

is relatively compact.

A mild solution of the system (1.1) is given by [26],

$$y(t) = \{C(t) - S(t)G\}\phi(0) - C(t)g(y) + S(t)y_0 + \int_0^t C(t-s)Gy(s)ds + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)v(s)ds \qquad (2.1)$$

where, $v \in S_{F,y,y'} = \{v \in L^1(J,X) : v(t) \in (F(t,y(t),y'(t))$ for a.e. $t \in J\}$.

Remark 2.3. (1) If dim $X < \infty$ and J is a compact real interval, then for each $y \in C(J,X), S_{F,y,y'} \neq \phi$ (see [23]).

(2) $S_{F,u,u'}$ is nonempty iff the function $y(.)$, and $v \in S_{F,y,y'}$, we define $y : J \to R$ by

$$Y(t) := \inf \left\{ \|v\| : v \in F(t,u,u') \right\} \in L^1(J,R)$$

(refer [22]).

Definition 2.4. System (1.1) is said to be infinite controllable on $J = [0,\infty)$ if for every $\phi(0) \in D(A), y_0 \in X_1$ and $y_1 \in X$, there exists a control $u \in L^2(J_b,U)$ such that the mild solution $y(.)$ of (1.1) satisfies $y(b) + g(y) = y_1$.

The following lemmas will be used in the proof of our main theorem.

Lemma 2.5. ([23]) *Let $I = J_b$ be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H5) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma o S_F : C(I,X) \to BCC(C(I,X)) \qquad y \to (\Gamma o S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.6. ([24]) *Let X be a locally convex space and $N_1 : X \to 2^X$ be a compact convex valued, u.s.c. multivalued map such that there exists a closed neighbourhood Up of 0 for which $N_1(Up)$is a relatively compact set for each $p \in N$. If the set*

$$\Omega := \left\{ y \in X : \lambda y \in N_1(y) \ \text{for some} \ \ \lambda > 1 \right\}$$

is bounded , then N_1 has a fixed point.

3. Main result

We now state and prove our main controllability result.

Theorem 3.1. *Assume that the hypotheses* $(H_1)-(H_8)$ *are satisfied . Let* $g : C(J,X) \to X$ *be a continuous function. Then the system (1.1) is controllable on* J.

Proof. For fixed $b \in N$, consider the space $Z = C^1(J,X)$ with norm

$$\|y\|_z = \max\{\|y\|, \|y'\|, t \in J\}.$$

Using the hypothesis (H5) for an arbitrary function $y(.)$, we define the control

$$u_y^b(t) = \widetilde{W}^{-1}\Bigg[y_1 - \{C(b) - S(b)G\}\phi(0) + C(b)g(y) - S(b)y_0 \\ - \int_0^b C(b-s)Gy(s)ds - \int_0^b S(b-s)v(s)ds\Bigg](t).$$

Using this control we shall show that the operator $N_1 : Z \to 2^Z$ defined by

$$N_1 y := \{h \in C(J,X) : h(t) \\ = \begin{cases} \{C(t) - S(t)G\}\phi(0) - C(t)g(y) + S(t)y_0 \\ + \int_0^t C(t-s)Gy(s)ds + \int_0^t S(t-\eta)B\widetilde{W}^{-1} \\ \Big[y_1 - \{C(b) - S(b)G\}\phi(0) + C(b)g(y) - S(b)y_0 \\ - \int_0^b C(b-s)Gy(s)ds - \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta \\ + \int_0^t S(t-s)v(s)ds : v \in S_{F,y,y'}; t \in J \end{cases}$$

where,

$$v \in S_{F,y,y'} = \Big\{v \in L^1(J,X) : v(t) \in F(t, y(t), y'(t)) \text{for a.e.} t \in J\Big\},$$

has a fixed point. This fixed point is then a solution of equation (2.1).

Clearly $y_1 - g(y) \in (N_1 y)(b)$, which means that the control u steers the system from initial state $\phi(0)$ to y_1 in time b, provided we obtain a fixed point of the nonlinear operator N_1.

In order to study the controllability problem for system (1.1), we apply fixed point theorem due to Ma [24] to the following system:

$$y''(t) \in \lambda^{-1}Ay(t) + \lambda^{-1}Bu(t) + \lambda^{-1}Gy'(t)\lambda^{-1}F(t, y(t), y'(t)), \quad (3.1) \\ y(0) + g(y) = \phi, \quad y'(0) = y_0, \;\; t \in J.$$

Let y be a mild solution of system (3.1). Then for some $\lambda \in (0,1)$,

$$\begin{aligned} y(t) &= \lambda^{-1}\{C(t)-S(t)G\}\phi(0)-\lambda^{-1}C(t)g(y)+\lambda^{-1}S(t)y_0 \\ &+\lambda^{-1}\int_0^t C(t-s)Gy(s)ds+\lambda^{-1}\int_0^t S(t-\eta)B\widetilde{W}^{-1} \\ &\Big[y_1-\{C(b)-S(b)G\}\phi(0)+C(b)g(y)-S(b)y_0- \\ &\int_0^b C(b-s)Gy(s)ds-\int_0^b S(b-s)v(s)ds\Big](\eta)d\eta \\ &+\lambda^{-1}\int_0^t S(t-s)v(s)ds, \quad v\in S_{F,y,y'};\ \ t\in J. \end{aligned} \tag{3.2}$$

We shall show that $N_1(U_q)$ is relatively compact for each neighbourhood U_q of $0 \in C(J,X)$ with $q \in N$ and the multivalued map N_1 has bounded, closed and convex values and it is u.s.c. The proof will be given in several steps.

Step 1. The set

$$\Omega := \{y \in C(J,X) : \lambda y \in N_1(y), \lambda > 1\}$$

is bounded.
For that we obtain a priori bounds for the equation (3.2).
we have

$$\begin{aligned} \|y(t)\|_b &\leq (M_1+M_1b\|G\|)\|\phi(0)\|+M_1L+M_1b\|y_0\|+M_1\|G\|\int_0^t\|y(s)\|ds \\ &+\int_0^t\|S(t-\eta)\|M_2M_3\Big[\|y_1\|+(M_1+M_1b\|G\|)\|\phi(0)\| \\ &+M_1L++M_1b\|y_0\|+M_1\|G\|\int_0^b\|y(s)\|ds \\ &+M_1b\int_0^b p(s)\psi(\|y(s)\|+\|y'(s)\|)ds\Big]d\eta \\ &+M_1b\int_0^t p(s)\psi(\|y(s)\|+\|y'(s)\|)ds \\ &\leq (M_1+M_1b\|G\|)\|\phi(0)\|+M_1L+M_1b\|y_0\| \\ &+M_1\|G\|\int_0^t\|y(s)\|ds+M_1Nb^2 \\ &+M_1b\int_0^t p(s)\psi(\|y(s)\|+\|y'(s)\|)ds. \end{aligned}$$

Denote by $r_1(t)$ the R.H.S. of the above inequality, we have

$$r_1(0) = (M_1 + M_1 b\|G\|)\|\phi(0)\| + M_1 L + M_1 b\|y_0\| + M_1 N b^2 \text{ and}$$
$$\|y(t)\| \leq r_1(t), t \in J_b.$$

Using the increasing charactor of ψ , we get

$$r_1'(t) \leq M_1\|G\|\|y(t)\| + M_1 b p(t)\psi(\|y(t)\| + \|y'(t)\|).$$

But

$$\begin{aligned} y'(t) &= \lambda\Big[(AS(t) - C(t)G)\phi(0) - AS(t)g(y) + C(t)y_0\Big] \\ &\quad +Gy(t) + \int_0^t AS(t-s)Gy(s)ds \end{aligned}$$

$$\begin{aligned} &+\lambda\int_0^t C(t-\eta)B\widetilde{W}^{-1}\Big[y_1 - \{C(b) - S(b)G\}\phi(0) + C(b)g(y) \\ &-S(b)y_0 - \int_0^b C(b-s)Gy(s)ds - \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta \\ &+\lambda\int_0^t C(t-s)v(s)ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \|y'(t)\|_b \leq\; & (M_1^* + M_1\|G\|)\|\phi(0)\| + M_1^* L + M_1\|y_0\| + \|G\|\|y(t)\| \\ & +M_1^*\|G\|\int_0^t \|y(s)\|ds + M_1M_2M_3 b\Big[\|y_1\| + (M_1 \\ & +M_1 b\|G\|)\|\phi(0)\| + M_1 L + M_1 b\|y_0\| + M_1\|G\|\int_0^b \|y(s)\|ds \\ & +M_1 b\int_0^b p(s)\psi(\|y(s)\| + \|y'(s)\|)ds\Big] \\ & +M_1 b\int_0^t p(s)\psi(\|y(s)\| + \|y'(s)\|)ds \\ \leq\; & (M_1^* + M_1 b\|G\|)\|\phi(0)\| + M_1^* L + M_1\|y_0\| \\ & +\|G\|\|y(t)\| + M_1^*\|G\|\int_0^t \|y(s)\|ds + M_1 N b \\ & +M_1 b\int_0^t p(s)\psi(\|y(s)\| + \|y'(s)\|)ds. \end{aligned}$$

Denote by $r_2(t)$ the R.H.S. of the above inequality, we have

$r_2'(t) \leq \|G\|\|y'(t)\| + M_1^*\|G\|\|y(t)\| + M_1p(t)\psi(\|y(t)\| + \|y'(t)\|)$ and $\|y'(t)\| \leq r_2(t);\ t \in J.$

Let $w(t) = r_1(t) + r_2(t)$, then

$$\begin{aligned} c = w(0) &= r_1(0) + r_2(0) \\ &= (M_1 + M_1^*)[\|\phi(0)\| + L] + \|G\|\|y(t)\| \\ &\quad +(1+b)\Big[M_1\|y(0)\| + M_1Nb + M_1\|G\|\|\phi(0)\|\Big]. \end{aligned}$$

Also,

$$\begin{aligned} w'(t) &= r_1'(t) + r_2'(t) \\ &\leq \Big[(M_1 + M_1^*)\|y(t)\| + \|y'(t)\|\Big]\|G\| \\ &\quad +(1+b)[M_1p(t)\psi(\|y(t)\| + \|y'(t)\|) \\ &\leq (M_1 + M_1^*)\|G\|r_1(t) + \|G\|r_2(t) + (1+b)[M_1p(t)\psi(r_1(t) + r_2(t)) \\ &= \widehat{p}(t)\Big(w(t) + \psi(w(t))\Big), t \in J, \text{ where } \widehat{p}(t) \\ &= max\{(1+b)M_1p(t), (M_1 + M_1^*)\|G\|, \|G\|\}. \end{aligned}$$

This implies that for each $t \in J_b$

$$\int_c^{w(t)} \frac{ds}{s + \psi s} \leq \int_0^b \widehat{p}(s)ds < \int_c^\infty \frac{ds}{s + \psi s} = \infty.$$

This inequality implies that there exists a constant K such that

$$r_1(t) + r_2(t) = w(t) \leq K,\ t \in J_b.$$

Then

$$\|y\|_z = max\{\|y(t)\|, \|y'(t)\|)\} \leq K$$

where, K depends only on b and on the functions p and ψ. This shows that Ω is bounded.

Step 2. N_1y is convex for each $y \in C(J, X)$.

Indeed if $h_1, h_2 \in N_1y$ then there exist $v_1, v_2 \in S_{F,y,y'}$ such that for each $t \in J$ we have

$$\begin{aligned} h_1(t) &= \{C(t) - S(t)G\}\phi(0) - C(t)g(y) + S(t)y_0 + \int_0^t C(t-s)Gy(s)ds \\ &\quad + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)v_1(s)ds \end{aligned}$$

and

$$\begin{aligned} h_2(t) &= \{C(t)-S(t)G\}\phi(0)-C(t)g(y)+S(t)y_0+\int_0^t C(t-s)Gy(s)ds \\ &\quad +\int_0^t S(t-s)Bu(s)ds+\int_0^t S(t-s)v_2(s)ds. \end{aligned}$$

Let $0 \leq \alpha \leq 1$. Then for each $t \in J$ we have

$$\begin{aligned} & (\alpha h_1+(1-\alpha)h_2)(t) \\ &= \{C(t)-S(t)G\}\phi(0)-C(t)g(y)+\int_0^t C(t-s)Gy(s)ds \\ & +S(t)y_0+\int_0^t S(t-s)[\alpha v_1(s)+(1-\alpha)v_2(s)]ds \\ & +\int_0^t S(t-\eta)B\widetilde{W}^{-1}\Big[y_1-\{C(b)-S(b)G\}\phi(0)-C(b)g(y) \\ & +S(b)y_0+\int_0^t C(t-s)Gy(s)ds \\ & -\int_0^b S(b-s)[\alpha v_1(s)+(1-\alpha)v_2(s)ds\Big](\eta)d\eta. \end{aligned}$$

Since $S_{F,y,y'}$ is convex as F is convex, then $v=\alpha h_1+(1-\alpha)h_2 \in S_{F,y,y'}$ and hence

$$\alpha h_1+(1-\alpha)h_2 \in N_1 y.$$

Step 3. $N_1(U_q)$ is bounded in $C(J,X)$ for each $q \in N$. Indeed, it is enough to show that there exists a positive constant l_1 such that for each $h \in N_1 y$, $y \in U_q = \Big\{y \in Z : \|y\|_\infty \leq q\Big\}$, one has $\|h\|_\infty \leq l_1$. In other words, we have to bound the sup-norm of both h and h'. If $h \in N_1 y$, then there exists $v \in S_{F,y,y'}$ such that for each $t \in J_b$ we have

$$\begin{aligned} h(t) = &\{C(t)-S(t)G\}\phi(0)-C(t)g(y)+S(t)y_0+\int_0^t C(t-s)Gy(s)ds \\ &+\int_0^t S(t-s)v(s)ds+\int_0^t S(t-\eta)B\widetilde{W}^{-1}\Big[y_1-\{C(b)-S(b)G\}\phi(0) \\ &+C(b)g(y)-S(b)y_0-\int_0^b C(b-s)Gy(s)ds-\int_0^b S(b-s)v(s)ds\Big](\eta)d\eta. \end{aligned}$$

By (H4), (H6)and (H7), we have for each $t \in J_b$ that

$$
\begin{aligned}
&\|h(t)\|_b \\
&\le \Big(\|C(t)\| + \|S(t)\|\Big)\|\phi(0)\| + \|C(t)\|\|g(y)\| + \|S(t)\|\|y_0\| \\
&\quad + \int_0^t \|C(t-s)Gy(s)\|ds + \int_0^t \|S(t-s)v(s)\|ds + \int_0^t \|S(t-\eta)\|\|B\|\|\widetilde{W}^{-1}\| \\
&\quad \Big[\|y_1\| + \Big(\|C(b)\| + \|S(b)\|\Big)\|\phi(0)\| + \|C(b)\|\|g(y)\| + \|S(b)\|\|y_0\| \\
&\quad + \int_0^b \|C(b-s)Gy(s)\|ds + \int_0^b \|S(b-s)v(s)\|ds\Big] d\eta \\
&\le (M_1 + M_1 bL)\|\phi(0)\| + M_1 L + M_1 b\|y_0\| + M_1\|G\| \int_0^t \|y(s)\|ds \\
&\quad + M_1 b \sup_{t\in[0,q]}\psi\Big(\|y(t)\| + \|y'(t)\|\Big)\left(\int_0^t p(s)ds\right) + \int_0^t \|S(t-\eta)\| M_2 M_3 \\
&\quad \Big[\|y_1\| + (M_1 + M_1 bL)\|\phi(0)\| + M_1 L + M_1 b\|y_0\| + M_1\|G\| \int_0^b \|y(s)\|ds \\
&\quad + M_1 b \sup_{t\in[0,q]}\psi\Big(\|y(t)\| + \|y'(t)\|\Big)\left(\int_0^b p(s)ds\right)\Big] d\eta.
\end{aligned}
$$

Then for each $h \in N_1(U_q)$ we have $\|h\|_\infty \le l_1$. Also,

$$
\begin{aligned}
\|h'(t)\|_b \le{}& (M_1^* + M_1 L)\|\phi(0)\| + M_1^*\|g(y)\| + M_1\|y_0\| + \|G\|\|y(t)\| \\
&+ M_1^* b\|G\| \int_0^t \|y(s)\|ds + M_1 M_2 M_3 b\Big[\|y_1\| + (M_1 + M_1 bL)\|\phi(0)\| \\
&+ M_1\|g(y)\| + M_1 b\|y_0\| + M_1\|G\| \int_0^b \|y(s)\|ds \\
&+ M_1 b \sup_{t\in[0,q]}\psi\Big(\|y(t)\| + \|y'(t)\|\Big)\left(\int_0^b p(s)ds\right)\Big] \\
&+ M_1 \sup_{t\in[0,q]}\psi\Big(\|y(t)\| + \|y'(t)\|\Big)\left(\int_0^t p(s)ds\right).
\end{aligned}
$$

Then for each $h \in N_1(U_q)$ we have $\|h'\|_\infty \le l_2$.

Step 4. $N_1(U_q)$ is equi-continuous sets of $U_q \in Z$ for each $q \in N$. That is the family $h \in N_1 y : y \in U_q$ is equi-continuous.

Let $t_1, t_2 \in J_b, 0 < t_1 < t_2 < b$ and U_q be a neighborhood of 0 in Z for $q \in N$. For each $y \in U_q$ and $h \in N_1 y$, we have

$$h(t) = \{C(t) - S(t)G\}\phi(0) - C(t)g(y) + S(t)y_0 + \int_0^t C(t-s)Gy(s)ds$$
$$+ \int_0^t S(t-s)v(s)ds + \int_0^t S(t-\eta)B\widetilde{W}^{-1}\Big[y_1 - \{C(b) - S(b)G\}\phi(0)$$
$$+C(b)g(y) - S(b)y_0 - \int_0^b C(b-s)Gy(s)ds - \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta.$$

Thus

$$\begin{aligned}
& \|h(t_1) - h(t_2)\| \\
\leq\; & \|C(t_1)\phi(0) - C(t_2)\phi(0)\| + \|S(t_1)G\phi(0) - S(t_2)G\phi(0)\| \\
& + \|C(t_1) - C(t_2)\|\|G\| + \|S(t_1)y_0 - S(t_2)y_0\| \\
& + \|\int_0^{t_1} [C(t_1 - s) - C(t_2 - s)]Gy(s)ds\| + \|\int_{t_1}^{t_2} C(t_2 - s)Gy(s)ds\| \\
& + \|\int_0^{t_1} [S(t_1 - s) - S(t_2 - s)]v(s)ds\| + \|\int_{t_1}^{t_2} S(t_2 - s)v(s)ds\| \\
& + \|\int_0^{t_1} [S(t_1 - \eta) - S(t_2 - \eta)]B\widetilde{W}^{-1}\Big[y_1 - g(y) - C(b)[\phi(0) - g(y)] \\
& + S(b)G\phi(0) - S(b)y_0 - \int_0^b C(b-s)Gy(s)ds + \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta\| \\
& + \|\int_{t_1}^{t_2} S(t_2 - \eta)B\widetilde{W}^{-1}\Big[y_1 - C(b)[\phi(0) - g(y)] + S(b)G\phi(0) - S(b)y_0 \\
& - \int_0^b C(b-s)Gy(s)ds + \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta\| \qquad (3.3) \\
\leq\; & \|C(t_1)\phi(0) - C(t_2)\phi(0)\| + \|C(t_1) - C(t_2)\|\|G\| + \|S(t_1)y_0 - S(t_2)y_0\| \\
& + M_1(t_1 - t_2)\int_0^b \|G\|\|y(s)\| + M_1(t_1 - t_2)\int_0^b \|v(s)\|ds \\
& + M_1 b\int_{t_1}^{t_2} \|v(s)\|ds + M_1(t_1 - t_2)\int_0^b M_2M_3\Big[\|y_1\| + M_1\big[\|\phi(0)\| + \|G\| \\
& + b\|y_0\| + b\|G\|\|\phi(0)\|\big] + M_1\int_0^b \|G\|\|y(s)\|ds + M_1 b\int_0^b \|v(s)\|ds\Big]d\eta \\
& + M_1 b\int_{t_1}^{t_2} M_2M_3[\|y_1\| + M_1[\|\phi(0)\| + \|G\| + b\|y_0\| + b\|G\|\|\phi(0)\|]
\end{aligned}$$

$$+M_1\int_0^b \|G\|\|y(s)\|ds + M_1 b\int_0^b \|v(s)\|ds]d\eta.$$

By using (H6), (H7) and continuity of $C(t)$ and $S(t)$, we see that the right-hand side of the above inequality tends to zero as $(t_2 - t_1) \longrightarrow 0$. The compactness of $C(t), S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$.

In an analogous way one can obtain a similar estimate for $\|h'(t_1) - h'(t_2)\|$. Thus $N_1(Uq)$ maps Uq into an equi-continuous family of functions. It is easy to see that the family $N_1(Uq)$ is uniformly bounded. The above estimate implies the required eqi-continuity. This also proves the relative compactness of $N_1(U_q)$. Now it remains to prove the upper-semicontinuity (u.s.c.)of N_1. By our discussion in Section 1, it is enough to prove that N_1 has a closed graph.

We do this in the next step using Lemma 2.5.

Step 5. N_1 has a closed graph.

Let $y_n \longrightarrow y^*, h_n \longrightarrow h^*$ and $h_n \in N_1(y_n)$. We shall prove that $h^* \in N_1 y^*$. Since $h_n \in N_1(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) &= C(t)[\phi(0) - g(y_n)] + S(t)y_0 + \int_0^t S(t-s)v_n(s)ds \\ &+ \int_0^t S(t-\eta)BW^{-1}\Big[y_1 - g(y) - C(b)\big[\phi(0) - g(y_n)\big] - S(b)y_0 \\ &- \int_0^b S(b-s)v_n(s)ds\Big](\eta)d\eta. \end{aligned}$$

We must prove that there exists $v^* \in S_{F,y^*}$ such that

$$\begin{aligned} h^*(t) &= C(t)\big[\phi(0) - g(y^*)\big] + S(t)y_0 + \int_0^t S(t-s)v^*(s)ds \\ &\quad + \int_0^t S(t-\eta)BW^{-1}\Big[y_1 - g(y) - C(b)[\phi(0) - g(y^*)] \\ &\quad - S(b)y_0 - \int_0^b S(b-s)v^*(s)ds\Big](\eta)d\eta \end{aligned} \tag{3.4}$$

Then the idea is to use the facts

(1) $h_n \longrightarrow h^*$;

(2) $h_n - C(t)\big[\phi(0) - g(y_n)\big] - S(t)y_0 \in \Gamma(S_{F,y_n})$, where
$\Gamma : L^1(J, X) \longrightarrow C(J, X)$, defined by

$$\begin{aligned}(\Gamma v)(t) = &\int_0^t S(t-s)v(s)ds + \int_0^t S(t-\eta)BW^{-1}\\ &\Big[y_1 - g(y) - C(b)\big[\phi(0) - g(y)\big] - S(b)y_0\\ &- \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta.\end{aligned}$$

If $\Gamma o S_F$ is a closed graph operator, we would be done. But we do not know whether $\Gamma o S_F$ is a closed graph operator. So we cut the functions $y_n, h_n - C(t)\phi(0) + C(t)g(y_n) - S(t)y_0, g_n$ and we consider them defined on the interval $[k, k+1]$ for any $k \in N \cup 0$. Then, using Lemma 2.5, in this case we are able to affirm that (3.4) is true on the compact interval $[k, k+1]$, i.e.,

$$\begin{aligned}h^*(t)|_{[k,k+1]} = &C(t)[\phi(0) - g(y^*)] + S(t)y_0 + \int_0^t S(t-s)v^{*^k}(s)ds\\ &+ \int_0^t S(t-\eta)BW^{-1}\Big[y_1 - g(y) - C(b)[\phi(0) - g(y^*)] - S(b)y_0\\ &- \int_0^b S(b-s)v^{*^k}(s)ds\Big](\eta)d\eta\end{aligned}$$

for a suitable L^1- selection v^{*^k} of $F(t, y^*(t))$ on the interval $[k, k+1]$.
At this point we can past the functions v^{*^k} obtaining the selection v^* defined by

$$v^*(t) = v^{*^k}(t), \qquad \text{for} \quad t \in [k, k+1).$$

We obtain then that v^* is an L^1- selection and (3.4) will be satisfied. We give now the details. Clearly we have that

$$\Big\| \Big(h_n - C(t)\phi(0) + C(t)g(y_n) - S(t)y_0\Big) - \Big(h^* - C(t)\Phi(0) + C(t)g(y^*) - S(t)y_0\Big) \Big\|_\infty$$

$\longrightarrow 0$ as $n \longrightarrow \infty$.

Now, we consider for all $k \in N \cup 0$, the mapping

$$S_F^k : C([k, k+1]; X) \longrightarrow L^1([k, k+1]; X)$$

$$u \mapsto S_{F,u}^k := \Big\{ f \in L^1\big([k, k+1]; X\big) : f(t) \in F(t, u(t)) \text{ for a.e. } t \in [k, k+1] \Big\}.$$

Also, we consider the linear continuous operators

$$\Gamma_k : L^1([k, k+1], X) \longrightarrow C([k, k+1]; X)$$

$$v \mapsto \Gamma_k(v)(t) \;=\; \int_0^t S(t-s)v(s)ds + \int_0^t S(t-\eta)BW^{-1} \Big[y_1 - g(y) - C(b)[\phi(0) - g(y)] - S(b)y_0 - \int_0^b S(b-s)v(s)ds\Big](\eta)d\eta.$$

From Lemma 2.5, it follows that $\Gamma_k o S_F^k$ is a closed graph operator for all $k \in N$. Moreover, we have that

$$\Big(h_n(t) - C(t)\phi(0) + C(t)g(y_n) - S(t)y_0\Big)|_{[k,k+1]} \in \Gamma_k(S^k_{F,y_n}).$$

Since $y_n \longrightarrow y^*$, it follows from Lemma 2.5 that

$$\Big(h^*(t) - C(t)\phi(0) + C(t)g(y^*) - S(t)y_0\Big)|_{[k,k+1]}$$
$$= \int_0^t S(t-s)v^{*^k}(s)ds + \int_0^t S(t-\eta)BW^{-1}\Big[y_1 - g(y) - C(b)[\phi(0) - g(y^*)] - S(b)y_0 - \int_0^b S(b-s)v^{*^k}(s)ds\Big](\eta)d\eta$$

for some $v^{*^k} \in S^k_{F,y^*}$. So the function v^* defined on J by

$$v^*(t) = v^{*^k}(t), \qquad \text{for} \quad t \in [k, k+1)$$

is in S_{F,y^*}, since $v^*(t) \in F(t, y^*(t))$ for a.e. $t \in J$.

Set $X := C(J,X)$. As a consequence of Lemma 2.6, we deduce that N_1 has a fixed point (in Z). This means that any fixed point of N_1 is a mild solution of (1.1) on J satisfying $(N_1y)(t) = y(t)$. Thus, system (1.1) is controllable on J. □

4. Example

Consider the following second-order partial differential inclusion:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}(x,t)\right) &\in z_{xx}(x,t) + \sigma(t, z(x,t)) + \mu(x,t) \\ z(0,t) + g(x,t) &= z(\pi,t) = 0 \quad \text{for} \quad t \geq 0 \\ \frac{\partial z}{\partial t}(x,0) &= z_0(x), \quad t \in J = [0,\infty) \qquad \text{for} \quad 0 < x < \pi \end{aligned}\right\} \tag{4.1}$$

Here $\sigma : J \times (0,\pi) \longrightarrow 2^{(0,\pi)}$ is strongly measurable and u.s.c.and $\mu : (0,\pi) \times J \longrightarrow (0,\pi)$ is continuous in t

Let $X = L^2[0,\pi]$ and $A : X \longrightarrow X$ be defined by

$$Aw = w", \qquad w \in D(A)$$

where $D(A) = \Big\{ w \in X : w, w'$are absolutely continuous, $w" \in X,\ w(0) = w(\pi) = 0 \Big\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2 (w, w_n) w_n, \quad w \in D(A)$$

where, $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, 3, \cdots$ is the orthogonal set of eigen functions of A.

It can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R$, in X and is given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt (w, w_n) w_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt (w, w_n) w_n, \quad w \in X.$$

Let $F : J \times X \longrightarrow X$ be defined by

$$F(t, y(t)) = \sigma(t, z(y, t)), w \in X, y \in (0, \pi).$$

Further, the function σ satisfies the following growth condition.
There exists a continuous function $p : J \longrightarrow [0, \infty)$ such that

$$\|\sigma(t, y)\| \leq p(t)\psi(\|y(t)\|, \qquad t \in J, w \in X;$$

where, $\psi : [0, \infty) \longrightarrow (0, \infty)$ is a continuous nondecreasing function such that

$$bM_1 \int_0^b p(s) ds < \int_c^{\infty} \frac{ds}{\psi(s)}, \quad \text{for each} \quad b > 0$$

and c is a known constant . Let $B : U \subset J \longrightarrow X$ be defined by

$$(B(u(t))(y) = \mu(y, t), \quad y \in (0, \pi)$$

such that it satisfies the condition $(H4)$. Also, $g : C(J, X) \longrightarrow X, J$ an unbounded real interval. Thus, all the conditons of Theorem (3.1) are satisfied. Hence, system (4.1) is controllable on J.

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