Nonlinear Functional Analysis and Applications
Vol. 14, No. 1 (2009), pp. 45–55

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A COMMON FIXED POINT THEOREM OF COMPATIBLE OF TYPE (γ) MAPS IN COMPLETE FUZZY METRIC SPACES

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Abstract. In this paper, we establish a common fixed point theorem of compatible of type γ maps in complete fuzzy metric spaces.

1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [17] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5] and Kramosil and Michalek [8] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [1, 2, 3, 4, 16]. Many authors [6, 11, 12] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

[0]Received January 4, 2007. Revised June 5, 2008.
[0]2000 Mathematics Subject Classification: 47H10; 54H25.
[0]Keywords: Compatible of type (γ) maps, complete fuzzy metric space, implicit relation, common fixed point theorem.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a \cdot b$ for all $a, b \in [0,1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Lemma 1.4. ([5]) *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 1.5. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t).$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$ i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.6. *Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.*

Proof. see proposition 1 of [10]. □

Definition 1.7. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X.$$

Definition 1.9. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible of type(γ) if $Ax_n = Sx_n = x$ implies that $Ax = Sx$, for $x \in X$.

Proposition 1.10. ([13])*Let self-mappings A and S of a fuzzy metric space $(X, M, *)$ be compatible. Then they are weak compatible.*

Throughout this section, a binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it satisfies the condition $t * s \geq ts$.

Three examples of such a continuous t-norm are

$$a * b = ab, \; a * b = \min(a, b)$$

and

$$a * b = \frac{ab}{\max\{a, b, \alpha\}}$$

for all $a, b \in [0, 1]$, where $\alpha \in (0, 1]$.

Lemma 1.11. *Let (X,M,*) be a fuzzy metric space. If sequence $\{x_n\}$ in X exist such that for every $n \in \mathbb{N}$.*

$$M(x_n, x_{n+1}, t) \geq 1 - k^n \alpha$$

for every $0 < k, \alpha < 1$, then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $m > n$ and $x_n, x_m \in X$, we have

$$\begin{aligned} M(x_n, x_m, t) &\geq M(x_n, x_{n+1}, \frac{t}{m-n}) * \cdots * M(x_{m-1}, x_m, \frac{t}{m-n}) \\ &\geq M(x_n, x_{n+1}, \frac{t}{m-n}). \cdots .M(x_{m-1}, x_m, \frac{t}{m-n}) \\ &\geq (1 - k^n \alpha).(1 - k^{n+1}\alpha). \cdots .(1 - k^{m-1}\alpha) \\ &\geq (1 - k^n \alpha)^{m-n} \\ &\geq 1 - (m-n)k^m \alpha \\ &> 1 - \epsilon. \end{aligned}$$

The last inequality indeed by inequality Bernoli, and for every $\epsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that for every $m > n \geq n_0$ we get $(m - n)k^m \alpha < \epsilon$. Hence sequence $\{x_n\}$ is Cauchy sequence. □

2. The main results

A class of implicit relation. Let Φ be the set of all continuous functions $\phi : [0, 1]^5 \longrightarrow [0, 1]$, increasing in any coordinate and $\phi(s, s, s, s^n, s^m) > s$ for every $s \in [0, 1)$ and $n, m \in \{0, 1, 2\}$ such that $n + m = 2$.

Example 2.1. Let $\phi : [0, 1]^5 \longrightarrow [0, 1]$ is define by

(i) $\phi_1(x_1, x_2, x_3, x_4, x_5) = \left(\min\left\{x_1, x_2, x_3, (x_4 x_5)^{1/2}\right\}\right)^h$ for some $0 < h < 1$.

(ii) $\phi_2(x_1, x_2, x_3, x_4, x_5) = (a(t)x_1 + b(t)x_2 + c(t)x_3 + d(t)(x_4 x_5)^{1/2})^h$ where $a, b, c, d : \mathbb{R}^+ \longrightarrow [0, 1]$, be four mappings such that $a(t)+b(t)+c(t)+d(t) = 1$, for all $t > 0$ and some $0 < h < 1$.

(iii) $\phi_3(x_1, x_2, x_3, x_4, x_5) = a(t)x_1 + b(t)x_2 + c(t)x_3 + d(t)x_4^{1/2} + e(t)x_5^{1/2}$ where $a, b, c, d, e : \mathbb{R}^+ \longrightarrow [0, 1]$, be five mappings such that $a(t) + b(t) + c(t) + d(t) + e(t) = 1$, for all $t > 0$.

Theorem 2.2. *Let f, g, S, T be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying that*

(i) $f(X) \subseteq T(X),\ g(X) \subseteq S(X)$,

(ii) *there exists a number $k \in (0, 1)$ such that*

$$\begin{aligned} &M(fx, gy, t) \\ &\geq 1 - k(1 - \phi \begin{pmatrix} M(Sx, Ty, t), & M(Sx, fx, t), \\ M(Ty, gy, t), & M(Sx, gy, qt), \\ M(Ty, fx, (3-q)t) & \end{pmatrix}) \end{aligned}$$

for every x, y in X, $q \in \{1, 2\}$, $t > 0$ and $\phi \in \Phi$,

(iii) *the pairs (f, S) and (g, T) are be weak compatible of type (γ).*

Then f, g, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point as $f(X) \subseteq T(X), g(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $fx_0 = Tx_1,\ gx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1},\ y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \cdots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $d_m(t) = M(y_m, y_{m+1}, t), t > 0$ we prove $\{d_m(t)\}$ is increasing w.r.t m. For $m = 2n$ and $q = 2$, we have

$$\begin{aligned}
& d_{2n}(t) \\
= \;& M(y_{2n}, y_{2n+1}, t) = M(fx_{2n}, gx_{2n+1}, t) \\
\geq \;& 1 - k(1 - \phi\begin{pmatrix} M(Sx_{2n}, Tx_{2n+1}, t), & M(Sx_{2n}, fx_{2n}, t), \\ M(Tx_{2n+1}, gx_{2n+1}, t), & M(Sx_{2n}, gx_{2n+1}, 2t), \\ M(Tx_{2n+1}, fx_{2n}, t) & \end{pmatrix}) \\
= \;& 1 - k(1 - \phi\begin{pmatrix} M(y_{2n-1}, y_{2n}, t), & M(y_{2n-1}, y_{2n}, t), \\ M(y_{2n}, y_{2n+1}, t), & M(y_{2n-1}, y_{2n+1}, 2t), \\ M(y_{2n}, y_{2n}, t) & \end{pmatrix}) \\
\geq \;& 1 - k(1 - \phi\begin{pmatrix} M(y_{2n-1}, y_{2n}, t), & M(y_{2n-1}, y_{2n}, t), \\ M(y_{2n}, y_{2n+1}, t), & M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t), \\ M(y_{2n}, y_{2n}, t) & \end{pmatrix}) \\
\geq \;& 1 - k(1 - \phi\begin{pmatrix} M(y_{2n-1}, y_{2n}, t), & M(y_{2n-1}, y_{2n}, t), \\ M(y_{2n}, y_{2n+1}, t), & M(y_{2n-1}, y_{2n}, t).M(y_{2n}, y_{2n+1}, t), \\ M(y_{2n}, y_{2n}, t) & \end{pmatrix}) \\
= \;& 1 - k(1 - \phi(d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), d_{2n-1}(t).d_{2n}(t), 1)).
\end{aligned}$$

Hence

$$d_{2n}(t) \geq 1 - k(1 - \phi(d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), d_{2n-1}(t).d_{2n}(t), 1)). \quad (2.1)$$

We claim that for every $n \in \mathbb{N}$, $d_{2n}(t) \geq d_{2n-1}(t)$. For if $d_{2n}(t) < d_{2n-1}(t)$ for some $n \in \mathbb{N}$, then since $d_{2n}(t).d_{2n-1}(t) \geq d_{2n}(t).d_{2n}(t)$ in inequality (2.1), we have

$$d_{2n}(t) \geq 1 - k(1 - \phi(d_{2n}(t), d_{2n}(t), d_{2n}(t), (d_{2n}(t))^2, 1)) > 1 - k(1 - d_{2n}(t)).$$

That is, $(1 - k)d_{2n}(t) > 1 - k$, a contradiction. Hence $d_{2n}(t) \geq d_{2n-1}(t)$ for every $n \in \mathbb{N}$ and $\forall t > 0$. Similarly for $m = 2n + 1$, we have $d_{2n+1}(t) \geq d_{2n}(t)$. Thus $\{d_n(t)\}$; is an increasing sequence in $[0, 1]$.
By inequality (2.1) and $d_n(t)$ is an increasing sequence, we have

$$\begin{aligned}
d_{2n}(t) \;&\geq\; 1 - k(1 - \phi(d_{2n-1}(t), d_{2n-1}(t), d_{2n-1}(t), (d_{2n-1}(t))^2, 1)) \\
&>\; 1 - k(1 - d_{2n-1}(t)).
\end{aligned}$$

Similarly for an odd integer $m = 2n+1$ and $q = 1$, we have $d_{2n+1}(t) \geq 1 - k(1 - d_{2n}(t))$. Thus

$$d_n(t) \geq 1 - k(1 - d_{n-1}(t)).$$

That is,

$$\begin{aligned} M(y_n, y_{n+1}, t) &= d_n(t) \\ &\geq 1 - k + kd_{n-1}(t) \\ &\geq \\ &\vdots \\ &\geq 1 - k^n + k^n M(y_0, y_1, t) \\ &= 1 - k^n(1 - M(y_0, y_1, t)) = 1 - k^n \alpha. \end{aligned}$$

Hence by Lemma 1.11, $\{y_n\}$ is Cauchy and the completeness of X, $\{y_n\}$ converges to y in X. That is, $\lim_{n\to\infty} y_n = y$. Hence

$$\begin{aligned} \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y. \end{aligned}$$

Since the pairs (f, S) and (g, T) are compatible of type (γ), hence we have $fy = Sy$ and $gy = Ty$. Now, we prove that $fy = y$. By (ii) for $q = 2$, we have

$$\begin{aligned} & M(fy, gx_{2n+1}, t) \\ \geq\ & 1 - k(1 - \phi \begin{pmatrix} M(Sy, Tx_{2n+1}, t), & M(Sy, fy, t), \\ M(Tx_{2n+1}, gx_{2n+1}, t), & M(Sy, gx_{2n+1}, 2t), \\ M(Tx_{2n+1}, fy, t) & \end{pmatrix}). \end{aligned}$$

By continuous M and ϕ, on making $n \longrightarrow \infty$ the above inequality, we get

$$\begin{aligned} & M(fy, y, t) \\ \geq\ & 1 - k(1 - \phi \begin{pmatrix} M(Sy, y, t), & M(Sy, fy, t), \\ M(y, y, t), & M(Sy, y, 2t), \\ M(y, fy, t) & \end{pmatrix}) \\ \geq\ & 1 - k(1 - \phi\big(M(Sy, y, t), 1, 1, M(Sy, y, t) * M(y, y, t), M(y, fy, t) \big)) \\ =\ & 1 - k(1 - \phi(M(fy, y, t), M(fy, y, t), M(fy, y, t), M(fy, y, t), M(fy, y, t))) \\ >\ & 1 - k + kM(fy, y, t). \end{aligned}$$

If $fy \neq y$, by above inequality we get $M(fy, y, t) > 1$ which is contradiction. Hence $M(fy, y, t) = 1$, i.e., $fy = y$. Thus $fy = Sy = y$.

Similarly, we prove that $gy = y$. For

$$\begin{aligned}
& M(y, gg, t) \\
= \;& M(fy, gy, t) \\
\geq \;& 1 - k(1 - \phi \begin{pmatrix} M(Sy, gy, t), & M(Sy, fy, t), \\ M(Ty, gy, t), & M(Sy, gy, 2t), \\ M(Ty, fy, t) & \end{pmatrix}) \\
\geq \;& 1 - k(1 - \phi \big(\; M(y, gy, t), 1, 1, M(y, gy, t) * M(gy, y, t), 1 \; \big)) \\
= \;& 1 - k(1 - \phi(M(y, gy, t), M(y, gy, t), M(y, gy, t), M^2(y, gy, t), 1)) \\
> \;& 1 - k + kM(y, gy, t).
\end{aligned}$$

We claim that $gy = y$. For if $gy \neq y$, then $M(y, gy, t) < 1$. On the above inequality we get

$$M(y, gy, t) > 1$$

a contradiction. Hence $fy = gy = Sy = Ty = y$. That is, y is a common fixed of f, g, S and T.

Uniqueness, let z be another common fixed point of f, g, S and$T$. Then $z = fz = gz = Sz = Tz$ and $M(z, y, t) < 1$, hence for $q = 2$, we have

$$\begin{aligned}
& M(y, z, t) \\
= \;& M(fy, gz, t) \\
\geq \;& 1 - k(1 - \phi \begin{pmatrix} M(Sy, Tz, t), & M(Sy, fy, t), \\ M(Tz, gz, t), & M(Sy, gz, 2t), \\ M(Tz, fy, t) & \end{pmatrix}) \\
\geq \;& 1 - k(1 - \phi \big(\; M(y, z, t), 1, 1, M(y, z, t) * M(z, z, t), M(y, z, t) \; \big)) \\
= \;& 1 - k(1 - \phi(M(y, z, t), M(y, z, t), M(y, z, t), M(y, z, t), M(y, z, t))) \\
> \;& 1 - k + kM(y, z, t).
\end{aligned}$$

That is $M(y, z, t) > 1$ is a contradiction. Therefore, y is the unique common fixed point of self-maps f, g, S and T. □

Corollary 2.3. *Let f, g, T, H, R and S be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying conditions:*

(i) $f(X) \subseteq TH(X)$, $g(X) \subseteq SR(X)$,
(ii) *there exists a number $k \in (0, 1)$ such that*

$$\begin{array}{ll} & M(fx,gy,t) \\ \geq & 1-k(1-\phi\left(\begin{array}{ll} M(SRx,THy,t), & M(SRx,fx,t), \\ M(THy,gy,t), & M(SRx,gy,qt), \\ M(THy,fx,(3-q)t) & \end{array}\right)) \end{array}$$

for every x, y in X, $q \in \{1,2\}$, $t > 0$ and $\phi \in \Phi$,

(iii) *the pairs (f, SR) and (g, TH) are be weak compatible of type (γ),*

(iv) $TH = HT, fR = Rf, gH = Hg$ *and* $SR = RS$.

Then f, g, H, R, S and T have a unique common fixed point in X.

Proof. By Theorem 2.2, f, g, TH and SR have a unique common fixed point in X. That is, there exists $y \in X$, such that $f(y) = g(y) = TH(y) = SR(y) = y$. We prove $R(y) = y$. By (ii), we get

$$\begin{array}{ll} & M(fRy,gy,t) \\ \geq & 1-k(1-\phi\left(\begin{array}{ll} M(SRRy,THy,t), & M(SRRy,fRy,t), \\ M(THy,gy,t), & M(SRRy,gy,qt), \\ M(THy,fRy,(3-q)t) & \end{array}\right)). \end{array}$$

For $q = 1$, we get

$$\begin{array}{ll} & M(Ry,y,t) \\ \geq & 1-k(1-\phi\left(\begin{array}{ll} M(Ry,y,t), & M(Ry,Ry,t), \\ M(y,y,t), & M(Ry,y,t), \\ M(y,Ry,2t) & \end{array}\right)) \\ = & 1-k(1-\phi\left(\ M(Ry,y,t),1,1,M(Ry,y,t),M(y,Ry,t)\ \right)) \\ > & 1-k+kM(Ry,y,t). \end{array}$$

Therefore it follows that $Ry = y$. Hence $S(y) = SR(y) = y$. Similarly, we get $T(y) = H(y) = y$. □

Corollary 2.4. *Let S, T and two sequences $\{f_i\}$, $\{g_j\}$ for every $i, j \in \mathbb{N}$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying conditions:*

(i) *there exists $i_0, j_0 \in \mathbb{N}$ such that $f_{i_0}(X) \subseteq T(X)$, $g_{j_0}(X) \subseteq S(X)$,*

(ii) *there exists a number $k \in (0,1)$ such that*

$$\begin{array}{ll} & M(f_ix,g_jy,t) \\ \geq & 1-k(1-\phi\left(\begin{array}{ll} M(Sx,Ty,t), & M(Sx,f_ix,t), \\ M(Ty,g_jy,t), & M(Sx,g_jy,qt), \\ M(Ty,f_ix,(3-q)t) & \end{array}\right)) \end{array}$$

for every x, y in X, $q \in \{1,2\}$, $t > 0$ and $\phi \in \Phi$,

(iii) *the pairs* (S, f_{i_0}) *and* (g_{j_0}, T) *are be weak compatible of type* (γ).

Then S, T *and* $\{f_i\}, \{g_j\}$ *have a unique common fixed point in* X *for every* $i, j = 1, 2, \cdots$.

Proof. By Theorem 2.2, S, T and f_{i_0} and g_{j_0} for some $i_0,\ j_0 \in \mathbb{N}$, have a unique common fixed point in X. That is, there exists a unique $x \in X$ such that

$$S(x) = T(x) = f_{i_0}(x) = g_{j_0}(x) = x.$$

Suppose there exists $i \in \mathbb{N}$ such that $i \neq i_0$. Then we have

$$\begin{aligned} M(f_i x, x, t) &= M(f_i x, g_{j_0} x, t) \\ &\geq 1 - k(1 - \phi \begin{pmatrix} M(Sx, Tx, t), & M(Sx, f_i x, t), \\ M(Tx, g_{j_0} x, t), & M(Sx, g_{j_0} x, qt), \\ M(Tx, f_i x, (3-q)t) & \end{pmatrix}). \end{aligned}$$

Hence for $q = 2$, we get

$$\begin{aligned} M(f_i x, x, t) &\geq 1 - k(1 - \phi \begin{pmatrix} M(x, x, t), & M(x, f_i x, t), \\ M(x, x, t), & M(x, x, 2t), \\ M(x, f_i x, t) & \end{pmatrix}) \\ &\geq 1 - k(1 - \phi(d, d, d, d, d)) \\ &> 1 - k(1 - d) \end{aligned}$$

it follows that $d = M(f_i x, x, t) > 1$ which is a contradiction. Hence for every $i \in \mathbb{N}$ it follows that $f_i x = x$.

Similarly, for every $j \in \mathbb{N}$, we get $g_j x = x$. Therefore for every $i, j \in \mathbb{N}$ we have

$$f_i x = g_j x = Sx = Tx = x.$$

□

Corollary 2.5. *Let* f, g, S, T *be self-mappings of a complete fuzzy metric space* $(X, M, *)$ *satisfying that*

(i) $f(X) \subseteq T(X),\ g(X) \subseteq S(X)$,

(ii) *there exists a number* $k \in (0, 1)$ *such that*

$$\begin{aligned} &M(fx, gy, t) \\ \geq\ & 1 - k(1 - \begin{pmatrix} a(t)M(Sx, Ty, t) - b(t)M(Sx, fx, t) - c(t)M(Ty, gy, t) \\ -d(t)(M(Sx, gy, qt).M(Ty, fx, (3-q)t))^{1/2} \end{pmatrix}^h) \end{aligned}$$

for every x, y *in* X, $q \in \{1, 2\}$, *where* $a, b, c, d : \mathbb{R}^+ \longrightarrow [0, 1]$, *be four mappings such that* $a(t) + b(t) + c(t) + d(t) = 1$, *for all* $t > 0$ *some* $0 < h < 1$ *and* $\phi \in \Phi$,

(iii) *the pairs* (f, S) *and* (g, T) *are be weak compatible of type* (γ).

Then f, g, S and T have a unique common fixed point in X.

Proof. It is enough in Theorem 2.2, define

$$\phi(x_1, x_2, x_3, x_4, x_5) = (a(t)x_1 + b(t)x_2 + c(t)x_3 + d(t)(x_4x_5)^{1/2})^h,$$

where $a, b, c, d : \mathbb{R}^+ \longrightarrow [0,1]$, be four mappings such that $a(t) + b(t) + c(t) + d(t) = 1$, for all $t > 0$ and some $0 < h < 1$. □

Corollary 2.6. *Let f and g be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying conditions:*

(i) there exists a number $k \in (0,1)$ such that

$$M(fx, gy, t) \geq 1 - k(1 - \phi\begin{pmatrix} M(x,y,t), & M(x,fx,t), \\ M(y,gy,t), & M(x,gy,qt), \\ M(y,fx,(3-q)t) & \end{pmatrix})$$

for every x, y in X, $q \in \{1,2\}$, $t > 0$ and $\phi \in \Phi$.
Then f and g have a unique common fixed point in X.

Proof. It is enough in Theorem2.2, we set $S = T = I$, where I is identity map. □

Corollary 2.7. *Let f and g be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying conditions:*

(i)*there exists a number $k \in (0,1)$ such that*

$$M(f^nx, g^my, t) \geq 1 - k(1 - \phi\begin{pmatrix} M(x,y,t), & M(x,f^nx,t), \\ M(y,g^my,t), & M(x,g^my,qt), \\ M(y,f^nx,(3-q)t) & \end{pmatrix})$$

for every x, y in X, for some $n, m \in \mathbb{N}$, $q \in \{1,2\}$, $t > 0$ and $\phi \in \Phi$.
If $f^ng = gf^n$ and $g^mf = fg^m$, then f and g have a unique common fixed point in X.

Proof. By Corollary 2.6, f^n and g^m have a unique common fixed point in X. That is, there exists a unique $x \in X$ such that $f^n(x) = g^m(x) = x$. Since $g(x) = g(g^m(x)) = g^m(g(x))$ and $g(x) = g(f^n(x)) = f^n(g(x))$, i.e., $g(x)$ is fixed point for f^n, g^m hence $g(x) = x$. Similarly, since $f(x) = f(f^n(x)) = f^n(f(x))$ and $f(x) = f(g^m(x)) = g^m(f(x))$, i.e., $f(x) = g(x) = x$. □

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