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COMMON FIXED POINT THEOREMS UNDER RATIONAL CONTRACTIONS IN COMPLEX VALUED EXTENDED b-METRIC SPACES

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Abstract. In this paper, we discuss the existence and uniqueness of fixed point and common fixed point theorems in complex valued extended $b-$metric spaces for a pair of mappings satisfying some rational contraction conditions which generalized and unify some well-known results in the literature.

1. Introduction

The fixed point theory is one of the important tools in nonlinear analysis, science and engineering. In 1992, Banach [3] introduced the famous Banach

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contraction theorem. After this remarkable contribution, many researchers proved generalization of Banach contraction theorem in metric spaces and generalized metric spaces [8, 12]. Bakhtin [2] and Czerwik [5] extended fixed point theorems in b-metric spaces which were generalizations of the Banach contraction principle. Subsequently, Azam et at. [1] introduced the concept of complex valued metric spaces. Then many authors [4, 6, 7, 9, 10, 11, 13, 14] presented fixed point theorems on single-valued and multi-valued mapping in b-metric spaces and complex valued metric spaces. In the article, we prove fixed point theorems in complex valued extended b-metric space using rational contractions.

2. Preliminaries

We give some definitions and their properties for our main results.

Definition 2.1. ([1]) Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Thus $z_1 \preceq z_2$ if one of the followings holds:

(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We will write$z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.

It follows that

(i) $0 \leq z_1 \precnsim z_2$ implies $|z_1| < |z_2|$;
(ii) $z_1 \preceq z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$;
(iii) $0 \leq z_1 \preceq z_2$ implies $|z_1| \leq |z_2|$;
(iv) if $a, b \in \mathbb{R}$, $0 \leq a \leq b$ and $z_1 \preceq z_2$, then $az_1 \preceq bz_2$ for all $z_1, z_2 \in \mathbb{C}$.

Definition 2.2. ([1]) Let W be a nonempty set. A function $d_{cv} : W \times W \to \mathbb{C}$ is called a complex valued metric on W, if for all $l, m, n \in W$, the following conditions are satisfied:

(i) $0 \preceq d_{cv}(l, m)$ and $d_{cv}(l, m) = 0$ if and only if $l = m$;
(ii) $d_{cv}(l, m) = d_{cv}(m, l)$;
(iii) $d_{cv}(l, m) \preceq d_{cv}(l, n) + d_{cv}(n, m)$.

Then, the pair (W, d_{cv}) is called a complex valued metric space.

Example 2.3. ([1]) Let $W=[0,1]$ and $l,m\in W$. Define $d_{cv}:W\times W\to\mathbb{C}$ by

$$d_{cv}(l,m)=\begin{cases}0 & \text{if } l=m,\\ \dfrac{i}{2} & \text{if } l\neq m.\end{cases} \tag{2.1}$$

Then d_{cv} is a complex valued metric and hence (W,d_{cv}) is a complex valued metric space.

Definition 2.4. ([1]) Let (W,d_{cv}) be a complex valued metric space.

(i) We say that a point $l\in W$ is an interior point of a set $M\subseteq W$, whenever there exists $0\prec r\in\mathbb{C}$ such that

$$B(l,r)=\{m\in W: d_{cv}(m,l)\prec r\},$$

(ii) We say that a point $l\in W$ is a limit point of a set $M\subseteq W$, whenever for every $0\prec r\in\mathbb{C}$ such that

$$B(l,r)\cap M-\{l\}\neq\emptyset.$$

Definition 2.5. ([2], [5]) Let W be a nonempty set and $s\geq 1$ be a given real number. A function $d_b:W\times W\to[0,\infty)$ is called a b-metric on W if for all $l,m,n\in W$, the following conditions are satisfied:

(b1) $d_b(l,m)=0$ if and only if $l=m$;
(b2) $d_b(l,m)=d_b(m,l)$;
(b3) $d_b(l,m)\leq s[d_b(l,n)+d_b(n,m)]$.

Then, the pair (W,d_b) is called a b-metric space.

Example 2.6. ([4]) Let $W=L_p[0,1]$ be the space of all real functions $l(t)$, $t\in[0,1]$ such that $\int_0^1|l(t)|^p dt<\infty$ with $0<p<1$. Define $d_b:W\times W\to\mathbb{R}^+$ as:

$$d_b(l,m)=\left(\int_0^1|l(t)-m(t)|^p dt\right)^{\frac{1}{p}}$$

then (W,d_b) is a b-metric space with coefficient $s=2^{\frac{1}{p}}$.

Definition 2.7. ([13]) Let W be a nonempty set and let $s\geq 1$ be a given real number. A function $d_{cvb}:W\times W\to\mathbb{C}$ is called a complex valued b-metric on W if for all $l,m,n\in W$, the following conditions are satisfied:

(i) $0\preceq d_{cvb}(l,m)$ and $d_{cvb}(l,m)=0$ if and only if $l=m$;
(ii) $d_{cvb}(l,m)=d_{cvb}(m,l)$;

(iii) $d_{cvb}(l,m) \preceq s[d_{cvb}(l,n) + d_{cvb}(n,m)]$.

Then, the pair (W, d_{cvb}) is called a complex valued b-metric space.

Example 2.8. ([13]) If $W = [0,1]$, define the mapping $d_{cvb} : W \times W \to \mathbb{C}$ by

$$d_{cvb}(l,m) = |l-m|^2 + i|l-m|^2$$

for all $l, m \in W$. Then (W, d_{cvb}) is a complex valued b-metric space with $s = 2$.

Definition 2.9. ([10]) Let W be a non-empty set and $\lambda : W \times W \to [1,\infty)$ be a function. Then $d_\lambda : W \times W \to [0,\infty)$ is called an extended b-metric if for all $l, m, n \in W$ it satisfies:

(i) $d_\lambda(l,m) = 0$ if and only if $l = m$;
(ii) $d_\lambda(l,m) = d_\lambda(m,l)$;
(iii) $d_\lambda(l,n) \leq \lambda(l,n)[d_\lambda(l,m) + d_\lambda(m,n)]$.

Then, the pair (W, d_λ) is called an extended b-metric space.

Example 2.10. Let $W = \{1,2,3\}$. Define $\lambda : W \times W \to [1,\infty)$ and $d_\lambda : W \times W \to \mathbb{R}^+$ as:

$$\begin{aligned}
&\lambda(l,m) = 1 + l + m,\\
&d_\lambda(1,1) = d_\lambda(2,2) = d_\lambda(3,3) = 0,\\
&d_\lambda(1,2) = d_\lambda(2,1) = 80,\ d_\lambda(1,3) = d_\lambda(3,1) = 1000,\\
&d_\lambda(2,3) = d_\lambda(3,2) = 600.
\end{aligned}$$

Then (W, d_λ) is an extended b-metric space.

Definition 2.11. ([10]) Let (W, d_λ) be an extended b-metric space.

(i) A sequence $\{l_n\}$ in W is said to converge to $l \in W$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(l_n, l) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} l_n = l$.
(ii) A sequence $\{l_n\}$ in W is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(l_m, l_n) < \epsilon$, for all $m, n \geq N$.
(iii) If every Cauchy sequence in W is convergent, then (W, d_λ) is said to be a complete extended b-metric space.

Lemma 2.12. ([10]) *Let (W, d_λ) be an extended b-metric space. If d_λ is continuous, then every convergent sequence has a unique limit.*

Definition 2.13. ([14]) Let W be a non-empty set and $\theta : W \times W \to [1,\infty)$ be a function. Then $d_\theta : W \times W \to \mathbb{C}$ is known as a complex valued extended b-metric space if the following conditions are satisfied for all $l, m, n \in W$:

(i) $0 \preceq d_\theta(l,m)$ and $d_\theta(l,m) = 0$ if and only if $l = m$;
(ii) $d_\theta(l,m) = d_\theta(m,l)$;
(iii) $d_\theta(l,n) \preceq \theta(l,n)[d_\theta(l,m) + d_\theta(m,n)]$.

Then the pair (W, d_θ) is called a complex valued extended b-metric space.

Example 2.14. If W be a nonempty set and $\theta : W \times W \to [1,\infty]$ be defined as:

$$\theta(l,m) = \frac{1+l+m}{l+m},$$

further, let

(i) $d_\theta(l,m) = \dfrac{i}{lm}$ for all $l,m \in (0,1]$;
(ii) $d_\theta(l,m) = 0 \Longleftrightarrow l = m$ for all $l,m \in [0,1]$;
(iii) $d_\theta(l,0) = d_\theta(0,l) = \dfrac{i}{l}$ for all $l \in (0,1]$.

Then the pair (W, d_θ) is known as a complex valued extended b-metric space.

Example 2.15. Let $W = [0,\infty)$. $\theta : W \times W \to [1,\infty)$ be a function defined by $\theta(l,m) = 1 + l + m$ and $d_\theta : W \times W \to \mathbb{C}$ be given as

$$d_\theta(l,m) = \left\{ \begin{array}{ll} 0 & \text{if } l = m \\ i & \text{if } l \neq m \end{array} \right\}.$$

Then (W, d_θ) is a complex valued extended b- metric space.

3. Main results

Now, we can state our main results.

Theorem 3.1. *Let (W, d_θ) be a complete complex valued extended b-metric space, let $\theta : W \times W \to [1,\infty)$ and let U, V be self-mappings from W into itself satisfy the following inequality:*

$$\begin{aligned} d_\theta(Ul,Vm) \preceq \mu_1 d_\theta(l,m) &+ \mu_2 \frac{d_\theta(l,Ul)d_\theta(m,Vm)}{d_\theta(l,Vm) + d_\theta(m,Ul) + d_\theta(l,m)} \\ &+ \mu_3 \left[\frac{d_\theta(l,Vl)d_\theta(m,Vm)\big[1 + d_\theta(m,Ul)d_\theta(l,m)\big]}{d_\theta(l,Vm) + d_\theta(l,m)} \right] \end{aligned} \tag{3.1}$$

for all $l,m \in W$ such that $l \neq m$, $d_\theta(l,Vm)+d_\theta(m,Ul)+d_\theta(l,m) \neq 0$, $d_\theta(l,Vm)+d_\theta(l,m) \neq 0$ where μ_1, μ_2 and μ_3 are non negative reals with $\mu_1+\theta(l_1,l_2)\big(\mu_2+\mu_3\big) < 1$, $\zeta = \mu_1 + (\mu_2+\mu_3)\,\theta(l_1,l_2)$ where $\zeta \in [0,\infty)$, $\lim\limits_{n,m\to\infty} \theta(l_n,l_m) < \frac{1}{\zeta}$ or $d_\theta(Ul,Vm) = 0$ if $d_\theta(l,Vm)+d_\theta(m,Ul)+d_\theta(l,m) = 0$, $d_\theta(l,Vm)+d_\theta(l,m) = 0$, then U and V have a unique common fixed point in W.

Proof. For any arbitrary point $l_0 \in W$, define a sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \text{ and } l_{2n+2} = Vl_{2n+1}, \quad \forall n \geq 0. \tag{3.2}$$

Now we prove that $\{l_n\}$ is a Cauchy sequence.

Let $l = l_0$, $m = l_1$ in (3.1). Then

$$\begin{aligned} d_\theta(l_1, l_2) &= d_\theta(Ul_0, Vl_1) \\ &\preceq \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, Ul_0) d_\theta(l_1, Vl_1)}{d_\theta(l_0, Vl_1) + d_\theta(l_1, Ul_0) + d_\theta(l_0, l_1)} \\ &\quad + \mu_3 \left[\frac{d_\theta(l_0, Vl_0) d_\theta(l_1, Vl_1) \big[1 + d_\theta(l_1, Ul_0) d_\theta(l_0, l_1)\big]}{d_\theta(l_0, Vl_1) + d_\theta(l_0, l_1)} \right] \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, l_1) d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_1, l_1) + d_\theta(l_0, l_1)} \\ &\quad + \mu_3 \left[\frac{d_\theta(l_0, l_1) d_\theta(l_1, l_2) \big[1 + d_\theta(l_1, l_1) d_\theta(l_0, l_1)\big]}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} \right] \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, l_1) d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} + \mu_3 \left[\frac{d_\theta(l_0, l_1) d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} |d_\theta(l_1, l_2)| = &\mu_1 |d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|} \\ &+ \mu_3 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}. \end{aligned}$$

Using triangle inequality

$$d_\theta(l_1, l_2) \leq \theta(l_1, l_2)[d_\theta(l_1, l_0) + d_\theta(l_0, l_2)].$$

Thus we have

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_1, l_2)|} |\theta(l_1, l_2)| \\ &\quad + \mu_3 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_1, l_2)|} |\theta(l_1, l_2)| \\ &= \big[\mu_1 + \big(\mu_2 + \mu_3\big)\theta(l_1, l_2)\big] |d_\theta(l_0, l_1)|, \end{aligned}$$

that is,

$$|d_\theta(l_1, l_2)| \leq \big[\mu_1 + \big(\mu_2 + \mu_3\big)\theta(l_1, l_2)\big] |d_\theta(l_0, l_1)|.$$

Since $|d_\theta(l_1,l_2)| < 1 + |d_\theta(l_1,l_2)|$, we have

$$|d_\theta(l_1,l_2)| \le \zeta |d_\theta(l_0,l_1)|,$$

$$|d_\theta(l_2,l_3)| \le \zeta^2 |d_\theta(l_0,l_1)|,$$

$$|d_\theta(l_3,l_4)| \le \zeta^3 |d_\theta(l_0,l_1)|,$$

$$\vdots$$

$$|d_\theta(l_n,l_{n+1})| \le \zeta^n |d_\theta(l_0,l_1)|.$$

Now, by triangle inequality, for any $m > n$, $m,n \in \mathbb{N}$, we have

$$\begin{aligned} d_\theta(l_n,l_m) \preceq\ & \theta(l_n,l_m)\zeta^n d_\theta(l_0,l_1) + \theta(l_n,l_m)\theta(l_{n+1},l_m)\zeta^{n+1} d_\theta(l_0,l_1) \dots \\ & + \theta(l_n,l_m)\theta(l_{n+1},l_m)\dots\theta(l_{m-2},l_m)\theta(l_{m-1},l_m)\zeta^{m-1} d_\theta(l_0,l_1). \end{aligned}$$

Then

$$\begin{aligned} d_\theta(l_n,l_m) \preceq\ & d_\theta(l_0,l_1)[\theta(l_n,l_m)\zeta^n + \theta(l_n,l_m)\theta(l_{n+1},l_m)\zeta^{n+1} \dots \\ & + \theta(l_n,l_m)\theta(l_{n+1},l_m)\dots\theta(l_{m-2},l_m)\theta(l_{m-1},l_m)\zeta^{m-1}]. \end{aligned}$$

Since $\lim_{n,m\to\infty} \theta(l_n,l_m)\zeta < 1$, so the series $\sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^{n} \theta(l_i,l_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^{n} \theta(l_i,l_m),\ S_n = \sum_{j=1}^{n} \zeta^j \prod_{i=1}^{j} \theta(l_i,l_m).$$

Thus, for $m > n$, the above can be written as

$$d_\theta(l_n,l_m) \preceq d_\theta(l_0,l_1)[S_{m-1} - S_n]$$

and

$$|d_\theta(l_n,l_m)| \le |d_\theta(l_0,l_1)|[S_{m-1} - S_n].$$

Letting $m \to \infty$, we obtain

$$|d_\theta(l_n,l_m)| \to 0.$$

Thus, $\{l_n\}$ is a Cauchy sequence in W. Since W is complete, there exists some $t \in W$ such that $l_n \to t$ as $n \to \infty$.

If not, then there exits $z \in W$ such that

$$|d_\theta(t,Ut)| = |z| > 0. \tag{3.3}$$

So, using the triangular inequality and (3.1), we have

$$\begin{aligned}
z &= d_\theta(t, Ut)\\
&\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(l_{2n+2}, Ut)\\
&= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(Vl_{2n+1}, Ut)\\
&\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)\Bigg[\mu_1 d_\theta(t, l_{2n+1})\\
&\quad + \mu_2 \frac{d_\theta(t, Ut)d_\theta(l_{2n+1}, Vl_{2n+1})}{d_\theta(t, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ut) + d_\theta(t, l_{2n+1})}\\
&\quad + \mu_3\left[\frac{d_\theta(l_{2n+1}, Ul_{2n+1})d_\theta(t, Ut)\big[1 + d_\theta(t, Vl_{2n+1})d_\theta(l_{2n+1}, t)\big]}{d_\theta(l_{2n+1}, Ut) + d_\theta(l_{2n+1}, t)}\right]\Bigg]\\
&= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \mu_1\theta(t, U_t)d_\theta(t, l_{2n+1})\\
&\quad + \mu_2\theta(t, U_t)\frac{d_\theta(t, Ut)d_\theta(l_{2n+1}, l_{2n+2})}{d_\theta(t, l_{2n+2}) + d_\theta(l_{2n+1}, Ut) + d_\theta(t, l_{2n+1})}\\
&\quad + \mu_3\theta(t, U_t)\left[\frac{d_\theta(l_{2n+1}, l_{2n+2})d_\theta(t, Ut)\big[1 + d_\theta(t, l_{2n+2})d_\theta(l_{2n+1}, t)\big]}{d_\theta(l_{2n+1}, Ut) + d_\theta(l_{2n+1}, t)}\right].
\end{aligned}$$

And so,

$$\begin{aligned}
|z| &= |d_\theta(t, Ut)|\\
&\leq \theta(t, U_t)|d_\theta(t, l_{2n+2})| + \mu_1\theta(t, U_t)|d_\theta(t, l_{2n+1})|\\
&\quad + \mu_2\theta(t, U_t)\frac{|d_\theta(t, Ut)||d_\theta(l_{2n+1}, l_{2n+2})|}{|d_\theta(t, l_{2n+2})| + |d_\theta(l_{2n+1}, Ut)| + |d_\theta(t, l_{2n+1})|}\\
&\quad + \mu_3\theta(t, U_t)\left[\frac{|d_\theta(l_{2n+1}, l_{2n+2})||d_\theta(t, Ut)|\big[1 + |d_\theta(t, l_{2n+2})||d_\theta(l_{2n+1}, t)|\big]}{|d_\theta(l_{2n+1}, Ut)| + |d_\theta(l_{2n+1}, t)|}\right].
\end{aligned}$$

As $n \to \infty$, we obtain that $|z| = |d_\theta(t, Ut)| \leq 0$, a contradiction. Thus, $|z| = 0$. Hence, $Ut = t$. Similarly, we obtain $Vt = t$.

Now, we show that U and V have a unique common fixed point. To prove this, assume that $t^{'} \neq t$ is another common fixed point of U and V. Then

$$\begin{aligned}
d_\theta(t, t') &= d_\theta(Ut, Vt')\\
&\preceq \mu_1 d_\theta(t, t') + \mu_2\frac{d_\theta(t, Ut)d_\theta(t', Vt')}{d_\theta(t, Vt') + d_\theta(t', Ut) + d_\theta(t, t')}\\
&\quad + \mu_3\frac{d_\theta(t, Vt)d_\theta(t', Vt')\big[1 + d_\theta(t', Ut)d_\theta(t, t')\big]}{d_\theta(t, Vt') + d_\theta(t, t')}.
\end{aligned} \tag{3.4}$$

And so, we have

$$|d_\theta(t,t')| \leq \mu_1|d_\theta(t,t')| + \mu_2\frac{|d_\theta(t,Ut)||d_\theta(t',Vt')|}{|d_\theta(t,Vt')|+|d_\theta(t',Ut)|+|d_\theta(t,t')|} + \mu_3\frac{|d_\theta(t,Vt)||d_\theta(t',Vt')|\big[1+|d_\theta(t',Ut)||d_\theta(t,t')|\big]}{|d_\theta(t,Vt')|+|d_\theta(t,t')|},$$

this implies that

$$|d_\theta(t,t')| \leq \mu_1|d_\theta(t,t')|,$$

which is a contradiction. Hence, $t = t'$ which shows the uniqueness of common fixed point in W.

Now, we consider the second case:

Step 1: $d_\theta(l,Vm) + d_\theta(m,Ul) + d_\theta(l,m) = 0$, $l = l_{2n}$ and $m = l_{2n+1}$. $d_\theta(l_{2n},Vl_{2n+1})+d_\theta(l_{2n+1},Ul_{2n})+d_\theta(l_{2n},l_{2n+1}) = 0$, $d_\theta(Ul_{2n},Vl_{2n+1}) = 0$. So that $l_{2n} = Ul_{2n} = l_{2n+1} = Vl_{2n+1} = l_{2n+2}$. Thus, we have $l_{2n+1} = Ul_{2n} = l_{2n}$, so there exist E_1 and f_1 such that

$$E_1 = Uf_1 = f_1,$$

where $E_1 = l_{2n+1}$ and $f_1 = l_{2n}$.

Using foregoing arguments, we show that there exist E_2 and f_2 such that

$$E_2 = Vf_2 = f_2,$$

where $E_2 = l_{2n+2}$ and $f_2 = l_{2n+1}$.

As, $d_\theta(f_1,Vf_2)+d_\theta(f_2,Uf_1)+d_\theta(f_1,f_2) = 0$ which implies $d_\theta(Uf_1,Vf_2) = 0$. Since $E_1 = Uf_1 = Vf_2 = E_2$, we obtain $E_1 = Uf_1 = UE_1$. Similarly, we have $E_2 = VE_2$.

As $E_1 = E_2$, then $UE_1 = VE_1 = E_1$. Hence $E_1 = E_2$ is common fixed point of U and V.

For uniqueness of common fixed point, assume that E_1' in W is another common fixed point of U and V. Then we have $UE_1' = VE_1' = E_1'$. As $d_\theta(E_1,VE_1')+d_\theta(E_1',UE_1)+d_\theta(E_1,E_2') = 0$, therefore, we have

$$d_\theta(E_1,E_1') = d_\theta(UE_1,VE_1') = 0.$$

This implies that $E_1 = E_1'$.

Step 2: $d_\theta(l,Vm)+d_\theta(l,m) = 0$. $l = l_{2n}$ and $m = l_{2n+1}$ in this expression, we get, $d_\theta(l_{2n},Vl_{2n+1}) + d_\theta(l_{2n},l_{2n+1}) = 0$, it implies that $d_\theta(Ul_{2n},Vl_{2n+1}) = 0$. So that $l_{2n} = Ul_{2n} = l_{2n+1} = Vl_{2n+1} = l_{2n+2}$. Thus,we have $l_{2n+1} = Ul_{2n} = l_{2n}$, so there exist E_3 and f_3 such that

$$E_3 = Uf_3 = f_3,$$

where $E_3 = l_{2n+1}$ and $f_3 = l_{2n}$.

Using foregoing arguments, we show that there exist E_4 and f_4 such that

$$E_4 = Vf_4 = f_4,$$

where $E_4 = l_{2n+2}$ and $f_4 = l_{2n+1}$. As, $d_\theta(f_3, Vf_4) + d_\theta(f_3, f_4) = 0$ which implies $d_\theta(Uf_3, Vf_4) = 0$. Hence $E_3 = Uf_3 = Vf_4 = E_4$. Thus we obtain $E_3 = Uf_3 = UE_3$. Similarly,we have $E_4 = VE_4$.

As $E_3 = E_4$, implies $UE_3 = VE_3 = E_3$. Hence $E_3 = E_4$ is common fixed point of U and V.

For uniqueness of common fixed point, assume that E'_3 in W is another common fixed point of U and V. Then we have $UE'_3 = VE'_3 = E'_3$. As $d_\theta(E_3, VE'_3) + d_\theta(E_3, E'_4) = 0$, therefore, we have

$$d_\theta(E_3, E'_3) = d_\theta(UE_3, VE'_3) = 0.$$

This implies that $E_3 = E'_3$. This completes the proof of the theorem. □

Corollary 3.2. *Let (W, d_θ) be a complete complex valued extended b-metric space, let $\theta : W \times W \to [1, \infty)$ and let V be a self-mapping from W into itself satisfy the following inequality:*

$$d_\theta(Vl, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta(l, Vl) d_\theta(m, Vm)}{d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m)} + \mu_3 \left[\frac{d_\theta(l, Vl) d_\theta(m, Vm)\big[1 + d_\theta(m, Vl) d_\theta(l, m)\big]}{d_\theta(l, Vm) + d_\theta(l, m)} \right] \tag{3.5}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m) \neq 0, d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ where μ_1, μ_2 and μ_3 are nonnegative reals with $\mu_1 + \theta(l_1, l_2)(\mu_2 + \mu_3) < 1$, $\zeta = \mu_1 + (\mu_2 + \mu_3)\,\theta(l_1, l_2)$ where $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Vl, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m) = 0$, $d_\theta(l, Vm) + d_\theta(l, m) = 0$, then V has a unique common fixed point in W.

Proof. We can prove this result by applying Theorem 3.1 with the condition $U = V$. □

Theorem 3.3. *Let (W, d_θ) be a complete complex valued extended b-metric space, let $\theta : W \times W \to [1, \infty)$ and let U, V be self-mappings from W into itself satisfy the following inequality,*

$$d_\theta(Ul, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d^2_\theta(l, Vm) + d^2_\theta(m, Ul)}{d_\theta(l, Vm) + d_\theta(m, Ul)} + \mu_3\big[d_\theta(l, Ul) + d_\theta(m, Vm)\big] + \mu_4\big[d_\theta(l, m) + d_\theta(m, Ul)\big] + \mu_5 \frac{d^2_\theta(m, Vm)}{d_\theta(l, Vm) + d_\theta(l, m)} \tag{3.6}$$

for all $l, m \in W$, *such that* $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Ul) \neq 0, d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ *where* $\mu_1, \mu_2, \mu_3, \mu_4$ *and* μ_5 *are nonnegative reals with* $\mu_1 + 2\theta(l_0, l_2)\mu_2 + 2\mu_3 + \mu_4 + \theta(l_0, l_2)\mu_5 < 1$, $\zeta(1 - (\mu_2 + \mu_5)\theta(l_0, l_2) - \mu_3) = (\mu_1 + \mu_2\theta(l_0, l_2) + \mu_3 + \mu_4)$ *where* $\zeta \in [0, \infty)$, $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. *or* $d_\theta(Ul, Vm) = 0$ *if* $d_\theta(l, Vm) + d_\theta(m, Ul) = 0$, $d_\theta(l, Vm) + d_\theta(l, m) = 0$, *then* U *and* V *have a unique common fixed point in* W.

Proof. For any arbitrary point $l_0 \in W$, define a sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \quad \text{and} \quad l_{2n+2} = Vl_{2n}, \quad \forall\, n \geq 0. \tag{3.7}$$

Now we prove that $\{l_n\}$ is a Cauchy sequence. Let $l = l_0$,$m = l_1$ in (3.6). Then, we have

$$\begin{aligned} d_\theta(l_1, l_2) &= d_\theta(Ul_0, Vl_1) \\ &\preceq \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{[d_\theta^2(l_0, Vl_1) + d_\theta^2(l_1, Ul_0)]}{d_\theta(l_0, Vl_1) + d_\theta(l_1, Ul_0)} \\ &\quad + \mu_3[d_\theta(l_0, Ul_0) + d_\theta(l_1, Vl_1)] \\ &\quad + \mu_4[d_\theta(l_0, l_1) + d_\theta(l_1, Ul_0)] + \mu_5 \frac{d_\theta^2(l_1, Vl_1)}{d_\theta(l_0, Vl_1) + d_\theta(l_0, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{[d_\theta^2(l_0, l_2) + d_\theta^2(l_1, l_1)]}{d_\theta(l_0, l_2) + d_\theta(l_1, l_1)} + \mu_3[d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] \\ &\quad + \mu_4[d_\theta(l_0, l_1) + d_\theta(l_1, l_1)] + \mu_5 \frac{d_\theta^2(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta^2(l_0, l_2)}{d_\theta(l_0, l_2)} + \mu_3[d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] \\ &\quad + \mu_4 d_\theta(l_0, l_1) + \mu_5 \frac{d_\theta^2(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)}. \end{aligned}$$

And so,

$$\begin{aligned} |d_\theta(l_1, l_2)| \leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta^2(l_0, l_2)|}{|d_\theta(l_0, l_2)|} + \mu_3[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ + \mu_4 |d_\theta(l_0, l_1)| + \mu_5 \frac{|d_\theta^2(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}, \end{aligned}$$

that is,

$$\begin{aligned} |d_\theta(l_1, l_2)| \leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 |d_\theta(l_0, l_2)| + \mu_3[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ + \mu_4 |d_\theta(l_0, l_1)| + \mu_5 \frac{|d_\theta^2(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}. \end{aligned}$$

Using the triangle inequality, we have

$$d_\theta(l_0,l_2) \le \theta(l_0,l_2)[d_\theta(l_0,l_1)+d_\theta(l_1,l_2)].$$

Thus, we have

$$\begin{aligned}|d_\theta(l_1,l_2)| &\le \mu_1|d_\theta(l_0,l_1)| + \mu_2\theta(l_0,l_2)[|d_\theta(l_0,l_1)|+|d_\theta(l_1,l_2)|]\\ &\quad + \mu_3[|d_\theta(l_0,l_1)|+|d_\theta(l_1,l_2)|]\\ &\quad + \mu_4|d_\theta(l_0,l_1)| + \mu_5\theta(l_0,l_2)|d_\theta(l_1,l_2)|\\ &= (\mu_1+\mu_2\theta(l_0,l_2)+\mu_3+\mu_4)|d_\theta(l_0,l_1)|\\ &\quad + ((\mu_2+\mu_5)\theta(l_0,l_2)+\mu_3)|d_\theta(l_1,l_2)|,\end{aligned}$$

it implies that

$$|d_\theta(l_1,l_2)| \le \frac{(\mu_1+\mu_2\theta(l_0,l_2)+\mu_3+\mu_4)}{(1-(\mu_2+\mu_5)\theta(l_0,l_2)-\mu_3)}|d_\theta(l_0,l_1)|.$$

Then, we obtain

$$|d_\theta(l_1,l_2)| \le \zeta|d_\theta(l_0,l_1)|.$$

Similarly, we have

$$\begin{aligned}|d_\theta(l_1,l_2)| &\le \zeta|d_\theta(l_0,l_1)|,\\ |d_\theta(l_2,l_3)| &\le \zeta^2|d_\theta(l_0,l_1)|,\\ |d_\theta(l_3,l_4)| &\le \zeta^3|d_\theta(l_0,l_1)|,\\ &\vdots\\ |d_\theta(l_n,l_{n+1})| &\le \zeta^n|d_\theta(l_0,l_1)|.\end{aligned}$$

Now, by triangle inequality, for any $m > n$, $m, n \in \mathbb{N}$ we have

$$\begin{aligned}d_\theta(l_n,l_m) &\preceq \theta(l_n,l_m)\zeta^n d_\theta(l_0,l_1) + \theta(l_n,l_m)\theta(l_{n+1},l_m)\zeta^{n+1}d_\theta(l_0,l_1)\dots\\ &\quad + \theta(l_n,l_m)\theta(l_{n+1},l_m)\dots\theta(l_{m-2},l_m)\theta(l_{m-1},l_m)\zeta^{m-1}d_\theta(l_0,l_1).\end{aligned}$$

Then

$$\begin{aligned}d_\theta(l_n,l_m) &\preceq d_\theta(l_0,l_1)[\theta(l_n,l_m)\zeta^n + \theta(l_n,l_m)\theta(l_{n+1},l_m)\zeta^{n+1}\dots\\ &\quad + \theta(l_n,l_m)\theta(l_{n+1},l_m)\dots\theta(l_{m-2},l_m)\theta(l_{m-1},l_m)\zeta^{m-1}].\end{aligned}$$

Since $\lim_{n,m\to\infty}\theta(l_n,l_m)\zeta < 1$, series $\sum_{n=1}^{\infty}\zeta^n\prod_{i=1}^{n}\theta(l_i,l_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let

$$S = \sum_{n=1}^{\infty}\zeta^n\prod_{i=1}^{n}\theta(l_i,l_m),\ S_n = \sum_{j=1}^{n}\zeta^j\prod_{i=1}^{j}\theta(l_i,l_m).$$

Then, for $m > n$, the above expression can be written as

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1)[S_{m-1} - S_n]$$

and

$$|d_\theta(l_n, l_m)| \leq |d_\theta(l_0, l_1)|[S_{m-1} - S_n].$$

Letting $n \to \infty$, we get

$$|d_\theta(l_n, l_m)| \to 0.$$

Thus, $\{l_n\}$ is a Cauchy sequence in W. Since W is complete, there exists some $t \in W$ such that $l_n \to t$ as $n \to \infty$. If not, then there exits $z \in W$ such that

$$|d_\theta(t, Ut)| = |z| > 0. \tag{3.8}$$

Using the triangle inequality, we have

$$\begin{aligned}
z &= d_\theta(t, Ut) \\
&\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(l_{2n+2}, Ut) \\
&= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(Vl_{2n+1}, Ut) \\
&\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)\mu_1 d_\theta(t, l_{2n+1}) \\
&\quad + \theta(t, U_t)\mu_2 \frac{[d_\theta^2(t, Vl_{2n+1}) + d_\theta^2(l_{2n+1}, Ut)]}{d_\theta(t, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ut)} \\
&\quad + \theta(t, U_t)\mu_3[d_\theta(t, Ut) + d_\theta(l_{2n+1}, Vl_{2n+1})] \\
&\quad + \theta(t, U_t)\mu_4[d_\theta(t, l_{2n+1}) + d_\theta(l_{2n+1}, Ut)] \\
&\quad + \theta(t, U_t)\mu_5 \frac{[d_\theta^2(l_{2n+1}, Vl_{2n+1})]}{d_\theta(t, Vl_{2n+1}) + d_\theta(t, l_{2n+1})} \\
&= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)\mu_1 d_\theta(t, l_{2n+1}) \\
&\quad + \theta(t, U_t)\mu_2 \frac{[d_\theta^2(t, l_{2n+2}) + d_\theta^2(l_{2n+1}, Ut)]}{d_\theta(t, l_{2n+2}) + d_\theta(l_{2n+1}, Ut)} \\
&\quad + \theta(t, U_t)\mu_3[d_\theta(t, Ut) + d_\theta(l_{2n+1}, l_{2n+2})] \\
&\quad + \theta(t, U_t)\mu_4[d_\theta(t, l_{2n+1}) + d_\theta(l_{2n+1}, Ut)] \\
&\quad + \theta(t, U_t)\mu_5 \frac{[d_\theta^2(l_{2n+1}, l_{2n+2})]}{d_\theta(t, l_{2n+2}) + d_\theta(t, l_{2n+1})}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}|z| = |d_\theta(t,Ut)| \le |\theta(t,Ut)|\Bigg(&|d_\theta(t,l_{2n+2})| + \mu_1|d_\theta(t,l_{2n+1})|\\ &+ \mu_2\frac{[|d_\theta^2(t,l_{2n+2})| + |d_\theta^2(l_{2n+1},Ut)|]}{|d_\theta(t,l_{2n+2})| + |d_\theta(l_{2n+1},Ut)|}\\ &+ \mu_3\big[|d_\theta(t,Ut)| + |d_\theta(l_{2n+1},l_{2n+2})|\big]\\ &+ \mu_4\big[|d_\theta(t,l_{2n+1})| + |d_\theta(l_{2n+1},Ut)|\big]\\ &+ \mu_5\frac{|d_\theta^2(l_{2n+1},l_{2n+2})|}{|d_\theta(t,l_{2n+2})| + |d_\theta(t,l_{2n+1})|}\Bigg).\end{aligned}$$

As $n\to\infty$, we obtain that $|z| = |d_\theta(t,Ut)| \le 0$, which is a contradiction. Thus, $|z| = 0$. Hence, $Ut = t$. Similarly, we obtain $Vt = t$.

Now, we show that U and V have a unique common fixed point. To prove this, assume that $t^{'} \neq t$ is another common fixed point of U and V. Then

$$\begin{aligned}d_\theta(t,t') &= d_\theta(Ut,Vt')\\ &\preceq \mu_1 d_\theta(t,t') + \mu_2\frac{[d_\theta^2(t,Vt') + d_\theta^2(t',Ut)]}{d_\theta(t,Vt') + d_\theta(t',Ut)} + \mu_3[d_\theta(t,Ut) + d_\theta(t',Vt')]\\ &\quad + \mu_4[d_\theta(t,t') + d_\theta(t',Ut)] + \mu_5\frac{d_\theta^2(t',Vt')}{d_\theta(t,Vt') + d_\theta(t,t')}.\end{aligned}$$

And also, we have

$$\begin{aligned}|d_\theta(t,t')| \le \mu_1|d_\theta(t,t')| &+ \mu_2\frac{[|d_\theta^2(t,Vt')| + |d_\theta^2(t',Ut)|]}{|d_\theta(t,Vt')| + |d_\theta(t',Ut)|}\\ &+ \mu_3[|d_\theta(t,Ut)| + |d_\theta(t',Vt')|]\\ &+ \mu_4[|d_\theta(t,t')| + |d_\theta(t',Ut)|] + \mu_5\frac{|d_\theta^2(t',Vt')|}{|d_\theta(t,Vt')| + |d_\theta(t,t')|},\end{aligned}$$

that is,

$$|d_\theta(t,t')| \le (\mu_1 + \mu_2 + 2\mu_4)|d_\theta(t,t')|,$$

which is a contradiction. Hence $t = t'$ which shows the uniqueness of common fixed point in W.

For the second case, $d_\theta(Ul,Vm) = 0$ if $d_\theta(l,Vm) + d_\theta(m,Ul) = 0$, the proof of uniqueness of common fixed point can be completed in the line of Theorem 3.1. This completes the proof of the theorem. □

Corollary 3.4. *Let* (W, d_θ) *be a complete complex valued extended b-metric space, let* $\theta : W \times W \to [1, \infty)$ *and let* V *be a self-mapping from* W *into itself satisfy the following inequality:*

$$\begin{aligned} d_\theta(Vl, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta^2(l, Vm) + d_\theta^2(m, Vl)}{d_\theta(l, Vm) + d_\theta(m, Vl)} \\ + \mu_3\big[d_\theta(l, Vl) + d_\theta(m, Vm)\big] + \mu_4\big[d_\theta(l, m) + d_\theta(m, Vl)\big] \\ + \mu_5 \frac{d_\theta^2(m, Vm)}{d_\theta(l, Vm) + d_\theta(l, m)} \end{aligned} \tag{3.9}$$

for all $l, m \in W$*, such that* $l \neq m$*,* $d_\theta(l, Vm) + d_\theta(m, Vl) \neq 0, d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ *where* $\mu_1, \mu_2, \mu_3, \mu_4$ *and* μ_5 *are nonnegative reals with* $\mu_1 + 2\theta(l_0, l_2)\mu_2 + 2\mu_3 + \mu_4 + \theta(l_0, l_2)\mu_5 < 1$*,* $\zeta(1 - (\mu_2 + \mu_5)\theta(l_0, l_2) - \mu_3) = (\mu_1 + \mu_2\theta(l_0, l_2) + \mu_3 + \mu_4)$ *where* $\zeta \in [0, \infty)$*,* $\lim_{n,m\to\infty} \theta(l_n, l_m) < \frac{1}{\zeta}$*. or* $d_\theta(Vl, Vm) = 0$ *if* $d_\theta(l, Vm) + d_\theta(m, Vl) = 0$*,* $d_\theta(l, Vm) + d_\theta(l, m) = 0$*, then* V *has a unique common fixed point in* W*.*

Proof. By using Theorem 3.3 with $U = V$, we can prove this result. □

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