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ESSENTIAL SPECTRUM OF A WEIGHTED GEOMETRIC REALIZATION

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Abstract. In this present article, we construct a new framework that's we call the weighted geometric realization of 2 and 3-simplexes. On this new weighted framework, we construct a nonself-adjoint 2-simplex Laplacian L and a self-adjoint 2-simplex Laplacian N. We propose general conditions to ensure sectoriality for our new nonself-adjoint 2-simplex Laplacian L. We show the relation between the essential spectra of L and N. Finally, we prove the absence of the essential spectrum for our 2-simplex Laplacians L and N.

1. Introduction

Recently, the sectorial operators are given special attention in view of later applications to the spectral theory and to the analytic and the asymptotic perturbation theory ([4], [9], [11]). The interest in spectral properties of non

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self-adjoint operators has already led to a variety of new results. The nonself-adjoint operators are more difficult to study than the self-adjoint ones. This can be explained by the complicated structure of the resolvent of such an operator seen as an analytic function and by the absence of the general spectral theorems.

In this article, we construct a new framework that's we call the weighted geometric realization of 2 and 3-simplexes. On this new weighted framework, we construct a nonself-adjoint 2-simplex Laplacian L and a self-adjoint 2-simplex Laplacian N. We propose general conditions to ensure sectoriality for our new non self-adjoint 2-simplex Laplacian L. We show the relation between the essential spectra of L and N. Finally, we prove the absence of the essential spectrum for our 2-simplex Laplacians L and N.

This current paper is organized as follows: In Section 1 (Introduction), we introduce what we want prove in this article. In Section 2 (Weighted geometric realization), we construct our new framework that's we call the weighted geometric realization of 2 and 3-simplexes. In Section 3 (Nonself-adjoint 2-simplex Laplacians), on the weighted geometric realization, we construct a new nonself-adjoint 2-simplex Laplacian and we define its Dirichlet 2-simplex Laplacian and its adjoint 2-simplex Laplacian. We give the Green's formula of our new nonself-adjoint 2-simplex Laplacian. In Section 4 (Sectoriality of the 2-simplex Laplacians), we prove that the nonself-adjoint 2-simplex Laplacian L is sectorial. After that, we characterize the essential spectrum by using the notion of sectoriality. In Section 5 (Absence of the essential spectrum), we present necessary conditions for the operator $\overline{L} + L'$ to be self-adjoint. After that, we study the relation between the essential spectra of L and N. Finally, we prove the absence of the essential spectrum for our 2-simplex Laplacians L and N by using the comparison Theorem of Lewis.

2. Weighted geometric realization

In this section, we construct a new framework that's we call the weighted geometric realization of 2 and 3-simplexes.

Let V the set of vertices at most countable, E the set of oriented edges and (V, E) a graph. We take E symmetric, that is, if $(x, y) \in E$, then $(y, x) \in E$. We take E irreflexive, that is, if $x \in E$, then $(x, x) \notin E$. Let (E^+, E^-) the partition of E. If $(x, y) \in E$, then $(x, y) \in E^+$ or $(x, y) \in E^-$. We have $(x, y) \in E^+$ if and only if $(y, x) \in E^-$. Orient the graph (V, E) means define the partition (E^+, E^-) of E. For $e = (x, y)$, we set $e^- = x$ and $e^+ = y$. The path between x and y is a finite set of oriented edges $e_1, e_2, e_3, ..., e_k$ such that $k \in \mathbb{N}^*, e_1^- = x, e_k^+ = y$ and for all $i \in \{1, 2, 3, ..., k-1\}, e_i^+ = e_{i+1}^-$. The simple path is a path where each edge appears only once time. The cycle is a path where the origin and the end are identical. The connected graph is a graph such that for all $x, y \in V$, there exists a path between x and y. The locally finite graph is a graph such that each vertice belongs to a finite number of edges.

In our paper, we work with a graph that's oriented, connected, irreflexive, symmetric and locally finite. The oriented 2-simplex is a surface surrounded by a simple cycle of length equals 3 and it is an element of V^3. $F = \left\{(x, y, z) \in V^3 \mid (x, y, z) \text{ is an oriented 2-simplex}\right\}$ is the set of oriented 2-simplexes. The oriented 3-simplex is a volume surrounded by four oriented 2-simplexes and it is an element of V^4. $T = \{(x, y, z, t) \in V^4 \mid (x, y, z, t)$ is an oriented 3-simplex$\}$ is the set of oriented 3-simplexes. The odd permutation means we change the positions of two vertices an odd number of times. The even permutation means we change the positions of two vertices an even number of times. Let $(\alpha, \beta) \in F^2$ or $(\alpha, \beta) \in T^2$. We have $\alpha = \beta$ if we use the even permutation to pass from α to β. We have $\alpha = -\beta$ if we use the odd permutation to pass from α to β. The geometric realization of 2 and 3-simplexes, denoted by R, is the pair (F, T). We define the weight on F by $w_F : F \to \mathbb{R}_+^*$. We define the weight on T by $w_T : T \to \mathbb{R}_+^*$. The weighted geometric realization of 2 and 3-simplexes, denoted by R_w, is the quadruplet (F, T, w_F, w_T) that's equals to (R, w_F, w_T).

3. Nonself-adjoint 2-simplex Laplacians

On our weighted geometric realization, we construct a new nonself-adjoint 2-simplex Laplacian L and we define its Dirichlet 2-simplex Laplacian and its adjoint 2-simplex Laplacian. After that, we establish the Green's formula associated to our new nonself-adjoint 2-simplex Laplacian L.

We define the following functional spaces associated to our weighted geometric realization R_w:

The cochains set of dimension 2, denoted by C^F, is defined as

$$C^F = \{f : F \to \mathbb{C} \mid f(-(x, y, z)) = -f(x, y, z)\}.$$

We set

$$C_s^F = \left\{f \in C^F \mid f \text{ has a finite support}\right\}.$$

For $(f, g) \in C^F \times C^F$, we define an inner product on C^F as

$$\langle f, g \rangle_F = \frac{1}{6} \sum_{(x,y,z) \in F} w_F(x, y, z) f(x, y, z) \overline{g(x, y, z)}.$$

Then

$$\|f\|_F = \sqrt{\langle f, f \rangle_F}.$$

The Hilbert space associated to F, denoted by $B(F)$, is given by

$$B(F) = \left\{ f \in C^F \mid \|f\|_F < \infty \right\}.$$

Now, we give the definition of our new nonself-adjoint 2-simplex Laplacian.

Definition 3.1. The 2-simplex Laplacian, denoted by L, is defined as

$$L : C_s^F \to C_s^F,$$

such that for all $f \in C_s^F$ and $(x, y, z) \in F$,

$$Lf(x,y,z) = \frac{1}{w_F(x,y,z)} \sum_{t \in V} w_T(x,y,z,t)\left(f(x,y,z) - f(y,z,t)\right).$$

Dirichlet operator: Let Ω be a subset of F, $f \in C_s^\Omega$ and $g : F \to \mathbb{C}$ be the extension of f to F by setting $g = 0$ outside Ω. For any operator A on C_s^Ω, the Dirichlet operator A_Ω^D is defined as

$$A_\Omega^D(f) = A(g)\mid_\Omega.$$

In the next Theorem, we introduce the adjoint 2-simplex Laplacian L' of the 2-simplex Laplacian L.

Theorem 3.2. *The adjoint* 2*-simplex Laplacian* L' *of the* 2*-simplex Laplacian* L *is given by*

$$L' : C_s^F \to C_s^F,$$

such that for all $f \in C_s^F$ *and* $(x, y, z) \in F$,

$$L'f(x,y,z) = \frac{1}{w_F(x,y,z)} \times \left(\sum \sum_{t \in V} w_T(x,y,z,t) f(x,y,z) - \sum_{t \in V} w_T(t,x,y,z) f(y,z,t) \right).$$

Proof. For all $f, g \in C_s^F$, we have

$$\begin{aligned} \langle Lg, f \rangle_F &= \frac{1}{6} \sum_{(x,y,z,t) \in T} w_T(x,y,z,t)\left(g(x,y,z) - g(y,z,t)\right) \overline{f(x,y,z)} \\ &= \frac{1}{6} \sum_{(x,y,z) \in F} g(x,y,z) \sum_{t \in V} w_T(x,y,z,t) \overline{f(x,y,z)} \\ &\quad - \frac{1}{6} \sum_{(t,x,y,z) \in T} w_T(t,x,y,z)\, g(x,y,z) \overline{f(y,z,t)} \end{aligned}$$

$$= \frac{1}{6} \sum_{(x,y,z)\in F} g(x,y,z)$$
$$\times \left(\sum_{t\in V} w_T(x,y,z,t)\, \overline{f(x,y,z)} - \sum_{(t,x,y,z)\in T} w_T(t,x,y,z)\, \overline{f(y,z,t)} \right).$$

Since

$$\langle Lg, f\rangle_F = \langle g, L'f\rangle_F ,$$

we obtain

$$L'f(x,y,z) = \frac{1}{w_F(x,y,z)}$$
$$\times \left(\sum_{t\in V} w_T(x,y,z,t)\, f(x,y,z) - \sum_{t\in V} w_T(t,x,y,z)\, f(y,z,t) \right).$$

This completes the proof. □

We develop a condition based on the weight of 3-simplexes F, this condition will be important for the sequel of our paper.

Condition 1: We take

$$w^+(x,y,z) = w^-(x,y,z), \forall (x,y,z) \in F$$

such that

$$w^+(x,y,z) = \sum_{t\in V} w_T(x,y,z,t) \text{ and } w^-(x,y,z) = \sum_{t\in V} w_T(t,x,y,z).$$

Corollary 3.3. *We suppose that the Condition* 1 *is satisfied. Then the* 2-*simplex Laplacian* L' *is simply given by*

$$L'f(x,y,z) = \frac{1}{w_F(x,y,z)} \sum_{t\in V} w_T(t,x,y,z)\,(f(x,y,z) - f(y,z,t)).$$

Next, we introduce the symmetric 2-simplex Laplacian Δ with a symmetric 3-simplex weight function.

Definition 3.4. The symmetric 2-simplex Laplacian, denoted by Δ, is defined as

$$\Delta : C_s^F \to C_s^F,$$

such that for all $f \in C_s^F$ and $(x,y,z) \in F$,

$$\begin{aligned}\Delta f(x,y,z) &= \left(L + L'\right) f(x,y,z) \\ &= \frac{1}{w_F\left(x,y,z\right)} \sum_{t \in V} a(x,y,z,t)\left(f\left(x,y,z\right) - f\left(y,z,t\right)\right),\end{aligned}$$

with

$$a(x,y,z,t) = w_T(x,y,z,t) + w_T(t,x,y,z)$$

the symmetric 3-simplex weight function.

For the symmetric 2-simplex Laplacian Δ, we define its quadratic form denoted by Q_L as

$$Q_L\left(f\right) = \langle Lf, f\rangle_F + \overline{\langle Lf, f\rangle_F},\ f \in C_s^F.$$

Moreover, we have

$$Q_L\left(f\right) = 2\mathcal{R}e\left\langle Lf, f\right\rangle_F.$$

Then, we obtain

$$\inf_{\|f\|_F = 1} Q_L\left(f\right) = \inf_{\|f\|_F = 1} 2\mathcal{R}e\left\langle Lf, f\right\rangle_F. \tag{3.1}$$

Theorem 3.5. (*Green's Formula*) *We suppose that the Condition* 1 *is satisfied. Then*

$$\begin{aligned}\langle Lf, g\rangle_F + \overline{\langle Lg, f\rangle_F} = \frac{1}{6} \sum_{(x,y,z,t) \in T} & w_T(x,y,z,t)\left(f\left(x,y,z\right) - f\left(y,z,t\right)\right) \\ & \times \overline{\left(g\left(x,y,z\right) - g\left(y,z,t\right)\right)},\end{aligned}$$

for all $f, g \in C_s^F$.

Proof. Let $f, g \in C_s^F$. We have

$$\begin{aligned}&\langle Lf, g\rangle_F + \overline{\langle Lg, f\rangle_F} \\ &= \langle \Delta f, g\rangle_F \\ &= \frac{1}{6} \sum_{(x,y,z,t) \in T} w_T(x,y,z,t)\left(f\left(x,y,z\right) - f\left(y,z,t\right)\right)\overline{g\left(x,y,z\right)} \\ &\quad + \frac{1}{6} \sum_{(t,x,y,z) \in T} w_T(t,x,y,z)\left(f\left(x,y,z\right) - f\left(y,z,t\right)\right)\overline{g\left(x,y,z\right)} \\ &= \frac{1}{6} \sum_{(x,y,z,t) \in T} w_T(x,y,z,t)\left(f\left(x,y,z\right) - f\left(y,z,t\right)\right)\overline{g\left(x,y,z\right)} \\ &\quad + \frac{1}{6} \sum_{(x,y,z,t) \in T} w_T(x,y,z,t)\left(f\left(x,y,z\right) - f\left(y,z,t\right)\right)\overline{g\left(y,z,t\right)}.\end{aligned}$$

Then, we get

$$\langle Lf,g\rangle_F+\overline{\langle Lg,f\rangle_F}=\frac{1}{6}\sum_{(x,y,z,t)\in T}w_T(x,y,z,t)\left(f\left(x,y,z\right)-f\left(y,z,t\right)\right)\\ \times\overline{\left(g\left(x,y,z\right)-g\left(y,z,t\right)\right)}.$$

This completes the proof. □

4. Sectoriality of the 2-simplex Laplacians

In this section, we propose general conditions to ensure sectoriality for our self-adjoint 2-simplex Laplacian L. After that, we analyze the essential spectrum via the notion of sectoriality. In order to study the notion of sectoriality for the 2-simplex Laplacian L, we present the following condition:

Condition 2: There exists a positive constant P such that for all $(x,y,z)\in F$, we have

$$\sum_{t\in V}\left|w_T\left(x,y,z,t\right)-w_T\left(t,x,y,z\right)\right|\leq Pw_F\left(x,y,z\right).$$

We define the following operator on C_s^F:

$$\left(L-L'\right)f\left(x,y,z\right)=\frac{1}{w_F\left(x,y,z\right)}\sum_{t\in V}\left(w_T\left(x,y,z,t\right)-w_T\left(t,x,y,z\right)\right)\\ \times\left(f\left(x,y,z\right)-f\left(y,z,t\right)\right),$$

for all $f\in C_s^F$ and $(x,y,z)\in F$.

Theorem 4.1. *We suppose that the Conditions* 1 *and* 2 *are satisfied. Then the operator* $(L-L')$ *extends to a unique bounded operator on* $B(F)$.

Proof. Let $f,g\in C_s^F$. We use the Condition 1, then we get

$$\langle Lf,g\rangle_F-\left\langle L'f,g\right\rangle_F=\frac{1}{6}\sum_{(x,y,z)\in F}w^+\left(x,y,z\right)f\left(x,y,z\right)\overline{g\left(x,y,z\right)}\\ -\frac{1}{6}\sum_{(x,y,z)\in F}w^-\left(x,y,z\right)f\left(x,y,z\right)\overline{g\left(x,y,z\right)}\\ +\frac{1}{6}\sum_{(x,y,z)\in F}\overline{g\left(x,y,z\right)}\\ \times\sum_{t\in V}\left(w_T\left(x,y,z,t\right)-w_T\left(t,x,y,z\right)\right)f\left(y,z,t\right).$$

By the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\left|\left\langle \left(L-L'\right) f, g\right\rangle_F\right| \\
&\leq \sum_{(x,y,z)\in F} \left|\overline{g\left(x,y,z\right)}\right| \sum_{t\in V} \left|w_T\left(t,x,y,z\right)-w_T\left(x,y,z,t\right)\right| \left|f\left(y,z,t\right)\right| \\
&\leq \sum_{(x,y,z)\in F} \left|\overline{g\left(x,y,z\right)}\right| \left(\sum_{t\in V} \left|w_T\left(t,x,y,z\right)-w_T\left(x,y,z,t\right)\right|\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{t\in V} \left|w_T\left(t,x,y,z\right)-w_T\left(x,y,z,t\right)\right| \left|f\left(y,z,t\right)\right|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

From the Condition 2, we obtain

$$
\begin{aligned}
&\left|\left\langle \left(L-L'\right) f, g\right\rangle_F\right| \\
&\leq \left(P \sum_{(x,y,z)\in F} w_F\left(x,y,z\right) \left|\overline{g\left(x,y,z\right)}\right|^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{(x,y,z)\in F} \sum_{t\in V} \left|w_T\left(t,x,y,z\right)-w_T\left(x,y,z,t\right)\right| \left|f\left(y,z,t\right)\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(P \sum_{(x,y,z)\in F} w_F\left(x,y,z\right) \left|\overline{g\left(x,y,z\right)}\right|^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{t\in V} \sum_{(x,y,z)\in F} \left|w_T\left(t,x,y,z\right)-w_T\left(x,y,z,t\right)\right| \left|f\left(y,z,t\right)\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(P \sum_{(x,y,z)\in F} w_F\left(x,y,z\right) \left|\overline{g\left(x,y,z\right)}\right|^2\right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{t\in V} \left|f\left(y,z,t\right)\right|^2 \sum_{(x,y,z)\in F} \left|w_T\left(t,x,y,z\right)-w_T\left(x,y,z,t\right)\right|\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \left(P \sum_{(x,y,z)\in F} w_F(x,y,z) \left|\overline{g(x,y,z)}\right|^2 \right)^{\frac{1}{2}}$$
$$\times \left(P \sum_{t\in V} w_F(y,z,t) \left|f(y,z,t)\right|^2 \right)^{\frac{1}{2}}.$$

Therefore, we find

$$\left\|\left(L-L'\right) f\right\|_F \leq P \left\|f\right\|_F, \ \ \forall f \in C_s^F.$$

Then, we conclude by the Hahn-Banach Theorem. □

Definition 4.2. (1) The numerical range of an operator T with domain $D(T)$, denoted by $W(T)$, is the nonempty set

$$W(T) = \{\langle Tf, f\rangle \mid f \in D(T) \ \text{ and } \ \|f\| = 1\}.$$

(2) Let B be a Hilbert space, an operator $T : D(T) \to B$ is said to be sectorial if $W(T)$ lies in a sector

$$S_{a,\theta} = \{z \in \mathbb{C} \mid \arg(z-a) \leq \theta\},$$

for some $a \in \mathbb{R}$ and $\theta \in \left[0, \frac{\pi}{2}\right)$.

In the next Theorem, we propose general conditions to ensure sectoriality for our self-adjoint 2-simplex Laplacian L.

Theorem 4.3. *We suppose that the Conditions* 1 *and* 2 *are satisfied. Then* 2*-simplex Laplacian* L *is sectorial.*

Proof. We apply the Green's formula, then we find

$$\begin{aligned} 2\mathcal{R}e \left\langle Lf, f\right\rangle_F &= \left\langle Lf, f\right\rangle_F + \left\langle L'f, f\right\rangle_F \\ &= \sum_{(x,y,z,t)\in T} w_T(x,y,z,t) \left|f(x,y,z) - f(y,z,t)\right|^2 \\ &\geq 0. \end{aligned}$$

We use the Theorem 4.1, we get that $\operatorname{Im}(f,f)$ is bounded by $\frac{P}{2}$. So, the imaginary part is bounded and the real part of the numerical range is positive. Then 2-simplex Laplacian L is sectorial such that a is any point in the open half line of the negative real part. □

Remark 4.4. A sectorial operator is closable as an operator if it is densely defined, see Theorem $V-3.4$ in [9].

We use the Theorem 4.1 with the Remark above, we get the following result.

Corollary 4.5. *If the Conditions* 1 *and* 2 *are satisfied, then* 2*-simplex Laplacian* L *is closable.*

5. Absence of the essential spectrum

In this section, we consider a self-adjoint 2-simplex Laplacian N. After that, we study the relation between the essential spectra of L and N. Finally, we show the absence of the essential spectrum for our 2-simplex Laplacians L and N.

Definition 5.1. The closure of L is the operator $\overline{L}$, defined by

- $D\left(\overline{L}\right) = \left\{ f \in B(F) \mid \exists (f_n)_{n\in\mathbb{N}} \in C_s^F, f_n \to f \text{ and } Lf_n \text{ converges} \right\}$,
- $\overline{L}(f) = \lim\limits_{n\to\infty} Lf_n, f \in D\left(\overline{L}\right)$ and $(f_n)_n \in C_s^F$ such that $f_n \to f$.

Definition 5.2. The 2-simplex Laplacian N is defined on $D\left(\overline{L}\right) \cap D\left(L'\right)$ as:

$$\begin{aligned} Nf(x,y,z) &= \frac{1}{2}\left(\overline{L} + L'\right) f(x,y,z) \\ &= \frac{1}{w_F(x,y,z)} \sum_{t\in V} \frac{w_T(x,y,z,t) + w_T(t,x,y,z)}{2} \\ &\quad \times \left(f(x,y,z) - f(y,z,t)\right), \end{aligned}$$

for all $f \in C_s^F$ and $(x,y,z) \in F$.

Theorem 5.3. *The* 2*-simplex Laplacian* N *is a symmetric extension of the symmetric operator* G *such that* $G = \frac{1}{2}\left(L + L'\right)$*.*

Proof. For all $f \in C_s^F$, we have

$$Nf = \overline{G} f$$

and

$$\overline{L}' + \left(L'\right)' \subset \left(\overline{L} + L'\right)'.$$

Since N' is an extension of 2-simplex Laplacian N, 2-simplex Laplacian N is symmetric. Moreover, we have

$$\begin{aligned} N &= \frac{1}{2}\left(\overline{L} + L'\right) \\ &= \frac{1}{2}\left(\left(L'\right)' + \overline{L}'\right) \\ &\subset \frac{1}{2}\left(L' + \overline{L}\right)' \\ &= N'. \end{aligned}$$

Therefore the 2-simplex Laplacian N is a symmetric extension of the symmetric operator G such that $G = \frac{1}{2}(L + L')$. $\square$

Proposition 5.4. *We suppose that the Conditions* 1 *and* 2 *are satisfied. Then*

$$D(\overline{L}) \subset D(L')$$

and $(N, D(\overline{L}))$ *is a closed operator.*

Proof. From the Condition 2, we get that $L - L'$ is extended to a unique bounded operator $\pi = \overline{L - L'}$ on $B(F)$. Since

$$L = (L - L') + L',$$

we obtain $\overline{L} = \pi + \overline{L'}$. So, we find

$$D(\overline{L}) \subset D(\overline{L'}) \subset D(L').$$

As $N = \overline{L} - \frac{1}{2}\pi$, we have N is closed. $\square$

Proposition 5.5. *We suppose that the Conditions* 1 *and* 2 *are satisfied. If* G *is essentially self-adjoint, then* $(N, D(\overline{L}))$ *is a self-adjoint operator.*

Proof. We have N is a symmetric closed extension of G. $\square$

Definition 5.6. The essential spectrum of a closed operator T, denoted by $\sigma_{ess}(T)$, is the set of all complex numbers λ for which the range $R(T - \lambda)$ is not closed or $\dim\ker(T - \lambda) = \infty$.

We define the following numbers:

$$m(T) = \inf\{\mathcal{R}e(\lambda) \,|\, \lambda \in W(T)\},$$
$$\beta^{ess}(T) = \inf\{\mathcal{R}e(\lambda) \,|\, \lambda \in \sigma_{ess}(T)\}.$$

If T is a bounded operator, then the spectrum is always a subset of the closure of the numerical range but this is not true in general. In fact, the essential spectrum of a closed operator is a subset of the closure of the numerical range. Therefore, for the 2-simplex Laplacian $\overline{L}$ we get

$$\beta^{ess}(\overline{L}) \geq m(\overline{L}).$$

The next theorem follows from part (IV), Theorem 1.11 in [4]. We compare the essential spectrum of L and the essential spectrum of its real part by using the sectoriality of L.

Theorem 5.7. *We suppose that the Conditions* 1 *and* 2 *are satisfied. If* G *is essentially self-adjoint, then*

$$\beta^{ess}(\overline{L}) \geq \inf \sigma_{ess}(T).$$

We define the Cheeger constants on $\Omega \subset F$ as:

$$h(\Omega) = \inf_{\substack{U \subset \Omega \\ \text{finite}}} \frac{w_T(\partial_T U)}{w_F(U)}$$

and

$$\widetilde{h}(\Omega) = \inf_{\substack{U \subset \Omega \\ \text{finite}}} \frac{w_T(\partial_T U)}{w^+(U)},$$

where for a subset U of F

$$w_T(\partial_T U) = \sum_{(x,y,z,t) \in \partial_T U} w_T(x,y,z,t),$$

$$w^+(U) = \sum_{(x,y,z) \in U} w^+(x,y,z) \quad \text{and} \quad w_F(U) = \sum_{(x,y,z) \in U} w_F(x,y,z)$$

$\partial_T U = \{(x,y,z,t) \in T \mid ((y,z,t) \in U$ and $(t,z,x),(x,y,t),(z,y,x) \notin U)$ or $((t,z,x) \in U$ and $(y,z,t),(x,y,t),(z,y,x) \notin U)$ or $((x,y,t) \in U$ and $(y,z,t),(t,z,x),(z,y,x) \notin U)$ or $((z,y,x) \in U$ and $(y,z,t),(t,z,x),(x,y,t) \notin U)\}$.

Definition 5.8. A weighted geometric realization $K = (F_K, T_K)$ is called a subweighted geometric realization of $R_w = (F_{R_w}, T_{R_w})$ if $F_K \subset F_{R_w}$ and $T_K = \{(x,y,z,t) \mid (y,z,t),(t,z,x),(x,y,t),(z,y,x) \in F_K\}$.

Definition 5.9. A filtration of $R_w = (F,T)$ is a sequence of finite subweighted geometric realizations $\{R_w^n = (F_n, T_n) \mid n \in \mathbb{N}\}$ such that $R_w^n \subset R_w^{n+1}$ and $\cup_{n \geq 1} F_n = F$.

Let R_w be a weighted geometric realization and $\{R_w^n \mid n \in \mathbb{N}\}$ a filtration of R_w. We set

$$P_{F_n^c} = \sup \left\{ \frac{w^+(x,y,z)}{w_F(x,y,z)} \mid (x,y,z) \in F_n^c \right\}.$$

The isoperimetric constant at infinity is defined by:

$$h_\infty = \lim_{n \to \infty} h(F_n^c).$$

Theorem 5.10. *The bottom of the real part of $W(L_\Omega^D)$ satisfies the following inequality:*

$$\frac{h^2(\Omega)}{8} \leq P_\Omega m\left(L_\Omega^D\right) \leq \frac{1}{2} P_\Omega h(\Omega), \tag{5.1}$$

with $\Omega \subset F$.

Proof. Let $\Omega \subset F$. We use the works of Dodziuk [3] and Grigoryan [7], we can deduce the following bounds of the symmetric quadratic form $Q_{L_\Omega^D}$ on C_s^Ω:

$$\frac{h^2(\Omega)}{8} \leq P_\Omega \inf_{\|f\|_F=1} Q_{L_\Omega^D}(f) \leq \frac{1}{2} P_\Omega h(\Omega).$$

We apply the equality (3.1), then, we obtain

$$\frac{h^2(\Omega)}{8} \leq P_\Omega m\left(L_\Omega^D\right) \leq \frac{1}{2} P_\Omega h(\Omega).$$

□

In the sequel, we suppose that the Conditions 1 and 2 are satisfied and G is essentially self-adjoint.

Lemma 5.11. *Let R_w be a weighted geometric realization. Then the bottom of the spectrum $\lambda_1\left(N_\Omega^D\right)$ of N_Ω^D satisfies*

$$\lambda_1\left(N_\Omega^D\right) = m\left(L_\Omega^D\right),$$

for all subset Ω of F.

Proof. Let Ω be a subset of F. We have $\lambda_1\left(N_\Omega^D\right)$ the bottom of the spectrum λ_1 of the Dirichlet 2-simplex Laplacian N_Ω^D on Ω, satisfies the variational definition:

$$\begin{aligned}
\lambda_1\left(N_\Omega^D\right) &= \lambda_1\left(\overline{G}_\Omega^D\right) \\
&= \inf_{\substack{f \in C_s^\Omega \\ \|f\|_F=1}} \left\langle G_\Omega^D f, f\right\rangle_F \\
&= \inf_{\substack{f \in C_s^\Omega \\ \|f\|_F=1}} \mathcal{R}e\left\langle L_\Omega^D f, f\right\rangle_F \\
&= m\left(L_\Omega^D\right).
\end{aligned}$$

Then

$$\lambda_1\left(N_\Omega^D\right) = m\left(L_\Omega^D\right),$$

for all subset Ω of F. □

In the next Theorem, we give the Cheeger's Theorem associated to the self-adjoint Laplacian N.

Theorem 5.12. *Let R_w be a weighted geometric realization. Then*

$$P_\Omega \lambda_1\left(N_\Omega^D\right) \geq \frac{h^2(\Omega)}{8},$$

for all subset Ω of F.

Proof. We apply the Lemma 5.11, we get

$$\lambda_1\left(N_\Omega^D\right) = m\left(L_\Omega^D\right).$$

By using inequality (5.1), we obtain that

$$P_\Omega \lambda_1\left(N_\Omega^D\right) \geq \frac{h^2(\Omega)}{8},$$

for all subset Ω of F. □

Our aim in the following is to express the dependence of the essential spectrum of N to the geometry at infinity.

Theorem 5.13. *Let R_w be a weighted geometric realization and $\{R_w^n \mid n \in \mathbb{N}\}$ be a filtration of R_w. Then, we get*

$$\lim_{n\to\infty}\left(\lim_{\substack{k\to\infty\\ k\geq n+1}} \lambda_1\left(N^D_{R_w^k \setminus R_w^n}\right)\right) = \lambda_1^{ess}(N).$$

Proof. We set

$$l = \lim_{n\to\infty}\left(\lim_{\substack{k\to\infty\\ k\geq n+1}} \lambda_1\left(N^D_{R_w^k \setminus R_w^n}\right)\right).$$

We have the sequences $\left(R_w^k \setminus R_w^n\right)_{k\geq n+1}$ and $((R_w^n)^c)_n$ are monotone. Then our limits exist. Moreover, we have for all $n \in \mathbb{N}$, $\left(R_w^k \setminus R_w^n\right)_{k\geq n+1}$ is a sequence of finite subweighted geometric realizations whose union is equal to $(R_w^n)^c$. By using Theorem 2.3.6 [1], we obtain

$$\lambda_1\left(N^D_{(R_w^n)^c}\right) = \lim_{k\to\infty}\left(\lambda_1\left(N^D_{R_w^k\setminus R_w^n}\right)\right).$$

We apply Proposition 1 [10], we get

$$l = \lambda_1^{ess}(N).$$

□

Now, we show the absence of the essential spectrum.

Theorem 5.14. *Let R_w be a weighted geometric realization and $\{R_w^n \mid n \in \mathbb{N}\}$ be a filtration of R_w. If there exists a sequence $(c_n)_n$ such that for all $k \geq n+1$*

$$\frac{h^2\left(R_w^k \setminus R_w^n\right)}{8P_{R_w^k\setminus R_w^n}} \geq c_n \text{ and } \lim_{n\to\infty} c_n = \infty \tag{5.2}$$

then $\sigma_{ess}(N)$ is empty.

Proof. We apply Theorem 5.12, we get

$$\lambda_1\left(N^D_{R^k_w\setminus R^n_w}\right)\geq c_n,$$

for all $k\geq n+1$. Therefore, we find

$$\lim_{n\to\infty}\left(\lim_{\substack{k\to\infty\\ k\geq n+1}}\lambda_1\left(N^D_{R^k_w\setminus R^n_w}\right)\right)\geq\lim_{n\to\infty}c_n.$$

Now, we use Theorem 5.13, we obtain that $\sigma_{ess}(N)$ is empty. □

In order to study the relationship between the essential spectrum of L and the essential spectrum of its real part. We will use The Comparison Theorem of Lewis [4]. Now, we begin by giving the definition considered by Lewis: The essential spectrum of a closed operator densely defined T is the set of all complex number λ for which $T-\lambda I$ has a singular sequence.

Theorem 5.15. (**Comparison Theorem of Lewis**) *Let T be a closed linear operator in a Hilbert space B with dense domain $D(T)$. Let A be a self-adjoint operator in B bounded from below and with $D(T)\subset D(A)$. If*

$$\mathcal{R}e\left\langle Tu,u\right\rangle\geq\left\langle Au,u\right\rangle,\forall u\in D(T)$$

then

$$\sigma_{ess}(T)\subseteq\left\{\lambda\in\mathbb{C}\mid\mathcal{R}e(\lambda)\geq\inf\sigma_{ess}(A)\right\}.$$

If $\sigma_{ess}(A)=\emptyset$, then $\sigma_{ess}(T)=\emptyset$.

We apply the Comparison Theorem of Lewis, we find the following consequence:

Proposition 5.16. *Let R_w be a weighted geometric realization. Then, we have*

$$\sigma_{ess}\left(\overline{L}\right)\subseteq\left\{\lambda\in\mathbb{C}\mid\mathcal{R}e(\lambda)\geq\inf\sigma_{ess}(N)\right\}$$

and if $\sigma_{ess}(N)=\emptyset$, then $\sigma_{ess}\left(\overline{L}\right)=\emptyset$.

Theorem 5.17. *Let R_w be a weighted geometric realization which satisfies the Hypothesis* (5.2)*. Then $\sigma_{ess}\left(\overline{L}\right)=\emptyset$.*

Proof. We use Theorem 5.14 and Proposition 5.16, we obtain the result. □

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