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STABILITY OF PICARD ITERATION IN METRIC SPACES

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Abstract. In this paper we establish a general result for the stability of Picard iteration. Several theorems in the literature are obtained as special cases.

1. Introduction

Let (X, d) be a complete metric space and the T be a a selfmapping of X. Let $x_{n+1} = f(T, x_n)$ be some iteration procedure in X. Suppose that $F(T)$, the fixed point set of T, is nonempty and that x_n converges to a point $q \in F(T)$. Let $\{y_n\} \subset X$, and define

$$\epsilon_n = d(y_{n+1}, f(T, y_n)).$$

If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$, then the iteration procedure

$$x_{n+1} = f(T, x_n)$$

is said to be T-stable. If these conditions hold for $x_{n+1} = Tx_n$; i.e., Picard iteration, then we shall say that Picard iteration is T-stable.

In this paper, we shall obtain sufficient conditions that Picard iteration is T stable for an arbitrary selfmap, and then demonstrate that a number of contractive conditions are Picard T-stable.

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2. Main results

We shall need the following lemma from [2].

Lemma 1. *Let* $\{x_n\}, \{\epsilon_n\}$ *be nonnegative sequences satisfying*

$$x_{n+1} \leq hx_n + \epsilon_n$$

for all $n \in \mathbb{N}, 0 \leq h < 1, \lim_{n\to\infty} \epsilon_n = 0$. *Then* $\lim_{n\to\infty} x_n = 0$.

Theorem 1. *Let* (X, d) *be a nonempty complete metric space,* T *a selfmap of* X *with* $F(T) \neq \emptyset$. *If there exist numbers* $L \geq 0, 0 \leq h < 1$ *such that*

$$d(Tx, q) \leq Ld(x, Tx) + hd(x, q) \tag{1}$$

for each $x \in X, q \in F(T)$, *and, in addition,* $\lim_{n\to\infty} d(x_{n+1}, Tx_n) = 0$ *and*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0, \tag{2}$$

then Picard iteration is T*-stable.*

Proof. Let $\{y_n\} \subset X$, $\epsilon_n = d(y_{n+1}, Ty_n)$, and $\lim_{n\to\infty} \epsilon_n = 0$. We need to show that $\lim_{n\to\infty} y_n = q$. From the conditions (1) and (2), we have

$$\begin{aligned} d(y_{n+1}, q) &\leq d(y_{n+1}, Ty_n) + d(Ty_n, q) \\ &\leq \epsilon_n + Ld(y_n, Ty_n) + hd(y_n, q). \end{aligned}$$

By Lemma 1, we have $\lim_{n\to\infty} y_n = q$. □

Theorem 2. *Let* (X, d) *be a nonempty complete metric space,* T *a selfmap of* X *satisfying: there exist* $0 \leq h < 1$, *and positive integers* p, q *such that, for each* $x, y \in X$,

$$\begin{aligned} d(T^p x, T^q y) \leq h \max\{&d(x, y), d(x, T^p x), d(y, T^q y), \\ &d(x, T^q y), d(y, T^p x)\}. \end{aligned} \tag{3}$$

Then Picard iteration is T*-stable.*

Proof. From Theorem 11 of [3], T has a unique fixed point. It remains to show that (2) is satisfied.

Let p_n be the diameter of the orbit of x_n;
that is, $p_n = \delta(O(x_n, Tx_n, \ldots))$. For any $i, j \geq n$, using (3),

$$\begin{aligned} d(T^p y_i, T^q y_j) &\leq h \max\{d(y_i, y_j), d(y_i, T^p y_i), d(y_j, T^q y_j), \\ &\qquad d(y_i, T^q y_j), d(y_j, T^p y_i)\} \\ &\leq hp_n. \end{aligned}$$

But

$$\begin{aligned} d(y_i, y_j) &\leq d(y_i, T^p y_{i-1}) + d(T^p y_{i-1}, T^q y_{j-1}) + d(T^q y_{j-1}, y_j) \\ &\leq \epsilon_{i-1} + hp_{n-1} + \epsilon_{i-1}, \end{aligned}$$

which implies that

$$p_n \leq 2\epsilon_{i-1} + hp_{n-1},$$

and $\lim_{n\to\infty} p_n = 0$ by Lemma 1. Since $d(y_n, Ty_n) \leq p_n$, $\lim_{n\to\infty} d(y_n, Ty_n) = 0$. The conclusion now follows from Theorem 1. □

Corollary 1. ([5], Theorem 1) *Let (X, d) be a nonempty complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \tag{4}$$

for all $x, y \in X$, where $L \geq 0, 0 \leq a < 1$. Suppose that T has a fixed point p. Let $x_0 \in X$ and $x_{n+1} = Tx_n$. Let $\{y_n\} \subset X$ and define $\epsilon_n = d(y_{n+1}, y_n)$. Then the Picard iteration is T-stable.

Proof. Since T satisfies (1) for all $x, y \in X$, T satisfies inequality (1) of our paper. From the proof of Theorem 1 of [5], $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Therefore, by our Theorem 1, Picard iteration is T-stable. □

Remark. Definition (3) of this paper is actually definition (74) of [3]. Therefore many contractive conditions are special cases of (3), and, for each of these, Picard iteration is T stable. For example, Theorems 1 and 2 of [1] and Theorem 1 of [4] are special cases of Theorem 2.

We shall not examine the analogues of Theorem 1 for Mann, Ishikawa, Kirk, or any other iteration scheme since, if one obtains convergence to a fixed point for a map using Picard iteration, there is no point in considering any other more complicated iteration procedure.

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