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A NOTE ON DEGENERATE LAH-BELL POLYNOMIALS ARISING FROM DERIVATIVES

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Abstract. Recently, Kim-Kim introduced Lah-Bell polynomials and numbers, and investigated some properties and identities of these polynomials and numbers. Kim studied Lah-Bell polynomials and numbers of degenerate version. In this paper, we study degenerate Lah-Bell polynomials arising from differential equations. Moreover, we investigate the phenomenon of scattering of the zeros of these polynomials.

1. Introduction

Definition 1.1. The unsigned Lah number $L_{n,k}$ counts the number of ways a set of n elements can be partioned into k nonempty linearly ordered subsets, and have an explicit formula (see [1, 13, 15, 16, 24, 25]).

$$L_{n,k} = \binom{n-1}{k-1}\frac{n!}{k!} = \binom{n}{k}\frac{(n-1)!}{(k-1)!}. \tag{1.1}$$

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The generating function of $L_{n,k}$ is defined by (see [13, 15, 24])

$$\frac{1}{k!}\left(\frac{t}{1-t}\right)^k=\sum_{n=k}^{\infty}L_{n,k}\frac{t^n}{n!}\ \ (k\geq 0). \tag{1.2}$$

Recently, Kim-Kim (see [13]) introduced the Lah-Bell polynomials as follows:

$$e^{\frac{tx}{1-t}}=\sum_{n=0}^{\infty}B_n^L(x)\frac{t^n}{n!}. \tag{1.3}$$

The degenerate exponential function is defined by (see [19–22])

$$e_\lambda^x(t)=(1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda(t):=e_\lambda^1(t)=(1+\lambda t)^{\frac{1}{\lambda}}, \tag{1.4}$$

where λ is a nonzero parameter.

Note that

$$\lim_{\lambda\to 0}e_\lambda^x(t)=\lim_{\lambda\to 0}(1+\lambda t)^{\frac{x}{\lambda}}=e^{xt}.$$

Since $e_\lambda^x(t)$ defined in (1.4) is infinitely differentiable at $t=0$, Taylor expansion of $e_\lambda^x(t)$ at $t=0$ gives the following series form (see [20–23]):

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \tag{1.5}$$

where $(x)_{n,\lambda}$ is defined by

$$(x)_{n,\lambda}=\begin{cases}1, & n=0,\\ \Big(x-(n-1)\lambda\Big)(x)_{n-1,\lambda}, & n\geq 1.\end{cases}$$

As is well known, the Stirling numbers of the first kind are given by (see [1–5, 10, 14])

$$(x)_n=\sum_{l=0}^{n}S_1(n,l)x^l, \tag{1.6}$$

where $(x)_n$ are defined by

$$(x)_n=\begin{cases}1, & n=0,\\ \Big(x+1-n\Big)(x)_{n-1}, & n\geq 1.\end{cases}$$

From (1.6), we easily get

$$\frac{1}{k!}\left(\ln(1+t)\right)^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!}. \tag{1.7}$$

As an inversion formula of (1.6), the Stirling numbers of the second kind are given by (see [1–5, 10])

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{1.8}$$

From (1.8), we get

$$\frac{1}{k!}\left(e^t - 1\right)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}. \tag{1.9}$$

Moreover, as degenerate version of (1.6) and (1.8), the degenerate Stirling numbers of the first and second kinds, respectively, are given by

$$\begin{aligned} (x)_n &= \sum_{l=0}^{n} S_{1,\lambda}(n,l)(x)_{l,\lambda}, \\ (x)_{n,\lambda} &= \sum_{l=0}^{n} S_{2,\lambda}(n,l)(x)_l \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \frac{1}{k!}\left(\ln_\lambda(1+t)\right)^k &= \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!}, \\ \frac{1}{k!}\left(e_\lambda(t) - 1\right)^k &= \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!}, \end{aligned} \tag{1.11}$$

where $\ln_\lambda(t) = \frac{t^\lambda - 1}{\lambda}$ (see [6–12, 17, 18]).

Recently, Kim-Kim (see [13]) introduced the degenerate Lah-Bell polynomials which are given by the generating function to be

$$e_\lambda^x\left(\frac{1}{1-t} - 1\right) = \sum_{n=0}^{\infty} B_{n,\lambda}^L(x)\frac{t^n}{n!}. \tag{1.12}$$

The rest of this paper is organized as follows. In the section 2, we study the differential equations on degenerate Lah-Bell polynomials. Using these differential equations, we derive some identities and properties of the degenerate Lah-Bell polynomials. In the section 3, we investigate the phenomenon of scattering of the zeros of those polynomials. Finally, in section 4, a summary of the Lah-Bell polynomials is given.

2. Some identities of the degenerate Lah-Bell polynomial

In this section, we derive some identities of the degenerate Lah-Bell polynomials. When $x = 1$, $B_{n,\lambda}^L := B_{n,\lambda}^L(1)$ are called the degenerate Lah-Bell numbers. The following theorem gives an explicit expression of the Lah-Bell polynomials.

Theorem 2.1. *For non-negative integer $n \geq 0$, we have*

$$B_{n,\lambda}^{L}(x) = \sum_{k=0}^{n} (x)_{k,\lambda}\ L_{n,k}. \tag{2.1}$$

Proof. Combining (1.12) and (1.5) and using the fact (1.2), one can obtain the following relation (see [13]):

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}^{L}(x)\frac{t^n}{n!} = e_{\lambda}^{x}\left(\frac{t}{1-t}\right) &= \sum_{k=0}^{\infty}(x)_{k,\lambda}\frac{1}{k!}\left(\frac{t}{1-t}\right)^{k} \\
&= \sum_{k=0}^{\infty}(x)_{k,\lambda}\sum_{n=k}^{\infty} L_{n,k}\frac{t^n}{n!} \qquad (2.2)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(x)_{k,\lambda}L_{n,k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Since the set $\left\{1, t, t^2, \cdots, t^n, \cdots\right\}$ are linearly independent, the relation (2.1) is satisfied. □

Combining Theorem 2 and the definition of Lah number (1.1), one can have the following corollary.

Corollary 2.2. *For non-negative integer $n \geq 0$, we have*

$$B_{n,\lambda}^{L}(x) = n!\sum_{k=0}^{n}\frac{(x)_{k,\lambda}}{k!}\binom{n-1}{n-k}.$$

Proof. Substituting (1.1) into (2.1) and simplifying it, one can complete the proof. □

The following theorem gives the relation between Lah-Bell polynomials and the Stirling numbers of first kind.

Theorem 2.3. *For non-negative integer $n \geq 0$, we have*

$$B_{n,\lambda}^{L}(x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\lambda^{l-k}x^{k}S_1(l,k)L_{n,l}.$$

In particular, for $x = 1$, we have

$$B_{n,\lambda}^{L} = \sum_{l=0}^{n}\sum_{k=0}^{l}\lambda^{l-k}S_1(l,k)L_{n,l}.$$

Proof. Using (1.4) and (1.7), we get

$$\begin{aligned}
e_\lambda^x\left(\frac{t}{1-t}\right) &= \left(1+\frac{\lambda t}{1-t}\right)^{\frac{x}{\lambda}} \\
&= e^{\frac{x}{\lambda}\ln\left(1+\frac{\lambda t}{1-t}\right)} \\
&= \sum_{k=0}^{\infty}\frac{\lambda^{-k}x^k}{k!}\left(\ln\left(1+\frac{\lambda t}{1-t}\right)\right)^k \\
&= \sum_{k=0}^{\infty}\lambda^{-k}x^k\sum_{l=k}^{\infty}S_1(l,k)\frac{\lambda^l}{l!}\left(\frac{t}{1-t}\right)^l \\
&= \sum_{k=0}^{\infty}\lambda^{-k}x^k\sum_{l=k}^{\infty}S_1(l,k)\lambda^l\sum_{n=l}^{\infty}L_{n,l}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\sum_{k=0}^{l}\lambda^{l-k}x^kS_1(l,k)L_{n,l}\Big)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

Hence, comparing Theorem 2.1 and (2.3) leads to completing of this proof. □

Theorem 2.4. *For non-negative integer $n \geq 0$, we have*

$$B_{n,\lambda}^L(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{l}{m}(x)_{l,\lambda}\frac{(-1)^{m-l}}{l!}\langle l\rangle_n,$$

where $\langle l\rangle_n$ is defined by

$$\langle l\rangle_n = \begin{cases} 1, & n=0, \\ (l+n-1)\langle l\rangle_{n-1}, & n\geq 1. \end{cases}$$

Proof. Combining (1.4) and (1.5), we can have

$$\begin{aligned}
\sum_{n=0}^{\infty}B_{n,\lambda}^L(x)\frac{t^n}{n!} &= e_\lambda^x\left(\frac{t}{1-t}\right) \\
&= \left(1+\frac{\lambda t}{1-t}\right)^{\frac{x}{\lambda}}
\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{1}{l!} \left(\frac{t}{1-t} \right)^{l} \\
&= \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{1}{l!} \sum_{m=0}^{l} \binom{l}{m} \left(\frac{1}{1-t} \right)^{m} (-1)^{l-m} \\
&= \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{1}{l!} \sum_{m=0}^{l} \binom{l}{m} \sum_{n=0}^{\infty} \langle m \rangle_{n} \frac{t^{n}}{n!} (-1)^{l-m} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{l}{m} (x)_{l,\lambda} \frac{(-1)^{l-m}}{l!} \langle m \rangle_{n} \frac{t^{n}}{n!}.
\end{aligned}
$$

By comparing the coefficients of the both sides of (2.4), we can finish this proof. □

Let $F(t,x)$ be a two variable function defined by

$$
F(t,x) := \sum_{n=0}^{\infty} B_{n,\lambda}^{L}(x) \frac{t^{n}}{n!}. \tag{2.5}
$$

Then, the k-th differentiation gives us the following:

$$
\begin{aligned}
\frac{\partial^{k}}{\partial t^{k}} F(t,x) &= \frac{\partial^{k}}{\partial t^{k}} \left(\sum_{n=0}^{\infty} B_{n,\lambda}^{L}(x) \frac{t^{n}}{n!} \right) \\
&= \sum_{n=k}^{\infty} B_{n,\lambda}^{L}(x) \frac{t^{n-k}}{(n-k)!} \qquad (2.6) \\
&= \sum_{n=0}^{\infty} B_{n+k,\lambda}^{L}(x) \frac{t^{n}}{n!}.
\end{aligned}
$$

Now, we observe that (see [16])

$$
\begin{aligned}
\frac{1}{1+\frac{\lambda t}{1-t}} &= \sum_{k=0}^{\infty} (-1)^{k} (\lambda)^{k} \left(\frac{t}{1-t} \right)^{k} \\
&= \sum_{k=0}^{\infty} (-1)^{k} \lambda^{k} \sum_{n=k}^{\infty} \binom{n-1}{n-k} t^{n} \qquad (2.7) \\
&= \sum_{n=0}^{\infty} n! \sum_{k=0}^{n} (-1)^{k} \lambda^{k} \binom{n-1}{n-k} \frac{t^{n}}{n!}
\end{aligned}
$$

and

$$(1-t)^{-2} = \sum_{l=0}^{\infty} \langle 2 \rangle_l \frac{t^l}{l!}. \tag{2.8}$$

From (2.7) and (2.8), we have

$$\begin{aligned}
&\frac{\partial}{\partial t} F(t,x) \\
&= \left(1 + \frac{\lambda t}{1-t}\right)^{\frac{x}{\lambda}-1} \frac{x}{(1-t)^2} \\
&= \left(\sum_{l=0}^{\infty} B_{l,\lambda}^{L}(x) \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} m! \sum_{k=0}^{m} \binom{m-1}{m-k} (-1)^k \lambda^k \frac{t^m}{m!}\right) \left(x \sum_{j=0}^{\infty} \langle 2 \rangle_j \frac{t^j}{j!}\right) \\
&= \left(\sum_{l=0}^{\infty} B_{l,\lambda}^{L}(x) \frac{t^l}{l!}\right) \left(\sum_{i=0}^{\infty} \left(\sum_{m=0}^{i} \sum_{k=0}^{m} \binom{i}{m} \binom{m-1}{m-k} m! (-1)^k \lambda^k x \langle 2 \rangle_{i-m} \frac{t^i}{i!}\right)\right) \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{m=0}^{i} \sum_{k=0}^{m} \binom{n}{i} \binom{i}{m} \binom{m-1}{m-k} m! (-1)^k \lambda^k x \langle 2 \rangle_{i-m} B_{n-i,\lambda}^{L}(x) \frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

Theorem 2.5. *For any real λ and non-negative integer $n \geq 0$, we have the following recurrence relation*

$$B_{n+1,\lambda}^{L}(x) = \sum_{i=0}^{n} \sum_{m=0}^{i} \sum_{k=0}^{m} \binom{n}{i} \binom{i}{m} \binom{m-1}{m-k} m! (-1)^k \lambda^k x \langle 2 \rangle_{i-m} B_{n-i,\lambda}^{L}(x).$$

Proof. Combining (2.6) and (2.9), we can complete this proof. □

Now, we observe that

$$\frac{\partial}{\partial x} F(t,x) = \frac{\partial}{\partial x} \left(1 + \frac{\lambda t}{1-t}\right)^{\frac{x}{\lambda}} = \frac{1}{\lambda} \left(1 + \frac{\lambda t}{1-t}\right)^{\frac{x}{\lambda}} \ln\left(1 + \frac{\lambda t}{1-t}\right).$$

By mathematical induction, we can obtain

$$\frac{\partial^k}{\partial x^k} F(t,x) = \frac{1}{\lambda^k} \left(1 + \frac{\lambda t}{1-t}\right)^{\frac{x}{\lambda}} \left(\ln\left(1 + \frac{\lambda t}{1-t}\right)\right)^k. \tag{2.10}$$

Since

$$\begin{aligned}\left(\ln\left(1+\frac{\lambda t}{1-t}\right)\right)^k &= k!\sum_{l=k}^{\infty} S_1(l,k)\frac{1}{l!}\left(\frac{t}{1-t}\right)^l \\ &= \sum_{m=k}^{\infty}\sum_{l=0}^{m}\frac{m!k!}{l!}S_1(l,k)\binom{m-1}{l-1}\frac{t^m}{m!},\end{aligned} \tag{2.11}$$

by (2.10) and (2.11), we get

$$\begin{aligned}&\frac{\partial^k}{\partial x^k}F(t,x) \\ &= \left(\frac{1}{\lambda^k}\sum_{m=k}^{\infty}\sum_{l=0}^{m}\frac{m!k!}{l!}S_1(l,k)\binom{m-1}{l-1}\frac{t^m}{m!}\right)\left(\sum_{j=0}^{\infty}B^L_{j,\lambda}(x)\frac{t^j}{j!}\right) \\ &= \sum_{n=k}^{\infty}\left(\sum_{m=0}^{n-k}\sum_{l=0}^{m}\frac{1}{\lambda^k}\binom{n}{m}\frac{m!k!}{l!}S_1(n,k)\binom{m-1}{l-1}B^L_{n-m,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

We obtain the following by k differentiations of the function $F(t,x)$ with respect to x:

$$\frac{\partial^k}{\partial x^k}F(t,x) = \frac{\partial^k}{\partial x^k}\sum_{m=0}^{\infty}B^L_{m,\lambda}(x)\frac{t^m}{m!} = \sum_{m=k}^{\infty}\frac{\partial^k}{\partial x^k}B^L_{m,\lambda}(x)\frac{t^m}{m!}. \tag{2.13}$$

Theorem 2.6. *For real number λ and non-negative integers n and k with $n \geq k$, we have*

$$\frac{\partial^k}{\partial x^k}B^L_{n,\lambda}(x) = \sum_{m=0}^{n-k}\sum_{l=0}^{m}\binom{n}{m}\frac{m!k!}{l!\lambda^k}S_1(n,k)\binom{m-1}{l-1}B^L_{n-m,\lambda}(x). \tag{2.14}$$

Proof. Comparing (2.12) and (2.13), one can proof the above recurrence relations. □

Theorem 2.7. *For $\lambda \in \mathbb{R}$ and non-negative integers $n, k \geq 0$, we have*

$$\frac{d^k}{dx^k}B^L_{n,\lambda}(x) = n!\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=0}^{l}\frac{1}{k!}\binom{n-1}{n-k}S_{2,\lambda}(k,l)S_1(l,m)(m)_k x^{m-k}.$$

Proof. Using (1.10) and Corollary 2.2, we have

$$\begin{aligned}
\frac{d^k}{dx^k}B_{n,\lambda}^L(x) &= \frac{d^k}{dx^k}n!\sum_{k=0}^{n}\frac{(x)_{k,\lambda}}{k!}\binom{n-1}{n-k} \\
&= n!\sum_{k=0}^{n}\frac{1}{k!}\binom{n-1}{n-k}\frac{d^k}{dx^k}(x)_{k,\lambda} \\
&= n!\sum_{k=0}^{n}\frac{1}{k!}\binom{n-1}{n-k}\frac{d^k}{dx^k}\sum_{l=0}^{k}S_{2,\lambda}(k,l)(x)_l \qquad (2.15) \\
&= n!\sum_{k=0}^{n}\frac{1}{k!}\binom{n-1}{n-k}\frac{d^k}{dx^k}\sum_{l=0}^{k}S_{2,\lambda}(k,l)\sum_{m=0}^{l}S_1(l,m)x^m \\
&= n!\sum_{k=0}^{n}\frac{1}{k!}\binom{n-1}{n-k}\sum_{l=0}^{k}\sum_{m=0}^{l}S_{2,\lambda}(k,l)S_1(l,m)(m)_k x^{m-k}.
\end{aligned}$$

This is the desired result of the theorem. □

3. Distribution of roots of the polynomials

Woon [26] has studied the distribution and structure of the zeros of Bernoulli polynomials. Hence, we will investigate the numerical patten of the zeros of the polynomials $B_{n,\lambda}^L(x)$. Using the mathematica tool, the polynomial $B_{n,\lambda}^L(x)$ can be determined explicitly. For example,

$$\begin{aligned}
B_{1,\lambda}^L(x) &= x, \\
B_{2,\lambda}^L(x) &= x^2 + (2-\lambda)x, \\
B_{3,\lambda}^L(x) &= x^3 + (6-3\lambda)x^2 + (6-6\lambda+2\lambda^2)x, \\
B_{4,\lambda}^L(x) &= x^4 + (12-6\lambda)x^3 + (36-36\lambda+11\lambda^2)x^2 \\
&\quad + (24-36\lambda+24\lambda^2-6\lambda^3)x.
\end{aligned}$$

From the definition of the Lah-Bell polynomials $B_{n,\lambda}^L(x)$, we can obtain the following properties of the roots:

- For any real number λ, the polynomials $B_{n,\lambda}^L(x)$ with $n = 1, 2$ have only real roots.
- For any real number λ and any positive integer n, all polynomials $B_{n,\lambda}^L(x)$ have a common root which is zero.

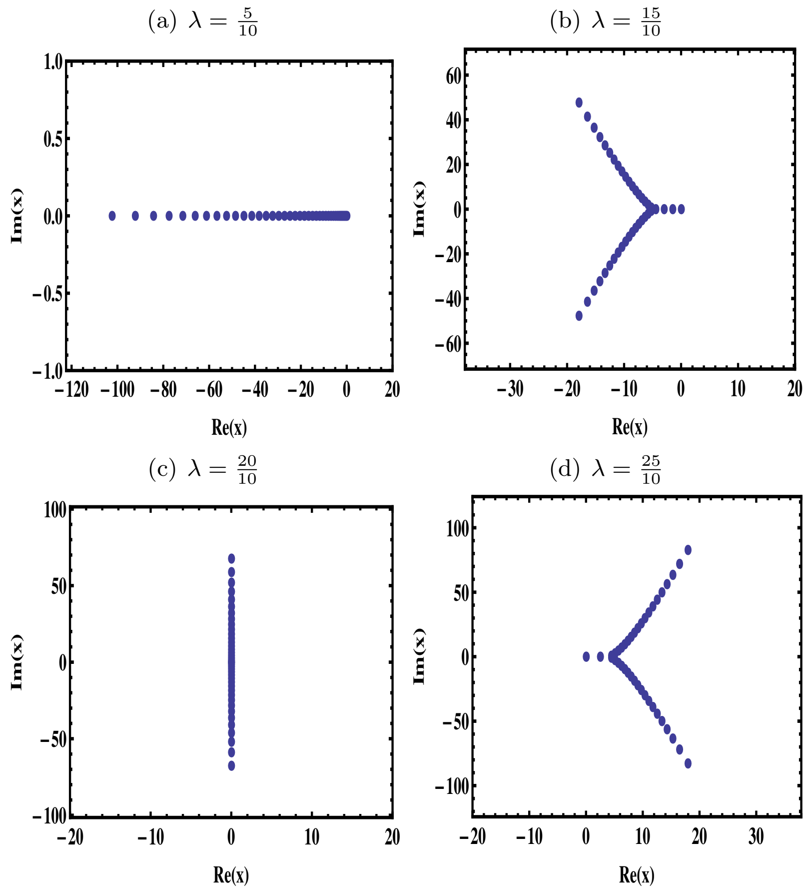

Fig. 1. The computed roots of $B^L_{40,\lambda}(x)$ with variable λ

Firstly, we observe the impact of λ on the distribution of the roots of the polynomials. For the propose, we fix the degree of polynomials as $n = 40$. Since the explicit form of the roots of $B^L_{n,\lambda}(x)$ is unknown, we calculate the roots by using the Mathematica tool with 100 working precision. The absolute numerical error is bounded as follows:

$$\sum_{i=1}^{40} |B_{40,\lambda}(x_i)| < 10^{-62},$$

where x_i denotes the root of polynomial. Hence, the results obtained from numerical computations are reliable. We compute the numerical roots of $B^L_{40,\lambda}(x)$ with four different parameters $\lambda = \frac{5}{10}, \frac{15}{10}, \frac{20}{10}$ and $\frac{25}{10}$ and the results are plotted in Fig. 1. As observed in Fig. 1, the roots of the Lah-Bell polynomials have four patterns.

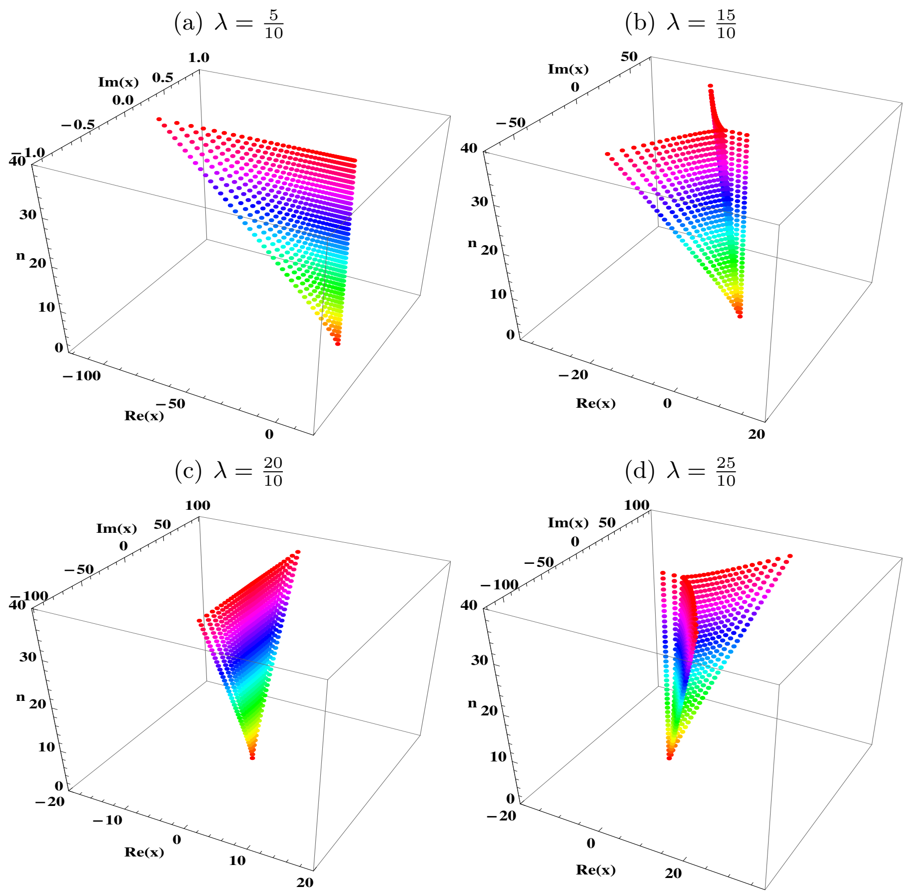

Fig. 2. The root distribution of $B^L_{n,\lambda}(x)$ with variable λ and different integer $n = 1, 2, \cdots, 40$.

Secondly, we investigate the impact of the degree of polynomials on the distribution of roots of the polynomials. We compute the numerical roots of the polynomials increasing the degree of polynomials from 1 to 40 and present in Fig. 2.

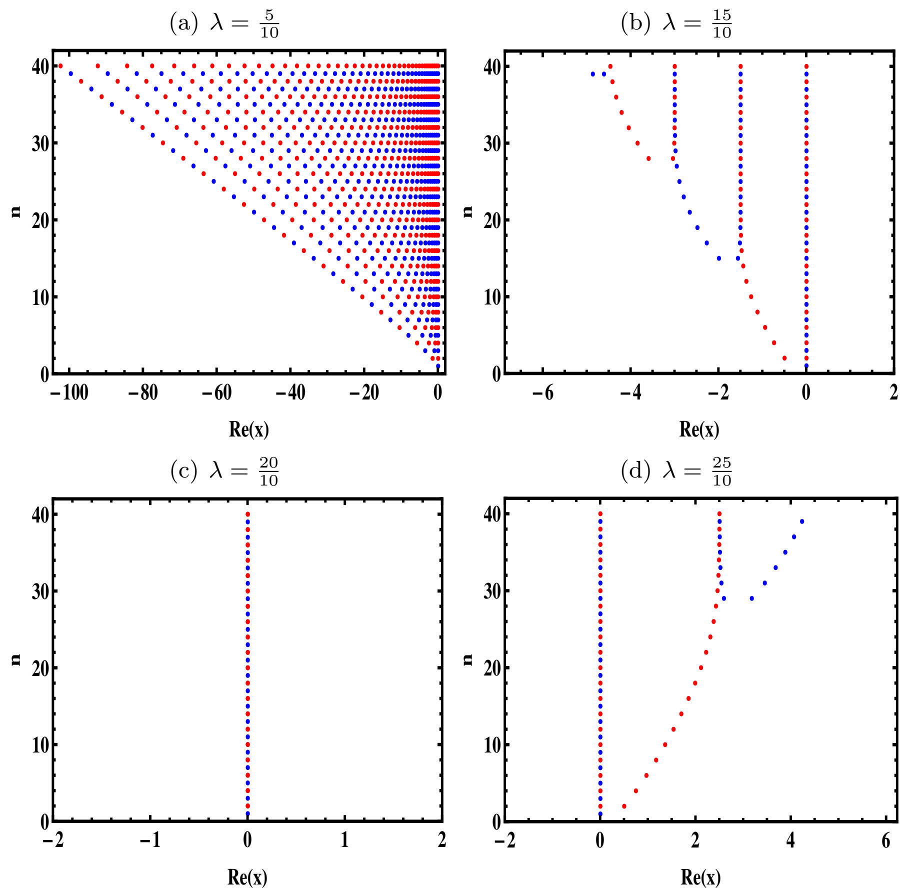

Fig. 3. Real zeros of $B^L_{n,\lambda}(x)$ for $\lambda = \frac{5}{10}, \frac{15}{10}, \frac{20}{10}, \frac{25}{10}$ and $1 \leq n \leq 40$.

Thirdly, to investigate the real roots distribution structure of $B^L_{n,\lambda}(x)$, we compute the real roots and display in Fig.3.

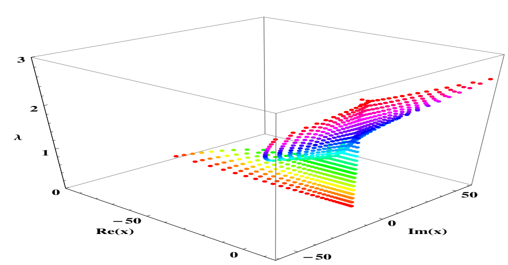

Fig. 4. Roots distribution structures vs $\lambda \in [0, \frac{25}{10}]$.

From the results of Fig.3, we can find a remarkably regular structure of the roots of the polynomials $B_{n,\lambda}(x)$. In order to find the roots structure, we count the number of real roots for $\lambda = \frac{5}{10}, \frac{15}{10}, \frac{20}{10}$ and $\frac{25}{20}$ and $n \in [1, 99]$ within $x \in [-1000, 1000]$ and summarize as follows:

- $\lambda = \frac{5}{10}$: the number of real roots=n.
- $\lambda = \frac{20}{10}$: the number of real roots=$\begin{cases} 1, & n = \text{odd}, \\ 2, & n = \text{even}. \end{cases}$
- $\lambda = \frac{15}{10}$: the number of real roots=$\begin{cases} 1, & n = 1, 3, \cdots, 13, \\ 2, & n = 2, 4, \cdots, 26, \\ 3, & n = 15, 17, \cdots, 37, \\ 4, & n = 28, 30, \cdots, 50, \\ 5, & n = 39, 41, \cdots, 63, \\ 6, & n = 52, 54, \cdots, 74, \\ 7, & n = 65, 67, \cdots, 87, \\ 8, & n = 76, 78, \cdots, 98, \\ 9, & n = 89, 91, \cdots, 99. \end{cases}$
- $\lambda = \frac{25}{10}$: the number of real roots=$\begin{cases} 1, & n = 1, 3, \cdots, 27, \\ 2, & n = 2, 4, \cdots, 50, \\ 3, & n = 29, 31, \cdots, 75, \\ 4, & n = 52, 54, \cdots, 100, \\ 5, & n = 77, 79, \cdots, 99. \end{cases}$

Finally, we compute the roots of the polynomials with fixed $n = 40$ and varying parameter $\lambda = \frac{k}{10}$, $k = 0, 1, \cdots, 25$. The numerical results are plotted in Fig. 4.

Summering the above discussion, we can obtain the properties of the roots of $B^L_{n,\lambda}(x)$.

- When $\lambda < 2$, the real parts of the roots of the polynomials $B^L_{n,\lambda}(x)$ are non-positive.
- When $\lambda = 2$, the polynomials $B^L_{n,\lambda}(x)$ have pure imaginary roots except for zero roots.
- When $\lambda > 2$, the real parts of the roots of the polynomials $B^L_{n,\lambda}(x)$ are non-negative.

4. Conclusion

In this paper, we review the Lah-Bell polynomials and numbers introduced by Kim-Kim and give an explicit formula for partial derivatives. In order to more accurate understanding the Lah-Bell polynomials, the distribution of roots was numerically investigated. Further, we count the number of real roots of $B^L_{n,\lambda}(x)$ with four different parameters $\lambda = \frac{5}{10}, \frac{15}{10}, \frac{20}{10}$ and $\frac{25}{10}$. Finally, we obtain the relation between the sign of real part of the root of $B^L_{n,\lambda}(x)$ and the value λ. In the next study, we will show theoretically the above facts.

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