**Nonlinear Functional Analysis and Applications**
**Vol. 26, No. 4 (2021), pp. 749-780**
**ISSN: 1229-1595(print), 2466-0973(online)**

https://doi.org/10.22771/nfaa.2021.26.04.07
http://nfaa.kyungnam.ac.kr/journal-nfaa


# CONVERGENCE AND STABILITY OF ITERATIVE ALGORITHM OF SYSTEM OF GENERALIZED IMPLICIT VARIATIONAL-LIKE INCLUSION PROBLEMS USING $(\theta, \varphi, \gamma)$-RELAXED COCOERCIVITY

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**Abstract.** In this paper, we give the notion of $M(.,.)$-$\eta$-proximal mapping for a nonconvex, proper, lower semicontinuous and subdifferentiable functional on Banach space and prove its existence and Lipschitz continuity. As an application, we introduce and investigate a new system of variational-like inclusions in Banach spaces. By means of $M(.,.)$-$\eta$-proximal mapping method, we give the existence of solution for the system of variational inclusions. Further, propose an iterative algorithm for finding the approximate solution of this class of variational inclusions. Furthermore, we discuss the convergence and stability analysis of the iterative algorithm. The results presented in this paper may be further expolited to solve some more important classes of problems in this direction.

[0]Received February 8, 2021. Revised June 5, 2021. Accepted June 8, 2021.
[0]2010 Mathematics Subject Classification: 47H05, 47H10, 47J25, 49J40.
[0]Keywords: Variational-like inclusion, $M(.,.)$-$\eta$-proximal mapping, 0-diagonally quasi-concave, iterative algorithm, convergence and stability.
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## 1. Introduction

Variational inequality theory has emerged as a powerful tool for studying a wide class of unrelated problems arising in various branches of physical, engineering, pure and applied sciences in a unified and general framework, see for example [2, 12, 14–16]. An important and useful generalization of variational inequalities is a variational inclusion. In 1994, Hassouni and Moudafi [17] discussed the approximation solvability of a new class of variational inequalities called variational inclusions. Since then Adly [1], Huang *et al.* [15], Ding [8,9], Ding and Feng [10], Ding and Lou [11], Ding and Xia [13] and Kazmi [16], have obtained some important extensions of the result [17].

In 2005, Kazmi and Bhat [20] introduced and studied some properties of $P$-$\eta$-proximal mapping, for a nonconvex, proper, lower semicontinuous and subdifferentiable functional on Banach space. Further, using $P$-$\eta$-proximal mapping and Wiener-Hopf equation technique, they discussed convergence and stability of an iterative algorithm for a generalized multi-valued variational inclusion.

In 2008, Sun *et al.* [30] introduced the notion of $M$-proximal mapping on Hilbert space. Again in 2009, Kazmi *et al.* [22] introduced $M$-proximal mapping, an extension of $P$-proximal mapping introduced in [13] and studied its some properties.

One of the important aspects in variational inequality theory is to study the convergence analysis and the stability of iterative algorithms. It is worth mentioning that in the recent past, stability of several iteration procedures for the functional equation of the type $Tu = f$ has been studied extensively by many authors, see for example Osilike [27] and the references cited therein. In 2000, Huang *et al.* [17] initiated the study of stability of iterative algorithms for a class of variational inequalities involving single-valued mappings in Hilbert space. Later, stability of iterative algorithms of various classes of variational inequalities (inclusions) have been discussed by many authors, see for example Liu *et al.* [26], Kazmi and Bhat [20], Kazmi *et al.* [21, 22], Kazmi and Khan [23] and the related references cited therein. As such the convergence and stability analysis of the iterative algorithms for some new classes of set-valued/multivalued variational inequalities (inclusions) is still an unexplored field.

Motivated and inspired by the above achievements, in this paper, we study a new system of generalized implicit variational-like inclusion problem in Banach spaces involving $M(.,.)$-$\eta$-proximal mapping for a nonconvex, proper, lower semicontinuous and subdifferentiable functional. We further construct an iterative algorithm with errors for approximating the solution of the system and discuss the convergence and stability of iterative sequence generated by the algorithm. The results presented in this paper improve and extend many known results in the literature, see for example [1, 3–6, 8–11, 13, 17–25, 29–31].

## 2. $M(.,.)$-$\eta$-proximal mapping and formulation of problem

Let $X$ be a real Banach space equipped with norm $\|.\|$, $X^\star$ be the topological dual space of $X, \langle\cdot,\cdot\rangle$ be the dual pair between $X$ and $X^\star$. Let $C(X)$ be the family of all nonempty compact subsets of $X$ and $2^X$ be the power set of $X$.

In the sequel, we need the following definitions and results from the literature.

**Definition 2.1.** $J : X \to 2^{X^\star}$ is said to be a normalized duality mapping, if it is defined by

$$J(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^2, \|x\| = \|f\|_{X^\star}\},\ \forall x \in X.$$

In the sequel, we shall denote a selection of normalized duality mapping $J$ by $j$. It is well known that if $X$ is smooth, then $J$ is single-valued and if $X \equiv H$, a real Hilbert space, then $J$ is an identity map.

**Definition 2.2.** ([6]) A Banach space $X$ is said to be *smooth,* if for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of $X$ is the function $\rho_X : [0,\infty) \to [0,\infty)$, defined by

$$\rho_X(\sigma) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X,\ \ \|x\| = 1,\ \ \|y\| = \sigma \right\}.$$

**Definition 2.3.** ([6]) A Banach space $X$ is said to be uniformly smooth if

$$\lim_{\sigma\to 0} \frac{\rho_X(\sigma)}{\sigma} = 0.$$

We note that a uniformly smooth Banach space is reflexive.

**Lemma 2.4.** ([6, 29]) *Let $X$ be a uniformly smooth Banach space and let $J : X \to X^\star$ be the normalized duality mapping. Then for all $x, y \in X$, we have*

(a) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle$;
(b) $\langle x-y, Jx-Jy\rangle \le 2d^2\rho_X(4\|x-y\|/d)$,
*where* $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$.

**Theorem 2.5.** (Nadler [23]) *Let $T : X \to CB(X)$ be a set-valued mapping on $X$ and $(X, d)$ be a complete metric space. Then we have*

(i) *for any given* $\mu > 0$, $x, y \in X$ *and* $u \in T(x)$, *there exists* $v \in T(y)$ *such that* $d(u,v) \leq (1+\mu)\mathcal{D}(T(x), T(y))$,

(ii) *if* $T : X \to C(X)$, *then (i) holds for* $\mu = 0$, *where* $C(X)$ *denotes the family of all nonempty compact subsets of* $X$.

**Lemma 2.6.** ([25]) *Let* $\{\zeta^n\}, \{\hbar^n\}$ *and* $\{c^n\}$ *be nonnegative sequences satisfying*

$$\zeta^{n+1} \leq (1-\omega^n)\zeta^n + \omega^n \hbar^n + c^n, \ \forall n \geq 0,$$

*where* $\{\omega^n\}_{n=0}^{\infty} \subset [0,1]$, $\sum\limits_{n=0}^{\infty} \omega^n = +\infty$, $\lim\limits_{n\to\infty} \hbar^n = 0$ *and* $\sum\limits_{n=0}^{\infty} c^n < \infty$. *Then* $\lim\limits_{n\to\infty} \zeta^n = 0$.

**Definition 2.7.** The Hausdorff metric $\mathcal{D}(\cdot,\cdot)$ on $CB(X)$, is defined by

$$\mathcal{D}(S,T) = \max\left\{\sup_{u\in S}\inf_{v\in T} d(u,v), \ \sup_{v\in T}\inf_{u\in S} d(u,v)\right\}, \ S, T \in CB(X),$$

where $d(\cdot,\cdot)$ is the induced metric on $X$ and $CB(X)$ denotes the family of all nonempty, closed and bounded subsets of $X$.

**Definition 2.8.** ([31]) A functional $f : X \times X \to R \cup \{+\infty\}$ is said to be 0-*diagonally quasi-concave* (in short, 0-$DQCV$) in the first argument, if for any finite set $\{x_1, \cdots, x_n\} \subset X$ and for any $y = \sum\limits_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum\limits_{i=1}^{n} \lambda_i = 1$, $\min\limits_{1\leq i\leq n} f(x_i, y) \leq 0$ holds.

**Lemma 2.9.** ([12]) *Let* $G$ *be a nonempty convex subset of a topological vector space and* $f : G \times G \to [-\infty, +\infty]$ *be such that*

(i) *for each* $x \in G$, $y \to f(x,y)$ *is lower semicontinuous on each compact subset of* $G$;

(ii) *for each* $y \in G$, $f(x,y)$ *is* 0-$DQCV$ *in* $x$;

(iii) *there exists a nonempty convex subset* $G_0$ *of* $G$ *and a nonempty compact subset* $K$ *of* $G$ *such that for each* $y \in G\backslash K$, *there exists* $x \in c_0(G_0 \cup \{y\})$ *satisfying* $f(x,y) > 0$, *where* $c_0(X)$ *denotes the convex hull of set* $X$.

*Then there exists* $\tilde{y} \in G$ *such that* $f(x, \tilde{y}) \leq 0$ *for all* $x \in G$.

**Definition 2.10.** ([9]) Let $\eta : X \times X \to X$ be a single-valued mapping. A proper functional $\phi : X \to R \cup \{+\infty\}$ is said to be $\eta$-*subdifferentiable* at a point $x \in X$, if there exists a point $f^\star \in X^\star$ such that

$$\phi(y) - \phi(x) \geq (f^\star, \eta(y,x)), \ \forall y \in X,$$

where $f^\star$ is called *$\eta$-subgradient* of $\phi$ at $x$. The set of all $\eta$-subgradients of $\phi$ at $x$ is denoted by $\partial\phi(x)$. The mapping $\partial\phi : X \to 2^{X^\star}$ defined by

$$\partial\phi(x) = \{f^\star \in X^\star : \phi(y) - \phi(x) \geq (f^\star, \eta(y,x))\,,\ \forall y \in X\}$$

is called *$\eta$-subdifferential* of $\phi$ at $x$.

**Definition 2.11.** ([22]) Let $X_1, X_2$ be real Banach spaces, let $S_1 : X_1 \times X_2 \to X_1$ and $S_2 : X_1 \times X_2 \to X_2$. Let $T : X_1 \times X_2 \to X_1 \times X_2$ be defined as $T(x,y) = (S_1(x,y), S_2(x,y))$ for any $(x,y) \in X_1 \times X_2$, and let $(x_0, y_0) \in X_1 \times X_2$. Assume that $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n) = (g(S_1, x_n, y_n), g(S_2, x_n, y_n))$ defines an iteration procedure which yields a sequence of points $\{(x_n, y_n)\} \in X_1 \times X_2$. Suppose that $F(T) = \{(x,y) \in X_1 \times X_2 : (x,y) = T(x,y)\} \neq \emptyset$ and $\{(x_n, y_n)\}$ converges to some $(p,q) \in F(T)$. Let $\{(u_n, v_n)\}$ be an arbitrary sequence in $X_1 \times X_2$ and $\epsilon_n = \|(u_{n+1}, v_{n+1}) - f(T, x_n, y_n)\|$, for all $n \geq 0$. If $\lim\limits_{n\to\infty} \epsilon_n = 0$ implies that $\lim\limits_{n\to\infty} (u_n, v_n) = (p,q)$, then the iteration procedure defined by $(x_{n+1}, y_{n+1}) = f(T, x_n, y_n)$ is said to be $T$-stable or stable with respect to T. If $\sum\limits_{n=0}^{\infty} \epsilon_n < +\infty$ implies that $\lim\limits_{n\to\infty} (u_n, v_n) = (p,q)$, then the iteration procedure $\{(x_n, y_n)\}$ is said to be almost $T$-stable.

**Definition 2.12.** Let $\eta : X \times X \to X$, $A, B : X \to X$ be single-valued mappings and let $M : X \times X \to X^\star$ be a nonlinear mapping. Then

(i) $M(A,.)$ is said to be *$\alpha$-strongly $\eta$-monotone* with respect to $A$ if there exists a constant $\alpha > 0$ satisfying

$$\langle M(Ax,u) - M(Ay,u), \eta(y,x)\rangle \geq \alpha\|x-y\|^2,\ \forall x,y,u \in X;$$

(ii) $M(.,B)$ is said to be *$\beta$-relaxed $\eta$-monotone* with respect to $B$ if there exists a constant $\beta > 0$ satisfying

$$\langle M(u,Bx) - M(u,By), \eta(y,x)\rangle \geq -\beta\|x-y\|^2,\ \forall x,y,u \in X;$$

(iii) $M(.,.)$ is said to be *$\alpha\beta$-symmetric $\eta$-monotone* with respect to $A$ and $B$ if $M(A,.)$ is *$\alpha$-strongly $\eta$-monotone* with respect to $A$ and $M(.,B)$ is *$\beta$-relaxed $\eta$-monotone* with respect to $B$ with $\alpha > \beta$ and $\alpha = \beta$ if and only if $x = y$ for all $x, y \in X$;

(iv) $M(.,.)$ is said to be *$(\xi_1, \xi_2)$-mixed Lipschitz continuous* if there exist constants $\xi_1, \xi_2 > 0$ satisfying

$$\|M(x,u) - M(y,v)\| \leq \xi_1\|x-y\| + \xi_2\|u-v\|,\ \forall x,y,u,v \in X.$$

**Definition 2.13.** Let $\eta : X \times X \to X$ and $A, B : X \to X$ be single-valued mappings. Let $\phi : X \to R \cup \{+\infty\}$ be a proper and $\eta$-subdifferentiable (not necessarily convex) functional and $M : X \times X \to X^\star$ be a nonlinear mapping.

If for any given point $x^\star \in X^\star$ and $\rho > 0$, there exists a unique point $x \in X$ satisfying

$$\langle M(Ax, Bx) - x^\star, \eta(y, x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0,\ \forall y \in X,$$

then the mapping $x^\star \to x$, denoted by $R^{\partial\phi, M(A,B)}_{\rho,\eta}(x^\star)$, is called $M(.,.)$-$\eta$-proximal mapping of $\phi$.

Clearly, we have $x^\star - M(Ax, Bx) \in \rho\partial\phi(x)$ and then it follows that

$$R^{\partial\phi, M(A,B)}_{\rho,\eta}(x^\star) = (M(A, B) + \rho\partial\phi)^{-1}(x^\star).$$

**Remark 2.14.** (i) If $\eta(y, x) = y - x$ for all $x, y \in X$ and $\phi$ is a proper and subdifferential functional on $X$ then the $M(.,.)$-$\eta$-proximal mapping of $\phi$ reduces to the $M(.,.)$-proximal mapping of $\phi$ discussed by Kazmi *et al.* [19].

(ii) If further in (i) above, $M(A, B) = P$, where $P : E \to E^\star$ is a nonlinear mapping, then $M(.,.)$-proximal mapping of $\phi$ reduces to $P$-proximal mapping of $\phi$ discussed in [13].

(iii) If $X \equiv H$, a Hilbert space, $\eta(y, x) = y - x$ for all $x, y \in X$ and $\phi$ is a proper, convex and lower semicontinuous functional on $X$ and $M(.,.)$ is the identity mapping on $H$, then the $M(.,.)$-proximal mapping of $\phi$ reduces to the usual proximal (resolvent) mapping of $\phi$ on Hilbert space.

Now we prove the following result which guarantees the existence of $M(.,.)$-$\eta$-proximal mapping of a proper, lower semicontinuous and subdifferentiable functional $\phi$ on Banach space.

**Theorem 2.15.** *Let $X$ be a reflexive Banach space. Let $\eta : X \times X \to X$ be $\tau$-Lipschitz continuous such that $\eta(y, y') + \eta(y', y) = 0$ for all $y, y' \in X$, let $M : X \times X \to X^\star$ be $\alpha\beta$-symmetric $\eta$- monotone continuous with respect to $A$ and $B$, let for any given $x^\star \in X^\star$, the function $h(y, x) = \langle x^\star - M(Ax, Bx), \eta(y, x)\rangle$ be 0-DQCV in $y$ and let $\phi : X \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable functional, which may not be convex. Then for any given constant $\rho > 0$ and $x^\star \in X^\star$, there exists a unique $x \in X$ such that*

$$\langle M(Ax, Bx) - x^\star, \eta(y, x)\rangle \geq \rho\phi(x) - \rho\phi(y),\ \forall y \in X, \tag{2.1}$$

*that is, $x = R^{\partial\phi, M(A,B)}_{\rho,\eta}(x^\star)$.*

*Proof.* For any given $M : X \times X \to X^\star$, $\rho > 0$ and $x^\star \in X^\star$, define a functional $f : X \times X \to R \cup \{+\infty\}$ by $f(y, x) = \langle x^\star - M(Ax, Bx), \eta(y, x)\rangle + \rho\phi(x) - \rho\phi(y)$, $\forall y, x \in X$. Since $M(.,.)$ and $\eta$ are continuous and $\phi$ is lower semicontinuous, for any $y \in X$, the mapping $x \mapsto f(y, x)$ is lower semicontinuous on $X$. Next, claim that $f(y, x)$ satisfies condition (ii) of Lemma 2.9. Indeed, let there exist a finite set $\{y_1, \cdots, y_m\} \subset X$ and $x_0 = \sum_{i=1}^{m} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\langle x^\star - M(Ax_0, Bx_0), \eta(y_i, x_0)\rangle + \rho\phi(x_0) - \rho\phi(y_i) > 0,\ \forall i = 1, 2, \cdots, m.$$

Since $\phi$ is $\eta$-subdifferentiable at $x_0$, there exists a point $f^\star \in X^\star$ such that

$$\rho\phi(y_i) - \rho\phi(x_0) \geq \rho\langle f^\star, \eta(y_i, x_0)\rangle,\ \forall i = 1, 2, \cdots, m.$$

Hence we must have

$$\langle x^\star - M(Ax_0, Bx_0) - \rho f^\star, \eta(y_i, x_0)\rangle > 0.$$

On the other hand, since $h(y, x_0) = \langle x^\star - M(Ax_0, Bx_0) - \rho f^\star, \eta(y, x_0)\rangle$ is 0-$DQCV$ in $y$, we have

$$\begin{aligned} 0 &< \sum_{n=0}^{\infty} \lambda_i \langle x^\star - M(Ax_0, Bx_0) - \rho f^\star, \eta(y_i, x_0)\rangle \\ &= \langle x^\star - M(Ax_0, Bx_0) - \rho f^\star, \eta(x_0, x_0)\rangle = 0, \end{aligned}$$

which is a contradiction. Hence $f(y, x)$ satisfies condition (ii) of Lemma 2.9.

Now, take a fixed $\tilde{y} \in \text{dom}\ \phi$. Since $\phi$ is $\eta$-subdifferentiable at $\tilde{y}$, there exists $f^\star \in X^\star$ such that

$$\begin{aligned} f(\tilde{y}, x) &= \langle x^\star - M(Ax, Bx), \eta(\tilde{y}, x)\rangle + \rho\phi(x) - \rho\phi(\tilde{y}) \\ &\geq \langle M(A\tilde{y}, B\tilde{y}) - M(Ax, B\tilde{y}), \eta(\tilde{y}, x)\rangle \\ &\quad + \langle M(Ax, B\tilde{y}) - M(Ax, Bx), \eta(\tilde{y}, x)\rangle \\ &\quad + \langle x^\star - M(A\tilde{y}, B\tilde{y}), \eta(\tilde{y}, x)\rangle + \rho\langle f^\star, \eta(x, \tilde{y})\rangle \\ &\geq \alpha\|\tilde{y} - x\|^2 - \beta\|\tilde{y} - x\|^2 - (\|x^\star\| + \|M(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)\|\eta(\tilde{y}, x)\| \\ &\geq (\alpha - \beta)\|\tilde{y} - x\|^2 - \tau(\|x^\star\| + \|M(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)\|\tilde{y} - x\| \\ &= \|\tilde{y} - x\|[(\alpha - \beta)\|\tilde{y} - x\| - \tau(\|x^\star\| + \|M(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)]. \end{aligned}$$

Let $r = \frac{\tau}{(\alpha - \beta)}(\|x^\star\| + \|M(A\tilde{y}, B\tilde{y})\| + \rho\|f^\star\|)$, and $K = \{x \in X, \|\tilde{y} - x\| \leq r\}$. Then $G_0 = \{\tilde{y}\}$ and $K$ are both weakly compact convex subsets of $X$ and for each $x \in X \backslash K$, there exists $\tilde{y} \in c_0(G_0 \cup \{\tilde{y}\})$ such that $f(\tilde{y}, x) > 0$. Hence all the conditions of Lemma 2.9 are satisfied. By Lemma 2.9, there exists $\tilde{x} \in X$ such that $f(y, \tilde{x}) \leq 0$ for all $y \in X$, that is, for any given $x^\star \in X^\star$,

$$\langle M(A\tilde{x}, B\tilde{x}) - x^\star, \eta(y, \tilde{x})\rangle \geq \rho\phi(\tilde{x}) - \rho\phi(y),\ \forall y \in X.$$

Next, we show that $\tilde{x}$ is a unique solution of problem (2.1). Suppose that $\tilde{x}_1, \tilde{x}_2 \in X$ are any two solutions of problem (2.1). Then we have, for any given $x^\star \in X^\star$,

$$\langle M(A\tilde{x}_1, B\tilde{x}_1) - x^\star, \eta(y, \tilde{x}_1)\rangle \geq \rho\phi(\tilde{x}_1) - \rho\phi(y),\ \forall y \in X \tag{2.2}$$

and

$$\langle M(A\tilde{x}_2, B\tilde{x}_2) - x^\star, \eta(y, \tilde{x}_2)\rangle \geq \rho\phi(\tilde{x}_2) - \rho\phi(y),\ \forall y \in X. \tag{2.3}$$

Taking $y = \tilde{x}_2$ in (2.2) and $y = \tilde{x}_1$ in (2.3) and then adding the resulting inequalities, we obtain

$$\langle M(A\tilde{x}_1, B\tilde{x}_1) - M(A\tilde{x}_2, B\tilde{x}_2), \eta(\tilde{x}_2, \tilde{x}_1)\rangle \geq 0.$$

Since $\eta(y, y') + \eta(y', y) = 0$ for all $y, y' \in X$ and $M : X \times X \to X^\star$ is $\alpha\beta$-symmetric $\eta$- monotone continuous with respect to $A$ and $B$, we have

$$\begin{aligned}&\langle M(A\tilde{x}_1, B\tilde{x}_2) - M(A\tilde{x}_2, B\tilde{x}_2), \eta(\tilde{x}_1, \tilde{x}_2)\rangle\\&+ \langle M(A\tilde{x}_1, B\tilde{x}_1) - M(A\tilde{x}_1, B\tilde{x}_2), \eta(\tilde{x}_1, \tilde{x}_2)\rangle \leq 0,\end{aligned}$$

thus

$$\alpha\|\tilde{x}_1 - \tilde{x}_2\|^2 - \beta\|\tilde{x}_1 - \tilde{x}_2\|^2 \leq 0.$$

That is $(\alpha - \beta)\|\tilde{x}_1 - \tilde{x}_2\|^2 \leq 0$, hence we have $\tilde{x}_1 = \tilde{x}_2$. This completes the proof. □

**Remark 2.16.** Theorem 2.15 shows that for any $\alpha\beta$-symmetric $\eta$-monotone mapping $M : X \times X \to X^\star$ and $\rho > 0$, the $M(.,.)$-$\eta$-proximal mapping $R^{\partial\phi, M(A,B)}_{\rho,\eta} : X^\star \to X$ of a proper, lower semicontinuous and $\eta$-subdifferentiable functional $\phi$ is well defined and for each $x^\star \in X^\star, x = R^{\partial\phi, M(A,B)}_{\rho,\eta}(x^\star)$ is the unique solution of problem (2.1).

Now, we give the following important result which guarantees the Lipschitz continuity of the $M(.,.)$-$\eta$-proximal mapping.

**Theorem 2.17.** *Let $\eta : X \times X \to X$ be $\tau$-Lipschitz continuous such that $\eta(y, y') + \eta(y', y) = 0$ for all $y, y' \in X$, let $M : X \times X \to X^\star$ be $\alpha\beta$-symmetric $\eta$-monotone continuous with respect to $A$ and $B$, let for any given $x^\star \in X^\star$, the function $h(y, x) = \langle x^\star - M(Ax, Bx), \eta(y, x)\rangle$ be 0-DQCV in $y$, let $\phi : X \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable functional and let $\rho > 0$ be any given constant. Then the $M(.,.)$-$\eta$-proximal mapping $R^{\partial\phi, M(A,B)}_{\rho,\eta}$ of $\phi$ is $L$-Lipschitz continuous, where $L = \dfrac{\tau}{(\alpha - \beta)}$, that is, for any $x_1^\star, x_2^\star \in X^\star$,*

$$\|R^{\partial\phi, M(A,B)}_{\rho,\eta}(x_1^\star) - R^{\partial\phi, M(A,B)}_{\rho,\eta}(x_2^\star)\| \leq L\|x_1^\star - x_2^\star\|.$$

*Proof.* For any given $x_1^\star, x_2^\star \in X^\star$, we have $x_1 = R_{\rho,\eta}^{\partial\phi,M(A,B)}(x_1^\star)$ and $x_2 = R_{\rho,\eta}^{\partial\phi,M(A,B)}(x_2^\star)$ such that

$$\langle M(Ax_1, Bx_1) - x_1^\star, \eta(y, x_1)\rangle \geq \rho\phi(x_1) - \rho\phi(y), \ \forall\ y \in X \tag{2.4}$$

and

$$\langle M(Ax_2, Bx_2) - x_2^\star, \eta(y, x_2)\rangle \geq \rho\phi(x_2) - \rho\phi(y), \ \forall\ y \in X. \tag{2.5}$$

Taking $y = x_2$ in (2.4) and $y = x_1$ in (2.5) and then adding the resulting inequalities, we obtain

$$\langle M(Ax_1, Bx_1) - M(Ax_2, Bx_2), \eta(x_1, x_2)\rangle \leq \langle x_1^\star - x_2^\star, \eta(x_1, x_2)\rangle,$$

which implies

$$\langle M(Ax_1, Bx_2) - M(Ax_2, Bx_2), \eta(x_1, x_2)\rangle$$
$$+ \langle M(Ax_1, Bx_1) - M(Ax_1, Bx_2), \eta(x_1, x_2)\rangle \leq \|x_1^\star - x_2^\star\|\|\eta(x_1, x_2)\|.$$

Since $M(.,.)$ is $\alpha\beta$-symmetric $\eta$-monotone continuous with respect to $A$ and $B$,

$$\alpha\|x_1 - x_2\|^2 - \beta\|x_1 - x_2\|^2 \leq \tau\|x_1^\star - x_2^\star\|\|x_1 - x_2\|$$

which implies

$$\|x_1 - x_2\| \leq L\|x_1^\star - x_2^\star\|. \tag{2.6}$$

Thus

$$\|R_{\rho,\eta}^{\partial\phi,M(A,B)}(x_1^\star) - R_{\rho,\eta}^{\partial\phi,M(A,B)}(x_2^\star)\| \leq L\|x_1^\star - x_2^\star\|.$$

□

**Definition 2.18.** Let $N : X_1 \times X_2 \times X_1 \times X_2 \to X_1, h : X_2 \to X_1$ be mappings. Then the mapping $N$ is called

(i) $\xi$-$h$-cocoercive in the second argument if there exists a constant $\xi > 0$ such that for all $x, y \in X_1, u, v, z \in X_2$,

$$\langle N(x, u, y, z) - N(x, v, y, z), h(u) - h(v)\rangle_1$$
$$\geq \xi\|N(x, u, y, z) - N(x, v, y, z)\|_1^2.$$

(ii) $(\theta, \varphi, \gamma)$-$h$-relaxed cocoercive in the fourth argument if there exist non-negative constants $\theta, \varphi$ and $\gamma$ such that for all $y, u, v \in X_2, x, z \in X_1$

$$\langle N(x, y, z, u) - N(x, y, z, v), h(u) - h(v)\rangle_1$$
$$\geq -\theta\|N(x, y, z, u) - N(x, y, z, v)\|_1^2 - \varphi\|h(u) - h(v)\|_1^2 + \gamma\|u - v\|_2^2.$$

(iii) Lipschitz continuous in the first argument if there exists a constant $\mu > 0$ such that for all $u, y, v \in X_1, x, z \in X_2$

$$\|N(u, x, y, z) - N(v, x, y, z)\|_1 \leq \mu\|u - v\|_1.$$

Similarly, we can define the Lipschitz continuity of $N$ in other arguments.

**Definition 2.19.** Let $X$ be *uniformly smooth* Banach space. Let $S : X \to X^\star, A, B : X \to X$, $\mathcal{H} : X \times X \to X^\star, \eta : X \times X \to X$ be single-valued mappings. Then

(i) $S$ is said to be $\eta$-*accretive*, if

$$\big\langle Sx - Sy, J\big(\eta(x, y)\big)\big\rangle \geq 0,\ \forall x, y \in X,$$

(ii) $A$ is said to be *strictly* $\eta$-*accretive*, if $A$ is $\eta$-*accretive* and equality holds if and only if $x = y$,

(iii) $\mathcal{H}(A, .)$ is said to be $\alpha$-*strongly* $\eta$-*accretive* with respect to $A$ if there exists a constant $\alpha > 0$ such that

$$\big\langle \mathcal{H}(Ax, z) - \mathcal{H}(Ay, z), J(\eta(x, y))\big\rangle \geq \alpha\|x - y\|^2,\ \forall x, y, z \in X,$$

(iv) $\mathcal{H}(., B)$ is said to be $\beta$-*relaxed* $\eta$-*accretive* with respect to $B$ if there exists a constant $\beta > 0$ such that

$$\big\langle \mathcal{H}(z, Bx) - \mathcal{H}(z, By), J(\eta(x, y))\big\rangle \geq -\beta\|x - y\|^2,\ \forall x, y, z \in X,$$

(v) $\mathcal{H}(., .)$ is said to be $d_1$-Lipschitz continuous with respect to $A$ if there exists a constant $d_1 > 0$ such that

$$\|\mathcal{H}(Ax, z) - \mathcal{H}(Ay, z)\| \leq d_1\|x - y\|,\ \forall x, y, z \in X,$$

(vi) $\eta$ is said to be $\tau$-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\big\|\eta(x, y)\big\| \leq \tau\|x - y\|,\ \forall x, y \in X.$$

Now, we formulate our main problem.

Let for each $i = 1, 2, j \in \{1, 2\}\setminus i$, $g_i : X_i \to X_i, \eta_i : X_i \times X_i \to X_i$ be single-valued mappings, $N_i : X_i^\star \times X_j^\star \times X_i^\star \times X_j^\star \to X_i^\star, Q_i : X_j^\star \times X_j^\star \to X_i^\star, E_i : X_i \times X_j \to X_j^\star, P_i : X_i \times X_j \to X_j^\star$ be single-valued mappings, let $S_i, T_i, G_i, F_i : X_i \to C(X_i^\star)$ be multi-valued mappings such that $u_i \in S_i(x_i), v_i \in T_i(x_i), w_i \in G_i(x_i), t_i \in F_i(x_i)$, let $\phi_i : X_i \to R \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta_i$-subdifferentiable and let $g_i(X_i) \cap \mathrm{dom}\partial\phi_i(.) \neq \emptyset$. We consider the following system of generalized implicit variational-like inclusion problem (SGIVLIP): Find $(x_i, u_i, v_i, w_i, t_i)$ such that

$$\left.\begin{aligned}
&\big\langle N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1, x_2), P_1(x_1, x_2)), \eta_1(y_1, g_1(x_1))\big\rangle\\
&\qquad\qquad \geq \rho_1(\phi_1(g_1(x_1)) - \phi_1(y_1)), \forall y_1 \in X_1,\ \rho_1 > 0,\\
&\big\langle N_2(u_2, v_1, w_2, t_1) - \rho_2 Q_2(E_2(x_2, x_1), P_2(x_2, x_1)), \eta_2(y_2, g_2(x_2))\big\rangle\\
&\qquad\qquad \geq \rho_2(\phi_2(g_2(x_2)) - \phi_2(y_2)),\ \forall y_2 \in X_2,\ \rho_2 > 0.
\end{aligned}\right\} \tag{2.7}$$

**Special Cases:** If in problem (2.7) $N_i : X_i^\star \times X_j^\star \to X_i^\star$, $Q_i : X_j^\star \times X_j^\star \to X_i^\star$ is an identity mapping such that $Q_i(E_i(x_i, x_j), P_i(x_i, x_j)) = -(E_i(x_i, x_j) + P_i(x_i, x_j))$. Then problem (2.7) reduces to the following problem:

$$\left.\begin{aligned}&\big\langle N_1(u_1, v_2) + \rho_1(E_1(x_1, x_2) + P_1(x_1, x_2)), \eta_1(y_1, g_1(x_1))\big\rangle \\ &\qquad\qquad \geq \rho_1(\phi_1(g_1(x_1)) - \phi_1(y_1)),\ \forall y_1 \in X_1, \\ &\big\langle N_2(u_2, v_1) + \rho_2(E_2(x_2, x_1) + P_2(x_2, x_1)), \eta_2(y_2, g_2(x_2))\big\rangle \\ &\qquad\qquad \geq \rho_2(\phi_2(g_2(x_2)) - \phi_2(y_2)),\ \forall y_2 \in X_2.\end{aligned}\right\} \tag{2.8}$$

Similar type of problem (2.8) has been considered by Kazmi *et al.* [21].

We remark that for the appropriate and suitable choices of mappings $N_i, Q_i, E_i, P_i, g_i, \eta_i, \phi_i, S_i, T_i, G_i, F_i$ and the underlying spaces $X_i$, we can obtain from SGIVLIP (2.7) many known and new classes of systems of generalized variational inequalities, see for example, [21, 27] and the relevant references cited therein.

## 3. Existence of solution

First, we give the following technical lemma.

**Lemma 3.1.** *For each $i = 1, 2$, let $X_i$ be a reflexive Banach space, let $\eta_i : X_i \times X_i \to X_i$ be a continuous mapping such that*

$$\eta_i(y_i, y_i') + \eta_i(y_i', y_i) = 0,\ \forall y_i, y_i' \in X_i,$$

*let $A_i$, $B_i : X_i \to X_i$ be nonlinear mappings, let the mapping $M_i : X_i \times X_i \to X_i^\star$ be $\alpha_i\beta_i$-symmetric $\eta_i$-monotone continuous with respect to $A_i$ and $B_i$, let for any given $x_i^\star \in X_i^\star$, the function $h_i(y_i, x_i) = \langle x_i^\star - M_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i)\rangle$ be 0-DQCV in $y_i$ and let $\phi_i : X_i \to R \cup \{\infty\}$ be a proper, lower semicontinuous and $\eta_i$-subdifferential functional. Then $(x_i, u_i, v_i, w_i, t_i)$ is a solution of SGIVLIP (2.7) if and only if*

$$\begin{aligned} g_1(x_1) = R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}&\{(M_1(A_1, B_1) \circ g_1)(x_1) \\ &- [N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1, x_2), P_1(x_1, x_2))]\}\end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned} g_2(x_2) = R^{\partial\phi_2, M_2(A_2,B_2)}_{\rho_2,\eta_2}&\{(M_2(A_2, B_2) \circ g_2)(x_2) \\ &- [N_2(u_2, v_1, w_2, t_1) - \rho_2 Q_2(E_2(x_1, x_2), P_2(x_1, x_2))]\},\end{aligned} \tag{3.2}$$

*where $M_i(A_i, B_i) \circ g_i$ denotes the composition $M_i(A_i, B_i)$ and $g_i$.*

*Proof.* For $x_i \in X_i, u_i \in S_i(x_i), v_i \in T_i(x_i), w_i \in G_i(x_i)$, $t_i \in F_i(x_i)$ and (3.1) satisfies. Then we have

$$g_1(x_1) = R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1) - [N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}$$

if and only if

$$g_1(x_1) = (M_1(A_1,B_1) + \rho_1\partial\phi_1)^{-1}\{(M_1(A_1,B_1)\circ g_1)(x_1) - [N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}.$$

This means that

$$(M_1(A_1,B_1)\circ g_1)(x_1) + \rho_1\partial\phi_1(g_1(x_1)) = (M_1(A_1,B_1)\circ g_1)(x_1) - [N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}.$$

It implies that

$$-[N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))] \in \rho_1\partial\phi_1(g_1(x_1)).$$

Therefore, we have

$$\rho_1\phi_1(y_1) - \rho_1\phi_1(g_1(x_1)) \geq \big\langle -[N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))], \eta_1(y_1,g_1(x_1))\big\rangle.$$

Hence we have

$$\big\langle N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2)), \eta_1(y_1,g_1(x_1))\big\rangle \geq \rho_1(\phi_1(g_1(x_1)) - \phi_1(y_1)).$$

Proceeding likewise by using (3.2), we have

$$\big\langle N_2(u_2,v_1,w_2,t_1) - \rho_2 Q_2(E_2(x_1,x_2),P_2(x_1,x_2)), \eta_2(y_2,g_2(x_2))\big\rangle \geq \rho_2(\phi_2(g_2(x_2)) - \phi_2(y_2)).$$

This means that $(x_i,u_i,v_i,w_i,t_i)$ is a solution of SGIVLIP (2.7). □

Now we give the following result for the existence of solution of SGIVLIP (2.7).

**Theorem 3.2.** *For $i \in \{1,2\}, j \in \{1,2\}\backslash\{i\}$, let $X_i$ be a uniformly smooth Banach space with $\rho_{X_i}(t) \leq c_i t^2$ for some $c_i > 0$, let $g_i : X_i \to X_i$ be $s_i$-strongly $\eta_i$-accretive and $L_{g_i}$-Lipschitz continuous, let $\eta_i : X_i \times X_i \to X_i$ be a continuous mapping such that $\eta_i(y_i,y_i') + \eta_i(y_i',y_i) = 0$, for all $y_i, y_i' \in X_i$ and $\eta_i$ be $L_{\eta_1}$-Lipschitz continuous. Let $A_i$, $B_i : X_i \to X_i$ be nonlinear mappings, let the mapping $M_i : X_i \times X_i \to X_i^\star$ be $\alpha_i\beta_i$-symmetric $\eta_i$-monotone continuous with respect to $A_i$ and $B_i$, let for any given $x_i^\star \in X_i^\star$ the function $h_i(y_i,x_i) = \langle x_i^\star\text{-}M_i(A_ix_i,B_ix_i), \eta_i(y_i,x_i)\rangle$ be 0-DQCV in $y_i$. Let $\phi_i : X_i \to R\cup\{+\infty\}$ be a proper, lower semicontinuous and $\eta_i$-subdifferential functional,* *let $Q_i : X_j^\star \times X_j^\star \to X_i^\star$ be such that $Q_i(E_i(.,x_j), P_i(.,x_j))$ is $\epsilon_i$-relaxed $\eta_i$-accretive with respect to $M_i(A_i,B_i)\circ g_i$ and $(L_{(Q_i,i)},\ L_{(Q_i,j)})$-mixed Lipschitz continuous, $P_i : X_i \times X_j \to X_j^\star$ be $(L_{(P_i,i)}, L_{(P_i,j)})$-mixed Lipschitz continuous. Let $(M_i(A_i,B_i)\circ g_i)$ be $L_{M_i}$-Lipschitz continuous and $E_i : X_i \times X_j \to X_j^\star$ be $(L_{(E_i,i)}, L_{(E_i,j)})$-mixed Lipschitz continuous, $N_i : X_i^\star \times X_j^\star \times X_i^\star \times X_j^\star \to X_i^\star$ be $L_{N_{i_1}}, L_{N_{i_2}}, L_{N_{i_3}}, L_{N_{i_4}}$-Lipschitz continuous in the first, second, third and fourth arguments, respectively and $\mu_i$-strongly $\eta_i$-accretive in the first argument, $\xi_i$-$Q_i(E_i(x_i',.), P_i(x_i',.))$-cocoercive in the second argument, $\omega_i$-relaxed $\eta_i$-accretive in the third argument and $(\theta_i,\varphi_i,\gamma_i)$-$Q_i(E_i(x_i',.), P_i(x_i',.))$-relaxed cocoercive in the fourth argument. Suppose $S_i, T_i, G_i, F_i : X_i \to C(X_i)$ are mappings such that $S_i$ is $L_{S_i}$-$\mathcal{D}$-Lipschitz continuous, $T_i$ is $L_{T_i}$-$\mathcal{D}$-Lipschitz continuous, $G_i$ is $L_{G_i}$-$\mathcal{D}$-Lipschitz continuous and $F_i$ is $L_{F_i}$-$\mathcal{D}$-Lipschitz continuous. Suppose that there are constants $\rho_1, \rho_2 > 0$ satisfying the following conditions:*

$$\left.\begin{aligned}
k_i &= b_i + d_j < 1, \text{ where,}\\
b_i : &= \Bigg\{\sqrt{(1-2s_i+2L_{g_i}\times(L_{\eta_i}+1)+64c_iL_{g_i}^2)}\\
&\quad + L_i\Bigg\{\Bigg[L_{M_i}^2 - 2\rho_i\epsilon_i + 2\rho_i\Big(L_{(Q_i,i)}L_{(E_i,i)} + L_{(Q_i,j)}L_{(P_i,i)}\\
&\quad \times(L_{M_i}(L_{\eta_i}+1))\Big) + 64c_i\rho_i^2\Big(L_{(Q_i,i)}^2L_{(E_i,i)}^2 + L_{(Q_i,j)}^2L_{(P_i,i)}^2\Big)\Bigg]^{\frac{1}{2}}\\
&\quad + \sqrt{\Big(1-2\mu_i+2L_{N_{i_1}}L_{S_i}\times(L_{\eta_i}+1)+64c_iL_{N_{i_1}}^2L_{S_i}^2\Big)}\\
&\quad + \sqrt{\Big(1-2\omega_i+2L_{N_{i_3}}L_{G_i}\times(L_{\eta_i}+1)+64c_iL_{N_{i_3}}^2L_{G_i}^2\Big)}\Bigg\}\Bigg\},\\
d_i : &= L_i\Bigg\{\rho_i^2\Big(L_{(Q_i,i)}^2L_{(E_i,j)}^2 + L_{(Q_i,j)}^2L_{(P_i,j)}^2\Big)\\
&\quad -2\Big(\rho_i\xi_iL_{N_{i_2}}^2L_{T_j}^2 + \Big(-\theta_iL_{N_{i_4}}^2L_{F_j}^2\\
&\quad -\rho_i^2\varphi_i\Big(L_{(Q_i,i)}^2L_{(E_i,j)}^2 + L_{(Q_i,j)}^2L_{(P_i,j)}^2\Big) + \gamma_i\Big)\Big)\\
&\quad +64c_i\Big(L_{N_{i_2}}L_{T_j} + L_{N_{i_4}}L_{F_j}\Big)^2\Bigg\}^{1/2};\ L_i := \frac{\tau_i}{\alpha_i-\beta_i}.
\end{aligned}\right\} \tag{3.3}$$

*Then SGIVLIP (2.7) has a solution.*

*Proof.* For each $(x_1,x_2)\in X_1\times X_2$, define a mapping $V : X_1\times X_2 \to X_1\times X_2$ by

$$V(x_1,x_2) = (K_1(x_1,x_2), K_2(x_1,x_2)),\ \forall (x_1,x_2)\in X_1\times X_2, \tag{3.4}$$

where $K_1 : X_1 \times X_2 \to X_1$ and $K_2 : X_1 \times X_2 \to X_2$ are respectively defined by

$$K_1(x_1, x_2) = x_1 - g_1(x_1) + R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1, B_1) \circ g_1)(x_1) - [N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1, x_2), P_1(x_1, x_2))]\} \tag{3.5}$$

and

$$K_2(x_1, x_2) = x_2 - g_2(x_2) + R^{\partial\phi_2, M_2(A_2,B_2)}_{\rho_2,\eta_2}\{(M_2(A_2, B_2) \circ g_2)(x_2) - [N_2(u_2, v_1, w_2, t_1) - \rho_2 Q_2(E_2(x_1, x_2), P_2(x_1, x_2))]\}. \tag{3.6}$$

For any $(x_1, x_2), (x'_1, x'_2) \in X_1 \times X_2$, it follows from (3.5), (3.6) and Lipschitz continuity of $R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}$ and $R^{\partial\phi_2, M_2(A_2,B_2)}_{\rho_2,\eta_2}$ that

$$\begin{aligned}
&\left\|K_1(x_1, x_2) - K_1(x'_1, x'_2)\right\|_1 \\
&\leq \Big\|x_1 - g_1(x_1) + R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1, B_1) \circ g_1)(x_1) \\
&\quad - [N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1, x_2), P_1(x_1, x_2))]\} \\
&\quad - \Big\{x'_1 - g_1(x'_1) + R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1, B_1) \circ g_1)(x'_1) \\
&\quad - [N_1(u'_1, v'_2, w'_1, t'_2) - \rho_1 Q_1(E_1(x'_1, x'_2), P_1(x'_1, x'_2))]\}\Big\}\Big\|_1 \\
&\leq \|(x_1 - x'_1) - (g_1(x_1) - g_1(x'_1))\|_1 \\
&\quad + \|R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1, B_1) \circ g_1)(x_1) \\
&\quad - [N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1, x_2), P_1(x_1, x_2))]\} \\
&\quad - R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1, B_1) \circ g_1)(x'_1) \\
&\quad - [N_1(u'_1, v'_2, w'_1, t'_2) - \rho_1 Q_1(E_1(x'_1, x'_2), P_1(x'_1, x'_2))]\}\|_1.
\end{aligned} \tag{3.7}$$

Since $g_i$ is $s_i$-strongly $\eta_i$-accretive and $L_{g_i}$-Lipschitz continuous and $\eta_i$ is $L_{\eta_i}$-Lipschitz continuous, using Lemma 2.4, we have

$$\begin{aligned}
&\left\|(x_1 - x'_1) - (g_1(x_1) - g_1(x'_1))\right\|_1^2 \\
&\leq \|x_1 - x'_1\|_1^2 - 2\langle g_1(x_1) - g_1(x'_1), J_1(\eta_1(x_1, x'_1))\rangle_1 \\
&\quad -2\langle g_1(x_1) - g_1(x'_1), J_1(x_1 - x'_1)\rangle_1 - J_1(\eta_1(x_1, x'_1)) \\
&\quad +2\langle g_1(x_1) - g_1(x'_1), J_1(x_1 - x'_1) - J_1(x_1 - x'_1 - (g_1(x_1) - g_1(x'_1)))\rangle_1 \\
&\leq \|x_1 - x'_1\|_1^2 - 2s_1\|x_1 - x'_1\|_1^2 \\
&\quad +2\|g_1(x_1) - g_1(x'_1)\| \times (\|x_1 - x'_1\| + \|\eta_1(x_1, x'_1)\|) + 64c_1L_{g_1}^2\|x_1 - x'_1\|_1^2 \\
&\leq \|x_1 - x'_1\|_1^2 - 2s_1\|x_1 - x'_1\|_1^2 \\
&\quad +2L_{g_1}\|x_1 - x'_1\| \times (\|x_1 - x'_1\| + L_{\eta_1}\|x_1 - x'_1\|) + 64c_1L_{g_1}^2\|x_1 - x'_1\|_1^2 \\
&= (1 - 2s_1 + 2L_{g_1} \times (1 + L_{\eta_1}) + 64c_1L_{g_1}^2)\|x_1 - x'_1\|_1^2.
\end{aligned}$$

This implies

$$\left\|(x_1 - x'_1) - (g_1(x_1) - g_1(x'_1))\right\|_1$$

$$\leq \sqrt{(1-2s_1+2L_{g_1}\times(1+L_{\eta_1})+64c_1L_{g_1}^2)}\ \|x_1-x_1'\|_1. \tag{3.8}$$

Now, using Theorem 2.17, we have the following estimate:

$$\begin{aligned}
&\Big\|R^{\partial\phi_1,M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1)-[N_1(u_1,v_2,w_1,t_2)\\
&\quad-\rho_1Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}-R^{\partial\phi_1,M_1(A_1,B_1)}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1')\\
&\quad-[N_1(u_1',v_2',w_1',t_2')-\rho_1Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))]\}\Big\|_1\\
&\leq L_1\Big\|\{(M_1(A_1,B_1)\circ g_1)(x_1)-[N_1(u_1,v_2,w_1,t_2)\\
&\quad-\rho_1Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}\\
&\quad-\{(M_1(A_1,B_1)\circ g_1)(x_1')-[N_1(u_1',v_2',w_1',t_2')\\
&\quad-\rho_1Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))]\}\Big\|_1\\
&\leq L_1\Big\|(M_1(A_1,B_1)\circ g_1)(x_1)-(M_1(A_1,B_1)\circ g_1)(x_1')\\
&\quad+\rho_1[Q_1(E_1(x_1,x_2),P_1(x_1,x_2))-Q_1(E_1(x_1',x_2),P_1(x_1',x_2))]\\
&\quad+\rho_1[Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))]\\
&\quad-[N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2',w_1',t_2')]\Big\|_1\\
&\leq L_1\Big\|(M_1(A_1,B_1)\circ g_1)(x_1)-(M_1(A_1,B_1)\circ g_1)(x_1')\\
&\quad+\rho_1[Q_1(E_1(x_1,x_2),P_1(x_1,x_2))-Q_1(E_1(x_1',x_2),P_1(x_1',x_2))]\Big\|_1\\
&\quad+L_1\Big\|[N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2',w_1',t_2')]\\
&\quad-\rho_1[Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))]\Big\|_1.
\end{aligned} \tag{3.9}$$

Since $Q_i(E_i(.,x_j),P_i(.,x_j))$ is $\epsilon_i$-relaxed $\eta_i$-accretive with respect to $(M_i(A_i,B_i)\circ g_i)$ and $(L_{(Q_i,i)},\ L_{(Q_i,j)})$-mixed Lipschitz continuous, $P_i : X_i \times X_j \to X_j^\star$ is $(L_{(P_i,i)},L_{(P_i,j)})$-mixed Lipschitz continuous, $(M_i(A_i,B_i)\circ g_i)$ is $L_{M_i}$-Lipschitz continuous and $E_i : X_i\times X_j \to X_j^\star$ is $(L_{(E_i,i)},L_{(E_i,j)})$-mixed Lipschitz continuous, from Lemma 2.4 we have

$$\begin{aligned}
&\|(M_1(A_1,B_1)\circ g_1)(x_1)-(M_1(A_1,B_1)\circ g_1)(x_1')\\
&\quad+\rho_1[Q_1(E_1(x_1,x_2),P_1(x_1,x_2))-Q_1(E_1(x_1',x_2),P_1(x_1',x_2))]\|_1^2\\
&\leq\|(M_1(A_1,B_1)\circ g_1)(x_1)-(M_1(A_1,B_1)\circ g_1)(x_1')\|_1^2\\
&\quad-2\rho_1\Big\langle Q_1(E_1(x_1,x_2),P_1(x_1,x_2))-Q_1(E_1(x_1',x_2),P_1(x_1',x_2)),\\
&\qquad J_1\Big(\eta_1((M_1(A_1,B_1)\circ g_1)(x_1),(M_1(A_1,B_1)\circ g_1)(x_1'))\Big)\Big\rangle_1
\end{aligned}$$

$$
\begin{aligned}
&- 2\rho_1\Big\langle Q_1(E_1(x_1,x_2),P_1(x_1,x_2)) - Q_1(E_1(x_1',x_2),P_1(x_1',x_2)),\\
&\quad J_1\Big((M_1(A_1,B_1)\circ g_1)(x_1) - (M_1(A_1,B_1)\circ g_1)(x_1')\Big)\\
&- J_1\Big(\eta_1((M_1(A_1,B_1)\circ g_1)(x_1) - (M_1(A_1,B_1)\circ g_1)(x_1'))\Big)\rangle_1\\
&- 2\rho_1\Big\langle Q_1(E_1(x_1,x_2),P_1(x_1,x_2)) - Q_1(E_1(x_1',x_2),P_1(x_1',x_2)),\\
&+ J_1\Big((M_1(A_1,B_1)\circ g_1)(x_1) - (M_1(A_1,B_1)\circ g_1)(x_1')\Big)\\
&- J_1\Big((M_1(A_1,B_1)\circ g_1)(x_1) - (M_1(A_1,B_1)\circ g_1)(x_1')\\
&+ \rho_1\Big[Q_1(E_1(x_1,x_2),P_1(x_1,x_2)) - Q_1(E_1(x_1',x_2),P_1(x_1',x_2))\Big]\Big)\Big\rangle_1\\
\le\ & L_{M_1}^2\|x_1-x_1'\|_1^2 - 2\rho_1\epsilon_1\|x_1-x_1'\|_1^2\\
&+ 2\rho_1\|Q_1(E_1(x_1,x_2),P_1(x_1,x_2)) - Q_1(E_1(x_1',x_2),P_1(x_1',x_2))\|\\
&\times\Big[\|(M_1(A_1,B_1)\circ g_1)(x_1) - (M_1(A_1,B_1)\circ g_1)(x_1')\|_1\Big]\\
&+ \Big\|\eta_1((M_1(A_1,B_1)\circ g_1)(x_1),(M_1(A_1,B_1)\circ g_1)(x_1'))\|_1\\
&+ 64c_1\rho_1^2\|Q_1(E_1(x_1,x_2),P_1(x_1,x_2)) - Q_1(E_1(x_1',x_2),P_1(x_1',x_2))\|^2\\
\le\ & L_{M_1}^2\|x_1-x_1'\|_1^2 - 2\rho_1\epsilon_1\|x_1-x_1'\|_1^2\\
&+ 2\rho_1\Big[\Big(L_{(Q_1,1)}\|E_1(x_1,x_2)-E_1(x_1',x_2)\|_2\\
&+ L_{(Q_1,2)}\|P_1(x_1,x_2)-P_1(x_1',x_2)\|_2\Big)\\
&\times(L_{M_1}\|x_1-x_1'\|_1) + L_{\eta_1}L_{M_1}\|x_1-x_1'\|_1\Big]\\
&+ 64c_1\rho_1^2\Big(L_{(Q_1,1)}^2L_{(E_1,1)}^2\|x_1-x_1'\|_1^2 + L_{(Q_1,2)}^2L_{(P_1,1)}^2\|x_1-x_1'\|_1^2\Big)\\
\le\ & \Big(L_{M_1}^2 - 2\rho_1\epsilon_1 + 2\rho_1\Big[\Big(L_{(Q_1,1)}L_{(E_1,1)} + L_{(Q_1,2)}L_{(P_1,1)}\Big)\times(L_{M_1}(1+L_{\eta_1}))\Big]\\
&+ 64c_1\rho_1^2\Big(L_{(Q_1,1)}^2L_{(E_1,1)}^2 + L_{(Q_1,2)}^2L_{(P_1,1)}^2\Big)\Big)\|x_1-x_1'\|_1^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|(M_1(A_1,B_1)\circ g_1)(x_1) - (M_1(A_1,B_1)\circ g_1)(x_1')\\
&\quad + \rho_1[Q_1(E_1(x_1,x_2),P_1(x_1,x_2)) - Q_1(E_1(x_1',x_2),P_1(x_1',x_2))]\|_1\\
&\le \Big[L_{M_1}^2 - 2\rho_1\epsilon_1 + 2\rho_1\Big(L_{(Q_1,1)}L_{(E_1,1)} + L_{(Q_1,2)}L_{(P_1,1)}\times(L_{M_1}(1+L_{\eta_1}))\Big)\\
&\quad + 64c_1\rho_1^2\Big(L_{(Q_1,1)}^2L_{(E_1,1)}^2 + L_{(Q_1,2)}^2L_{(P_1,1)}^2\Big)\Big]^{\frac{1}{2}}\|x_1-x_1'\|_1. \qquad (3.10)
\end{aligned}
$$

Now,

$$\begin{aligned}
&\|[N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2',w_1',t_2')]\\
&\quad-\rho_1[Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))]\|_1\\
&\le\|[N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2)]-(x_1-x_1')\|_1\\
&\quad+\|[N_1(u_1',v_2',w_1,t_2)-N_1(u_1',v_2',w_1',t_2)]+(x_1-x_1')\|_1\\
&\quad+\|N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)\\
&\quad+N_1(u_1',v_2',w_1',t_2)-N_1(u_1',v_2',w_1',t_2')\\
&\quad-\rho_1[Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))]\|_1. \qquad (3.11)
\end{aligned}$$

Since, $N_i$ is $L_{N_{i_1}}$-Lipschitz continuous in the first argument and $\mu_i$-strongly $\eta_i$-accretive in the first argument and $S_i$ is $L_{S_i}$-$\mathcal{D}$-Lipschitz continuous, we have from Lemma 2.4,

$$\begin{aligned}
&\|[N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2)]-(x_1-x_1')\|_1^2\\
&\le\|x_1-x_1'\|_1^2-2\Big\langle N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2),J_1(\eta_1(x_1,x_1'))\Big\rangle_1\\
&\quad-2\Big\langle N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2),J_1(x_1-x_1')-J_1(\eta_1(x_1,x_1'))\Big\rangle_1\\
&\quad-2\Big\langle N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2),J_1(x_1-x_1')\\
&\quad-J_1\Big((x_1-x_1')-\Big(N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2)\Big)\Big)\Big\rangle_1\\
&\le\|x_1-x_1'\|_1^2-2\mu_1\|x_1-x_1'\|_1^2\\
&\quad+2\|N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2)\|_1\times(\|x_1-x_1'\|_1+\|\eta_1(x_1,x_1')\|_1)\\
&\quad+64c_1\|N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2)\|_1^2\\
&\le\|x_1-x_1'\|_1^2-2\mu_1\|x_1-x_1'\|_1^2+2L_{N_{1_1}}\|u_1-u_1'\|_1\times(1+L_{\eta_1})\|x_1-x_1'\|_1\\
&\quad+64c_1L_{N_{1_1}}^2\|u_1-u_1'\|_1^2\\
&\le\|x_1-x_1'\|_1^2-2\mu_1\|x_1-x_1'\|_1^2+2L_{N_{1_1}}\mathcal{D}(S_1(x_1),S_1(x_1'))_1\times(1+L_{\eta_1})\|x_1-x_1'\|_1\\
&\quad+64c_1L_{N_{1_1}}^2\mathcal{D}(S_1(x_1),S_1(x_1'))_1^2\\
&\le\Big(1-2\mu_1+2L_{N_{1_1}}L_{S_1}\times(1+L_{\eta_1})+64c_1L_{N_{1_1}}^2L_{S_1}^2\Big)\ \|x_1-x_1'\|_1^2.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|[N_1(u_1,v_2,w_1,t_2)-N_1(u_1',v_2,w_1,t_2)]-(x_1-x_1')\|_1\\
&\quad\le\sqrt{\big(1-2\mu_1+2L_{N_{1_1}}L_{S_1}\times(1+L_{\eta_1})+64c_1L_{N_{1_1}}^2L_{S_1}^2\big)}\ \|x_1-x_1'\|_1. \qquad (3.12)
\end{aligned}$$

Again, since, $N_i$ is $L_{N_{i_3}}$-Lipschitz continuous in the third argument and $\omega_i$-relaxed $\eta_i$-accretive in the third argument and $G_i$ is $L_{G_i}$-$\mathcal{D}$-Lipschitz continuous, we have from Lemma 2.4,

$$\begin{aligned}
&\|[N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2)] + (x_1 - x_1')\|_1^2 \\
&\le \|x_1 - x_1'\|_1^2 + 2\Big\langle N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2), J_1(\eta_1(x_1, x_1'))\Big\rangle_1 \\
&\quad - 2\Big\langle N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2), J_1((x_1 - x_1') - J_1(\eta_1(x_1, x_1'))\Big\rangle_1 \\
&\quad - 2\Big\langle N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2), J_1(x_1 - x_1') \\
&\quad - J_1\Big((x_1 - x_1') + \Big(N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2))\Big)\Big)\Big\rangle_1 \\
&\le \|x_1 - x_1'\|_1^2 - 2\omega_1\|x_1 - x_1'\|_1^2 \\
&\quad + 2\|N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2)\|_1 \times (\|x_1 - x_1'\|_1 - \|\eta_1(x_1, x_1')\|_1) \\
&\quad + 64c_1\|N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2)\|_1^2 \\
&\le \|x_1 - x_1'\|_1^2 - 2\omega_1\|x_1 - x_1'\|_1^2 \\
&\quad + 2L_{N_{1_3}}\|w_1 - w_1'\|_1 \times (1 + L_{\eta_1})\|x_1 - x_1'\|_1 + 64c_1L_{N_{1_3}}^2\|w_1 - w_1'\|_1^2 \\
&\le \|x_1 - x_1'\|_1^2 - 2\omega_1\|x_1 - x_1'\|_1^2 + 2L_{N_{1_3}}\mathcal{D}(G_1(x_1), G_1(x_1'))_1 \times (1 + L_{\eta_1})\|x_1 - x_1'\|_1 \\
&\quad + 64c_1L_{N_{1_3}}^2\mathcal{D}(G_1(x_1), G_1(x_1'))_1^2 \\
&\le \Big(1 - 2\omega_1 + 2L_{N_{1_3}}L_{G_1} \times (1 + L_{\eta_1}) + 64c_1L_{N_{1_3}}^2L_{G_1}^2\Big)\|x_1 - x_1'\|_1^2.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|[N_1(u_1', v_2', w_1, t_2) - N_1(u_1', v_2', w_1', t_2)] + (x_1 - x_1')\|_1 \\
&\quad \le \sqrt{\big(1 - 2\omega_1 + 2L_{N_{1_3}}L_{G_1} \times (1 + L_{\eta_1}) + 64c_1L_{N_{1_3}}^2L_{G_1}^2\big)}\ \|x_1 - x_1'\|_1. \qquad (3.13)
\end{aligned}$$

Again, from Lemma 2.4, we have

$$\begin{aligned}
&\|N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2) + N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2') \\
&\quad - \rho_1[Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))]\|_1^2 \\
&\le \rho_1^2\|Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))\|_1^2 \\
&\quad - 2\Big\langle N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2) \\
&\quad + N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2'), \\
&\qquad J_1\Big(\rho_1[Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))] \\
&\quad - [N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2)
\end{aligned}$$

$$
\begin{aligned}
&+ N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2')]\Big)\Big\rangle_1\\
&\le \rho_1^2 \|Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))\|_1^2\\
&\quad - 2\Big\langle N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2)\\
&\quad + N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2'),\\
&\quad\ J_1\Big(\rho_1[Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))]\Big)\Big\rangle_1\\
&\quad + 2\Big\langle N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2)\\
&\quad + N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2'),\\
&\quad\ J_1\Big(\rho_1[Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))]\Big)\\
&\quad - J_1\Big(\rho_1[Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))]\\
&\quad - \Big(N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2)\\
&\quad + N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2')\Big)\Big)\Big\rangle_1. \qquad (3.14)
\end{aligned}
$$

Again, since $Q_i$ is $(L_{(Q_i,i)}, L_{(Q_i,j)})$-mixed Lipschitz continuous, $P_i$ is $(L_{(P_i,i)}, L_{(P_i,j)})$-mixed Lipschitz continuous and $E_i$ is $(L_{(E_i,i)}, L_{(E_i,j)})$-mixed Lipschitz continuous, from Lemma 2.4 we have

$$
\begin{aligned}
&\|Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))\|_1^2\\
&\le L_{(Q_1,1)}^2 \|E_1(x_1', x_2) - E_1(x_1', x_2')\|_2^2 + L_{(Q_1,2)}^2 \|P_1(x_1', x_2) - P_1(x_1', x_2')\|_2^2\\
&\le L_{(Q_1,1)}^2 L_{(E_1,2)}^2 \|x_2 - x_2'\|_2^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2 \|x_2 - x_2'\|_2^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))\|_1^2\\
&\le \Big(L_{(Q_1,1)}^2 L_{(E_1,2)}^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2\Big)\|x_2 - x_2'\|_2^2. \qquad (3.15)
\end{aligned}
$$

Again, since $N_i$ is $\xi_i$-$Q_i(E_i(x_i', .), P_i(x_i', .))$-cocoercive in the second argument and $(\theta_i, \varphi_i, \gamma_i)$-$Q_i(E_i(x_i', .), P_i(x_i', .))$-relaxed cocoercive in the fourth argument, $L_{N_{i_2}}, L_{N_{i_4}}$-Lipschitz continuous in the second and fourth arguments, respectively and $T_i$ is $L_{T_i}$-$\mathcal{D}$-Lipschitz continuous, $F_i$ is $L_{F_i}$-$\mathcal{D}$-Lipschitz continuous, we have

$$
\begin{aligned}
&-2\Big\langle N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2) + N_1(u_1', v_2', w_1', t_2) - N_1(u_1', v_2', w_1', t_2'),\\
&\quad J_1\Big(\rho_1\Big(Q_1(E_1(x_1', x_2), P_1(x_1', x_2)) - Q_1(E_1(x_1', x_2'), P_1(x_1', x_2'))\Big)\Big)\Big\rangle_1\\
&\le -2\Big\langle N_1(u_1', v_2, w_1, t_2) - N_1(u_1', v_2', w_1, t_2),
\end{aligned}
$$

$$
\begin{aligned}
&\quad J_1\Big(\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2)) - Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\Big)\Big\rangle_1 \\
&\quad - 2\Big\langle N_1(u_1',v_2',w_1',t_2) - N_1(u_1',v_2',w_1',t_2'), \\
&\quad J_1\Big(\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2)) - Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\Big)\Big\rangle_1 \\
&\le -2\rho_1\xi_1\|N_1(u_1',v_2,w_1,t_2) - N_1(u_1',v_2',w_1,t_2)\|_1^2 \\
&\quad - 2\Big( - \theta_1\|N_1(u_1',v_2',w_1',t_2) - N_1(u_1',v_2',w_1',t_2')\|_1^2 \\
&\quad - \varphi_1\rho_1^2\|Q_1(E_1(x_1',x_2),P_1(x_1',x_2)) - Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\|_1^2 \\
&\quad + \gamma_1\|x_2 - x_2'\|_2^2\Big) \\
&\le -2\rho_1\xi_1 L_{N_{1_2}}^2\|v_2 - v_2'\|_2^2 - 2\Big( - \theta_1 L_{N_{1_4}}^2\|t_2 - t_2'\|_2^2 \\
&\quad - \rho_1^2\varphi_1\Big(L_{(Q_1,1)}^2\|E_1(x_1',x_2) - E_1(x_1',x_2')\|_2^2 \\
&\quad + L_{(Q_1,2)}^2\|P_1(x_1',x_2) - P_1(x_1',x_2')\|_2^2\Big) + \gamma_1\|x_2 - x_2'\|_2^2\Big) \\
&\le -2\rho_1\xi_1 L_{N_{1_2}}^2\|v_2 - v_2'\|_2^2 - 2\Big( - \theta_1 L_{N_{1_4}}^2\|t_2 - t_2'\|_2^2 \\
&\quad - \rho_1^2\varphi_1\Big(L_{(Q_1,1)}^2 L_{(E_1,2)}^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2\Big)\|x_2 - x_2'\|_2^2 + \gamma_1\|x_2 - x_2'\|_2^2\Big) \\
&\le -2\rho_1\xi_1 L_{N_{1_2}}^2\mathcal{D}(T_2(x_2),T_2(x_2'))_2^2 - 2\Big( - \theta_1 L_{N_{1_4}}^2\mathcal{D}(F_2(x_2),F_2(x_2'))_2^2 \\
&\quad - \rho_1^2\varphi_1\Big(L_{(Q_1,1)}^2 L_{(E_1,2)}^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2\Big)\|x_2 - x_2'\|_2^2 + \gamma_1\|x_2 - x_2'\|_2^2\Big) \\
&\le -2\rho_1\xi_1 L_{N_{1_2}}^2 L_{T_2}^2\|x_2 - x_2'\|_2^2 - 2\Big( - \theta_1 L_{N_{1_4}}^2 L_{F_2}^2\|x_2 - x_2'\|_2^2 \\
&\quad - \rho_1^2\varphi_1\Big(L_{(Q_1,1)}^2 L_{(E_1,2)}^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2\Big)\|x_2 - x_2'\|_2^2 + \gamma_1\|x_2 - x_2'\|_2^2\Big) \\
&\le \Big( - 2\rho_1\xi_1 L_{N_{1_2}}^2 L_{T_2}^2 - 2\Big( - \theta_1 L_{N_{1_4}}^2 L_{F_2}^2 \\
&\quad - \rho_1^2\varphi_1\Big(L_{(Q_1,1)}^2 L_{(E_1,2)}^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2\Big) + \gamma_1\Big)\Big)\|x_2 - x_2'\|_2^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&- 2\Big\langle N_1(u_1',v_2,w_1,t_2) - N_1(u_1',v_2',w_1,t_2) + N_1(u_1',v_2',w_1',t_2) - N_1(u_1',v_2',w_1',t_2'), \\
&\quad J_1\Big(\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2)) - Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\Big)\Big\rangle_1 \\
&\quad \le \Big( - 2\rho_1\xi_1 L_{N_{1_2}}^2 L_{T_2}^2 - 2\Big( - \theta_1 L_{N_{1_4}}^2 L_{F_2}^2 \\
&\qquad - \rho_1^2\varphi_1\Big(L_{(Q_1,1)}^2 L_{(E_1,2)}^2 + L_{(Q_1,2)}^2 L_{(P_1,2)}^2\Big) + \gamma_1\Big)\Big)\|x_2 - x_2'\|_2^2. \qquad (3.16)
\end{aligned}
$$

Again,

$$
\begin{aligned}
&2\Big\langle N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)+N_1(u_1',v_2',w_1',t_2)-N_1(u_1',v_2',w_1',t_2'),\\
&\quad J_1\Big(\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\Big)\\
&\quad -J_1\Big(\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\\
&\quad -\Big(N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)+N_1(u_1',v_2',w_1',t_2)\\
&\quad -N_1(u_1',v_2',w_1',t_2')\Big)\Big)\Big\rangle_1\\
&\le 64c_1\|N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)+N_1(u_1',v_2',w_1',t_2)\\
&\quad -N_1(u_1',v_2',w_1',t_2')\|_1^2\\
&\le 64c_1\Big(\|N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)\|_1+\|N_1(u_1',v_2',w_1',t_2)\\
&\quad -N_1(u_1',v_2',w_1',t_2')\|_1\Big)^2\\
&\le 64c_1\Big(L_{N_{1_2}}\|v_2-v_2'\|_2+L_{N_{1_4}}\|t_2-t_2'\|_2\Big)^2\\
&\le 64c_1\Big(L_{N_{1_2}}\mathcal{D}(T_2(x_2),T_2(x_2'))_2+L_{N_{1_4}}\mathcal{D}(F_2(x_2),F_2(x_2'))_2\Big)^2\\
&\le 64c_1\Big(L_{N_{1_2}}L_{T_2}\|x_2-x_2'\|_2+L_{N_{1_4}}L_{F_2}\|x_2-x_2'\|_2\Big)^2\\
&\le 64c_1\Big(L_{N_{1_2}}L_{T_2}+L_{N_{1_4}}L_{F_2}\Big)^2\|x_2-x_2'\|_2^2. \qquad (3.17)
\end{aligned}
$$

It follows from (3.14)-(3.17), that

$$
\begin{aligned}
&\|N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)+N_1(u_1',v_2',w_1',t_2)-N_1(u_1',v_2',w_1',t_2')\\
&\quad -\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\|_1^2\\
&\le \rho_1^2\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)}+L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big)\|x_2-x_2'\|_2^2\\
&\quad -2\Big(\rho_1\xi_1L^2_{N_{1_2}}L^2_{T_2}+\Big(-\theta_1L^2_{N_{1_4}}L^2_{F_2}\\
&\quad -\rho_1^2\varphi_1\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)}+L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big)+\gamma_1\Big)\Big)\|x_2-x_2'\|_2^2\\
&\quad +64c_1\Big(L_{N_{1_2}}L_{T_2}+L_{N_{1_4}}L_{F_2}\Big)^2\|x_2-x_2'\|_2^2.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|N_1(u_1',v_2,w_1,t_2)-N_1(u_1',v_2',w_1,t_2)+N_1(u_1',v_2',w_1',t_2)-N_1(u_1',v_2',w_1',t_2')\\
&\quad -\rho_1\Big(Q_1(E_1(x_1',x_2),P_1(x_1',x_2))-Q_1(E_1(x_1',x_2'),P_1(x_1',x_2'))\Big)\|_1
\end{aligned}
$$

$$\leq \Big[\rho_1^2\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)} + L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big) - 2\Big(\rho_1\xi_1 L^2_{N_{1_2}}L^2_{T_2} + \Big(-\theta_1 L^2_{N_{1_4}}L^2_{F_2} - \rho_1^2\varphi_1\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)} + L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big) + \gamma_1\Big)\Big) + 64c_1\Big(L_{N_{1_2}}L_{T_2} + L_{N_{1_4}}L_{F_2}\Big)^2\Big]^{1/2}\|x_2 - x_2'\|_2. \tag{3.18}$$

From (3.5), (3.7)-(3.18), we have

$$\begin{aligned}
&\Big\|K_1(x_1,x_2) - K_1(x_1',x_2')\Big\|_1 \\
&\leq \Big\{\sqrt{(1 - 2s_1 + 2L_{g_1}\times(1 + L_{\eta_1}) + 64c_1L^2_{g_1})}\| \\
&\quad + L_1\Big\{\Big[L^2_{M_1} - 2\rho_1\epsilon_1 + 2\rho_1\Big(\Big[L_{(Q_1,1)}L_{(E_1,1)} + L_{(Q_1,2)}L_{(P_1,1)}\Big]\times(L_{M_1}(1 + L_{\eta_1}))\Big) \\
&\quad + 64c_1\rho_1^2\Big(L^2_{(Q_1,1)}L^2_{(E_1,1)} + L^2_{(Q_1,2)}L^2_{(P_1,1)}\Big)\Big]^{\frac{1}{2}} \\
&\quad + \sqrt{\Big(1 - 2\mu_1 + 2L_{N_{1_1}}L_{S_1}\times(1 + L_{\eta_1}) + 64c_1L^2_{N_{1_1}}L^2_{S_1}\Big)} \\
&\quad + \sqrt{\Big(1 - 2\omega_1 + 2L_{N_{1_3}}L_{G_1}\times(1 + L_{\eta_1}) + 64c_1L^2_{N_{1_3}}L^2_{G_1}\Big)}\Big\}\Big\}\|x_1 - x_1'\|_1 \\
&\quad + L_1\Big\{\rho_1^2\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)} + L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big) \\
&\quad - 2\Big(\rho_1\xi_1L^2_{N_{1_2}}L^2_{T_2} + \Big(-\theta_1L^2_{N_{1_4}}L^2_{F_2} - \rho_1^2\varphi_1\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)} + L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big) + \gamma_1\Big)\Big) \\
&\quad + 64c_1\Big(L_{N_{1_2}}L_{T_2} + L_{N_{1_4}}L_{F_2}\Big)^2\Big\}^{1/2}\|x_2 - x_2'\|_2 \\
&\leq b_1\|x_1 - x_1'\|_1 + d_1\|x_2 - x_2'\|_2.
\end{aligned} \tag{3.19}$$

Similarly, we infer that

$$\Big\|K_2(x_1,x_2) - K_2(x_1',x_2')\Big\|_2 \leq b_2\|x_2 - x_2'\|_2 + d_2\|x_1 - x_1'\|_1. \tag{3.20}$$

From (3.19) and (3.20), we have

$$\begin{aligned}
&\|K_1(x_1,x_2) - K_1(x_1',x_2')\Big\|_1 + \Big\|K_2(x_1,x_2) - K_2(x_1',x_2')\Big\|_2 \\
&\leq k_1\|x_1 - x_1'\|_1 + k_2\|x_2 - x_2'\|_2 \\
&\leq k\{\|x_1 - x_1'\|_1 + \|x_2 - x_2'\|_2\},
\end{aligned} \tag{3.21}$$

where $k_1 = b_1 + d_2, k_2 = b_2 + d_1$ and $k = \max\{k_1, k_2\}$.

Now, define the norm $\|.\|_\star$ on $X_1 \times X_2$ by

$$\Big\|(x_1,x_2)\Big\|_\star = \Big\|x_1\Big\|_1 + \Big\|x_2\Big\|_2,\ \forall (x_1,x_2) \in X_1 \times X_2. \tag{3.22}$$

Then we know that $(X_1 \times X_2, \|.\|_\star)$ is a Banach space. Hence, it follows from (3.4), (3.21) and (3.22) that

$$\begin{aligned}
&\Big\|V(x_1,x_2) - V(x_1',x_2')\Big\|_\star \\
&\le \Big\|(K_1(x_1,x_2),K_2(x_1,x_2)) - (K_1(x_1',x_2'),K_2(x_1',x_2'))\Big\|_\star \\
&\le \Big\|K_1(x_1,x_2) - K_1(x_1',x_2'), K_2(x_1,x_2) - K_2(x_1',x_2')\Big\|_\star \\
&\le \Big\|K_1(x_1,x_2) - K_1(x_1',x_2')\Big\|_1 + \Big\|K_2(x_1,x_2) - K_2(x_1',x_2')\Big\|_2 \\
&\le k\Big\{\Big\|x_1 - x_1'\Big\|_1 + \Big\|x_2 - x_2'\Big\|_2\Big\}. \qquad (3.23)
\end{aligned}$$

Since $k$ =max $\{k_1,k_2\} < 1$ by (3.3), it follows from (3.23) that $V$ is a contraction mapping. Hence, by Banach contraction principle, it admits a unique fixed point $(x_1,x_2) \in X_1 \times X_2$ such that $V(x_1,x_2) = (x_1,x_2)$, which implies that

$$\left.\begin{array}{l}
g_1(x_1) = R^{\partial\phi_1}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1) \\
-[N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}, \\
\\
g_2(x_2) = R^{\partial\phi_2}_{\rho_2,\eta_2}\{(M_2(A_2,B_2)\circ g_2)(x_2) \\
-[N_2(u_2,v_1,w_2,t_1) - \rho_2 Q_2(E_2(x_1,x_2),P_2(x_1,x_2))]\}.
\end{array}\right\}$$

It follows from Lemma 3.1, that $(x_i,u_i,v_i,w_i,t_i)$ is a solution of SGIVLIP (2.7). This completes the proof. □

## 4. Iterative algorithm, convergence and stability analysis

Lemma 3.1 is very important from the numerical point of view as it along with Nadler [23] allows us to suggest the following iterative algorithm for finding the approximate solution of SGIVLIP (2.7).

**Algorithm 4.1.** *For each $i = 1,2, j \in \{1,2\} \setminus i$, given $(x_i^0,u_i^0,v_i^0,w_i^0,\ t_i^0)$, where $x_i^0 \in X_i, u_i^0 \in S_i(x_i^0), v_i^0 \in T_i(x_i^0), w_i^0 \in G_i(x_i^0), t_i^0 \in F_i(x_i^0)$ such that $S_i,T_i,G_i,F_i : X_i \to C(X_i)$ compute the sequences $\{x_i^n\},\{u_i^n\},\{v_i^n\},\{w_i^n\},\{t_i^n\}$ by the iterative schemes:*

$$\begin{aligned}
x_1^{n+1} = (1-a^n)x_1^n + a^n\Big\{&x_1^n - g_1(x_1^n) + R^{\partial\phi_1^n,M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}\{(M_1(A_1,B_1)\circ g_1)(x_1^n) \\
&- [N_1(u_1^n,v_2^n,w_1^n,t_2^n) - \rho_1 Q_1(E_1(x_1^n,x_2^n),P_1(x_1^n,x_2^n))]\Big\} + a^n e_1^n,
\end{aligned}$$

$$x_2^{n+1} = (1-a^n)x_2^n + a^n\Big\{x_2^n - g_2(x_2^n) + R^{\partial\phi_2^n,M_2^n(A_2^n,B_2^n)}_{\rho_2,\eta_2^n}\{(M_2(A_2,B_2)\circ g_2)(x_2^n)$$

$$- [N_2(u_2^n, v_1^n, w_2^n, t_1^n) - \rho_2 Q_2(E_2(x_1^n, x_2^n), P_1(x_1^n, x_2^n))]\Big\} + a^n e_2^n,$$

$$\begin{aligned} u_i^n &\in S_i(x_i^n) : \|u_i^{n+1} - u_i^n\|_i \le \mathcal{D}(S_i(x_i^{n+1}), S_i(x_i^n))_i; \\ v_i^n &\in T_i(x_i^n) : \|v_i^{n+1} - v_i^n\|_i \le \mathcal{D}(T_i(x_i^{n+1}), T_i(x_i^n))_i, \\ w_i^n &\in G_i(x_i^n) : \|w_i^{n+1} - w_i^n\|_i \le \mathcal{D}(G_i(x_i^{n+1}), G_i(x_i^n))_i; \\ t_i^n &\in F_i(x_i^n) : \|t_i^{n+1} - t_i^n\|_i \le \mathcal{D}(F_i(x_i^{n+1}), F_i(x_i^n))_i, \end{aligned}$$

*where $n = 0, 1, 2, \cdots$, $\rho_i > 0$ are constants, $M_i^n$ are $\alpha_i^n \beta_i^n$- symmetric $\eta_i^n$-monotone continuous with respect to $A_i^n$ and $B_i^n$ and $\{e_1^n, e_2^n\}_{n \ge 0}$ is sequence in $X_1 \times X_2$ introduced to take into account possible inexact computation which satisfies $\lim_{n\to\infty} \|e_1^n\| = \lim_{n\to\infty} \|e_2^n\| = 0$ and $\{a^n\}$ is a sequence of real numbers such that $a^n \in [0, 1]$ and $\sum_{n=0}^{\infty} a^n = +\infty$.*

First, we give the following conditions:

**Condition 4.2.** *Let for each $n \ge 0, \eta^n, \eta : X \times X \to X$ be $\tau^n$-Lipschitz continuous such that $\eta^n(y, y') + \eta^n(y', y) = 0$ and $\tau$-Lipschitz continuous such that $\eta(y, y') + \eta(y', y) = 0$, for all $y', y \in X$, respectively, let $M^n(A^n, B^n) : X \times X \to X^\star$ be $\alpha^n\beta^n$-symmetric $\eta^n$-monotone, let $M(A, B) : X \times X \to X^\star$ be $\alpha\beta$-symmetric $\eta$-monotone, let for any given $x^\star \in X^\star$, the functions*

$$h^n(y, x) = \langle x^\star - M^n(A^n x, B^n x), \eta^n(y, x)\rangle$$

*and*

$$h(y, x) = \langle x^\star - M(Ax, Bx), \eta(y, x)\rangle$$

*be 0-DQCV in $y$, let $\phi^n : X \to R \cup \{\infty\}$ be a proper, lower semicontinuous and $\eta^n$-subdifferentiable function, and let $\phi : X \to R \cup \{\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable functional.*

**Condition 4.3.** *Let Condition 4.2 hold. The sequence $\{\partial\phi^n\}$ which approximates $\{\partial\phi\}$ in the following sense:*

$$\lim_{n\to\infty} R_{\rho,\eta^n}^{\partial\phi^n, M^n(A^n,B^n)}(x^\star) = R_{\rho,\eta}^{\partial\phi, M(A,B)}(x^\star), \ \forall x^\star \in X^\star.$$

Now, we discuss the convergence analysis of iterative Algorithm 4.1, and we give the following result for the existence of solution of SGIVLIP (2.7).

**Theorem 4.4.** *For each $i \in \{1, 2\}$, let $X_i$ be a uniformly smooth Banach space with $\rho_{X_i}(t) \le c_i t^2$ for some $c_i > 0$. For $i \in \{1, 2\}$, $j \in \{1, 2\}\backslash\{i\}$, let the mappings $\eta_i^n, \eta_i : X_i \times X_i \to X_i$, $A_i^n$, $B_i^n$, $A_i$, $B_i : X_i \to X_i$, $M_i^n(A_i^n, B_i^n)$,*

*$M_i(A_i, B_i) : X_i \to X_i^\star$, $\phi_i^n$, $\phi_i : X_i \to R \cup \{+\infty\}$, and for any given $x_i^\star \in X_i^\star$, the functions*

$$h_i^n(y_i, x_i) = \langle x_i^\star - M_i^n(A_i^n x_i, B_i^n x_i), \eta_i^n(y_i, x_i)\rangle$$

*and*

$$h_i(y_i, x_i) = \langle x_i^\star - M_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i)\rangle$$

*satisfy the Conditions (4.2)-(4.3). Let $Q_i : X_j^\star \times X_j^\star \to X_i^\star$ be such that $Q_i(E_i(., x_j), P_i(., x_j))$ is $\epsilon_i$-relaxed $\eta_i$-accretive with respect to $M_i(A_i, B_i) \circ g_i$ and $(L_{(Q_i,i)}, L_{(Q_i,j)})$-mixed Lipschitz continuous, $P_i : X_i \times X_j \to X_j^\star$ be $(L_{(P_i,i)}, L_{(P_i,j)})$-mixed Lipschitz continuous. Let $(M_i(A_i, B_i) \circ g_i)$ be $L_{M_i}$-Lipschitz continuous and $E_i : X_i \times X_j \to X_j^\star$ be $(L_{(E_i,i)}, L_{(E_i,j)})$-mixed Lipschitz continuous, $N_i : X_i^\star \times X_j^\star \times X_i^\star \times X_j^\star \to X_i^\star$ be $L_{N_{i_1}}, L_{N_{i_2}}, L_{N_{i_3}}, L_{N_{i_4}}$ Lipschitz continuous in the first, second, third and fourth arguments, respectively and $\mu_i$-strongly $\eta_i$-accretive in the first argument, $\omega_i$-relaxed $\eta_i$-accretive in the third argument and $\xi_i$-$Q_i(E_i(x_i', .), P_i(x_i', .))$-cocoercive in the second argument and $(\theta_i, \varphi_i, \gamma_i)$-$Q_i(E_i(x_i', .), P_i(x_i', .))$-relaxed cocoercive in the fourth argument and $S_i, T_i, G_i, F_i : X_i \to C(X_i)$ be such that $S_i$ is $L_{S_i} - \mathcal{D}$-Lipschitz continuous, $T_i$ is $L_{T_i} - \mathcal{D}$-Lipschitz continuous, $G_i$ is $L_{G_i} - \mathcal{D}$-Lipschitz continuous and $F_i$ is $L_{F_i} - \mathcal{D}$-Lipschitz continuous. Suppose that there exist constants $\rho_1, \rho_2 > 0$, such that the following conditions are satisfied:*

$$\left.\begin{aligned}
k_i^n &= b_i^n + d_j^n < 1, \text{ where,}\\
b_i^n : &= \Big\{\sqrt{(1 - 2s_i + 2L_{g_i} \times (1 + L_{\eta_i}) + 64c_i L_{g_i}^2)}\\
&\quad + L_i^n\Big\{\Big[L_{M_i}^2 - 2\rho_i\epsilon_i + 2\rho_i\Big(L_{(Q_i,i)}L_{(E_i,i)} + L_{(Q_i,j)}L_{(P_i,i)}\\
&\quad \times(L_{M_i}(L_{\eta_i} + 1))\Big) + 64c_i\rho_i^2\Big(L_{(Q_i,i)}^2 L_{(E_i,i)}^2 + L_{(Q_i,j)}^2 L_{(P_i,i)}^2\Big)\Big]^{\frac{1}{2}}\\
&\quad + \sqrt{\Big(1 - 2\mu_i + 2L_{N_{i_1}} L_{S_i} \times (1 + L_{\eta_i}) + 64c_i L_{N_{i_1}}^2 L_{S_i}^2\Big)}\\
&\quad + \sqrt{\Big(1 - 2\omega_i + 2L_{N_{i_3}} L_{G_i} \times (1 + L_{\eta_i}) + 64c_i L_{N_{i_3}}^2 L_{G_i}^2\Big)}\Big\}\Big\},\\
d_i^n : &= L_i^n\Big\{\rho_i^2\Big(L_{(Q_i,i)}^2 L_{(E_i,j)}^2 + L_{(Q_i,j)}^2 L_{(P_i,j)}^2\Big)\\
&\quad - 2\Big(\rho_i\xi_i L_{N_{i_2}}^2 L_{T_j}^2 + \Big(-\theta_i L_{N_{i_4}}^2 L_{F_j}^2\\
&\quad - \rho_i^2\varphi_i\Big(L_{(Q_i,i)}^2 L_{(E_i,j)}^2 + L_{(Q_i,j)}^2 L_{(P_i,j)}^2\Big) + \gamma_i\Big)\Big)\\
&\quad + 64c_i\Big(L_{N_{i_2}} L_{T_j} + L_{N_{i_4}} L_{F_j}\Big)^2\Big\}^{1/2};\ L_i^n := \frac{\tau_i^n}{\alpha_i^n - \beta_i^n}.
\end{aligned}\right\} \quad (4.1)$$

*Then for each $i = 1, 2$, the sequences $\{x_i^n\}$, $\{u_i^n\}$, $\{v_i^n\}$, $\{w_i^n\}$, $\{t_i^n\}$ generated by Algorithm (4.1) converges strongly to $x_i, u_i, v_i, w_i, t_i$, respectively, where $(x_1, x_2, u_1, u_2, v_1, v_2, w_1, w_2, t_1, t_2)$ is a solution of SGIVLIP (2.7).*

*Proof.* It follows from Theorem 3.2 that $(x_1, x_2, u_1, u_2, v_1, v_2, w_1, w_2, t_1, t_2)$ is a solution of SGIVLIP (2.7) and hence further it follows from Lemma 3.1 that

$$x_1 = (1-a^n)x_1 + a^n\Big\{x_1 - g_1(x_1) + R^{\partial\phi_1}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1) - [N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\},$$

$$x_2 = (1-a^n)x_2 + a^n\Big\{x_2 - g_2(x_2) + R^{\partial\phi_2}_{\rho_2,\eta_2}\{(M_2(A_2,B_2)\circ g_2)(x_2) - [N_2(u_2,v_1,w_2,t_1) - \rho_2 Q_2(E_2(x_1,x_2),P_2(x_1,x_2))]\}. \quad (4.2)$$

From Algorithm 4.1, we estimate

$$\|x_1^{n+1} - x_1\|_1 \le (1-a^n)\|x_1^n - x_1\|_1 + a^n\|x_1^n - x_1 - (g_1(x_1^n) - g_1(x_1))\|_1 + a^n\|R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\mathcal{Y}_1^n) - R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}(\mathcal{Y}_1)\|_1 + a^n e_1^n,$$

where

$$\mathcal{Y}_1^n = \{(M_1(A_1,B_1)\circ g_1)(x_1^n) - [N_1(u_1^n,v_2^n,w_1^n,t_2^n) - \rho_1 Q_1(E_1(x_1^n,x_2^n),P_1(x_1^n,x_2^n))]\}$$

and

$$\mathcal{Y}_1 = \{(M_1(A_1,B_1)\circ g_1)(x_1) - [N_1(u_1,v_2,w_1,t_2) - \rho_1 Q_1(E_1(x_1,x_2),P_1(x_1,x_2))]\}.$$

Using Theorem 3.2, we have

$$\begin{aligned}\|x_1^{n+1} - x_1\|_1 &\le (1-a^n)\|x_1^n - x_1\|_1 + a^n\|x_1^n - x_1 - (g_1(x_1^n) - g_1(x_1))\|_1 \\ &\quad + a^n\|R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\mathcal{Y}_1^n) - R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\mathcal{Y}_1)\|_1 \\ &\quad + a^n\Phi_1^n + a^n e_1^n \\ &\le (1-a^n)\|x_1^n - x_1\|_1 + a^n\|x_1^n - x_1 - (g_1(x_1^n) - g_1(x_1))\|_1 \\ &\quad + a^n L_1^n\|\mathcal{Y}_1^n - \mathcal{Y}_1\|_1 + a^n\Phi_1^n + a^n e_1^n, \end{aligned} \quad (4.3)$$

where

$$\Phi_1^n := \|R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\mathcal{Y}_1) - R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}(\mathcal{Y}_1)\|_1. \quad (4.4)$$

Using the same arguments used in estimating (3.8)-(3.23), we have

$$\begin{aligned}&\|x_1^{n+1} - x_1\|_1 \\ &\le (1-a^n)\|x_1^n - x_1\|_1 + a^n\Big[\sqrt{(1 - 2s_1 + 2L_{g_1}\times(1+L_{\eta_i}) + 64c_1 L_{g_1}^2)} \\ &\quad + L_1^n\Big\{\Big[L_{M_1}^2 - 2\rho_1\epsilon_1 + 2\rho_1\Big(L_{(Q_1,1)}L_{(E_1,1)} + L_{(Q_1,2)}L_{(P_1,1)}\times(L_{M_1}(1+L_{\eta_i}))\Big) \\ &\quad + 64c_1\rho_1^2\Big(L_{(Q_1,1)}^2 L_{(E_1,1)}^2 + L_{(Q_1,2)}^2 L_{(P_1,1)}^2\Big)\Big]^{\frac{1}{2}} \\ &\quad + \sqrt{\Big(1 - 2\mu_1 + 2L_{N_{1_1}}L_{S_1}\times(1+L_{\eta_i}) + 64c_1 L_{N_{1_1}}^2 L_{S_1}^2\Big)}\end{aligned}$$

$$+\sqrt{\Big(1-2\omega_1+2L_{N_{1_3}}L_{G_1}\times(1+L_{\eta_i})+64c_1L^2_{N_{1_3}}L^2_{G_1}\Big)}\Big\}\Big]\|x_1^n-x_1\|_1$$
$$+a^nL_1^n\Big[\rho_1^2\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)}+L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big)$$
$$-2\Big(\rho_1\xi_1L^2_{N_{1_2}}L^2_{T_2}+\Big(-\theta_1L^2_{N_{1_4}}L^2_{F_2}-\rho_1^2\varphi_1\Big(L^2_{(Q_1,1)}L^2_{(E_1,2)}+L^2_{(Q_1,2)}L^2_{(P_1,2)}\Big)+\gamma_1\Big)\Big)$$
$$+64c_1\Big(L_{N_{1_2}}L_{T_2}+L_{N_{1_4}}L_{F_2}\Big)^2\Big]^{1/2}\|x_2^n-x_2\|_2+a^n\Phi_1^n+a^ne_1^n$$
$$\le(1-a^n)\|x_1^n-x_1\|_1$$
$$+a^n\{b_1^n\|x_1^n-x_1\|_1+d_1^n\|x_2^n-x_2\|_2\}+a^n\Phi_1^n+a^n\|e_1^n\|_1. \tag{4.5}$$

Similarly, we infer that

$$\|x_2^{n+1}-x_2\|_2\le(1-a^n)\|x_2^n-x_2\|_2+a^n\{b_2^n\|x_2^n-x_2\|_2+d_2^n\|x_1^n-x_1\|_1\}$$
$$+a^n\Phi_2^n+a^n\|e_2^n\|_2. \tag{4.6}$$

From (4.5) and (4.6), we have

$$\|x_1^n-x_1\|_1+\|x_2^{n+1}-x_2\|_2$$
$$\le[1-a^n(1-k_1^n)]\Big\|x_1^n-x_1\Big\|_1+[1-a^n(1-k_2^n)]\Big\|x_2^n-x_2\Big\|_2$$
$$+a^n(\Phi_1^n+\Phi_2^n)+a^n\Big(\|e_1^n\|_1+\|e_2^n\|_2\Big)$$
$$\le[1-a^n(1-\max\{k_1^n,k_2^n\})]\Big(\Big\|x_1^n-x_1\Big\|_1+\Big\|x_2^n-x_2\Big\|_2\Big)$$
$$+a^n(1-\max\{k_1^n,k_2^n\})\frac{(\Phi_1^n+\Phi_2^n+\|e_1^n\|_1+\|e_2^n\|_2)}{(1-\max\{k_1^n,k_2^n\})}, \tag{4.7}$$

where $k_1^n=b_1^n+d_2^n;\ k_2^n=b_2^n+d_1^n$.

If $\zeta^n=\Big\|x_1^n-x_1\Big\|_1+\Big\|x_2^n-x_2\Big\|_2,\ \hbar^n=\dfrac{\Big\{(\Phi_1^n+\Phi_2^n+\|e_1^n\|_1+\|e_2^n\|_2\Big\}}{(1-\max\{k_1^n,k_2^n\})}$ and

$$\omega^n=a^n(1-\max\{k_1^n,k_2^n\}),$$

then, we have

$$\zeta^{n+1}\le(1-\omega^n)\zeta^n+\omega^n\hbar^n.$$

Using Lemma 2.6, we have $\zeta^n\to0$ as $n\to\infty$. This implies $x_1^n\to x_1, x_2^n\to x_2$ as $n\to\infty$. Since $S_i$ is $L_{S_i}-\mathcal{D}$-Lipschitz continuous, it follows from Algorithm 4.1 that

$$\Big\|u_i^n-u_i\Big\|_i\le\mathcal{D}(S_i(x_i^n),S_i(x_i))_i\le L_{S_i}\Big\|x_i^n-x_i\Big\|_i.$$

This implies that $u_i^n \to u_i$ as $n \to \infty$. Further, we claim that $u_i \in S_i(x_i)$,

$$\begin{aligned} d(u_i, S_i(x_i)) &\leq \|u_i - u_i^n\|_i + d(u_i^n, S_i(x_i))_i \\ &\leq \|u_i - u_i^n\|_i + \mathcal{D}(S_i(x_i^n), S_i(x_i))_i \\ &\leq \|u_i - u_i^n\|_i + L_{S_i}\|x_i^n - x_i\|_i \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $S_i(x_i)$ is compact, we have $u_i \in S_i(x_i)$. Similarly, we can prove that $v_i \in T_i(x_i), w_i \in G_i(x_i), t_i \in F_i(x_i)$. So the approximate solution $(x_i^n, u_i^n, v_i^n, w_i^n, t_i^n)$ generated by Algorithm 4.1 converges strongly to $(x_i, u_i, v_i, w_i, t_i)$ which is a solution of (2.7). □

Now, we discuss the stability analysis of Algorithm 4.1.

**Theorem 4.5.** *Let, for $i \in \{1,2\}$, $j \in \{1,2\}\backslash\{i\}$, $X_i$, $\eta_i^n$, $\eta_i$, $A_i^n$, $B_i^n$, $A_i$, $B_i$, $M_i^n(A_i^n, B_i^n)$, $M_i(A_i, B_i)$, $\phi_i^n$, $\phi_i$, $h_i^n$, $h_i$, $N_i$, $Q_i$, $E_i$, $P_i$, $S_i$, $T_i$, $G_i$, $F_i$, $g_i$, $M_i(A_i, B_i) \circ g_i$ be same as in Theorem 4.4 and let (4.2) of Theorem 4.4 hold. Let Conditions (4.2) and (4.3) hold and let $\{(\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n, \bar{t}_i^n)\}_{n\geq 0}$ be any sequence in $X_i$ and define $\epsilon^n = \epsilon_1^n + \epsilon_2^n$ for $n \geq 0$ by*

$$\begin{aligned} \epsilon_1^n = \Big\|\bar{x}_1^{n+1} - \Big[(1-a^n)\bar{x}_1^n + a^n\Big\{\bar{x}_1^n - g_1(\bar{x}_1^n) + R^{\partial\phi_1}_{\rho_1,\eta_1^n}\{(M_1(A_1,B_1)\circ g_1)(\bar{x}_1^n) \\ - \Big(N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n, \bar{t}_2^n) - \rho_1 Q_1(E_1(\bar{x}_1^n, \bar{x}_2^n), P_1(\bar{x}_1^n, \bar{x}_2^n))\Big)\Big\} + a^n e_1^n\Big]\Big\|_1, \\ \epsilon_2^n = \Big\|\bar{x}_2^{n+1} - \Big[(1-a^n)\bar{x}_2^n + a^n\Big\{\bar{x}_2^n - g_2(\bar{x}_2^n) + R^{\partial\phi_2}_{\rho_2,\eta_2^n}\{(M_2(A_2,B_2)\circ g_2)(\bar{x}_2^n) \\ - \Big(N_2(\bar{u}_2^n, \bar{v}_1^n, \bar{w}_2^n, \bar{t}_1^n) - \rho_2 Q_2(E_2(\bar{x}_1^n, \bar{x}_2^n), P_2(\bar{x}_1^n, \bar{x}_2^n))\Big)\Big\} + a^n e_2^n\Big]\Big\|_2, \end{aligned}$$

$$\begin{aligned} \bar{u}_i^n \in S_i(\bar{x}_i^n) &: \|\bar{u}_i^{n+1} - \bar{u}_i^n\|_i \leq \mathcal{D}(S_i(\bar{x}_i^{n+1}), S_i(\bar{x}_i^n))_i, \\ \bar{v}_i^n \in T_i(\bar{x}_i^n) &: \|\bar{v}_i^{n+1} - \bar{v}_i^n\|_i \leq \mathcal{D}(T_i(\bar{x}_i^{n+1}), T_i(\bar{x}_i^n))_i, \\ \bar{w}_i^n \in G_i(\bar{x}_i^n) &: \|\bar{w}_i^{n+1} - \bar{w}_i^n\|_i \leq \mathcal{D}(G_i(\bar{x}_i^{n+1}), G_i(\bar{x}_i^n))_i, \\ \bar{t}_i^n \in F_i(\bar{x}_i^n) &: \|\bar{t}_i^{n+1} - \bar{t}_i^n\|_i \leq \mathcal{D}(F_i(\bar{x}_i^{n+1}), F_i(\bar{x}_i^n))_i. \end{aligned} \tag{4.8}$$

*where $\rho_1, \rho_2$ are positive constants. Then, for any sequences $\{\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n, \bar{t}_i^n\}$, $\lim_{n\to\infty}(\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n, \bar{t}_i^n) = (x_i, u_i, v_i, w_i, t_i)$ if and only if $\lim_{n\to\infty}\epsilon^n = 0$, where $\epsilon^n = \epsilon_1^n + \epsilon_2^n$, for all $n \geq 0$.*

*Proof.* By Theorem 4.4, there exists a solution $(x_i, u_i, v_i, w_i, t_i)$ of SGIVLIP (2.7). From Lemma 3.1, iterative algorithm 4.1, and using the same arguments used in estimating (3.7)-(3.15), (4.2), we have

$$\begin{aligned} &\|\bar{x}_1^{n+1} - x_1\|_1 \\ &= \Big\|\bar{x}_1^{n+1} - \Big[(1-a^n)\bar{x}_1^n + a^n\Big\{\bar{x}_1^n - g_1(\bar{x}_1^n) + R^{\partial\phi_1^n}_{\rho_1,\eta_1^n}\{(M_1(A_1,B_1)\circ g_1)(\bar{x}_1^n) \end{aligned}$$

$$
\begin{aligned}
&- [N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n, \bar{t}_2^n) - \rho_1 Q_1(E_1(\bar{x}_1^n, \bar{x}_2^n), P_1(\bar{x}_1^n, \bar{x}_2^n))]\}\Big\} + a^n e_1^n\Big]\Big\|_1 \\
&+ \Big\|\Big[(1-a^n)\bar{x}_1^n + a^n\Big\{\bar{x}_1^n - g_1(\bar{x}_1^n) + R^{\partial\phi_1^n}_{\rho_1,\eta_1^n}\{(M_1(A_1,B_1)\circ g_1)(\bar{x}_1^n) \\
&- [N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n, \bar{t}_2^n) - \rho_1 Q_1(E_1(\bar{x}_1^n, \bar{x}_2^n), P_1(\bar{x}_1^n, \bar{x}_2^n))]\}\Big\} + a^n e_1^n\Big] - x_1\Big\|_1 \\
\le\ & \epsilon_1^n + \Big\|\Big[(1-a^n)\bar{x}_1^n + a^n\Big\{\bar{x}_1^n - g_1(\bar{x}_1^n) + R^{\partial\phi_1^n}_{\rho_1,\eta_1^n}\{(M_1(A_1,B_1)\circ g_1)(\bar{x}_1^n) \\
&- [N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n, \bar{t}_2^n) - \rho_1 Q_1(E_1(\bar{x}_1^n, \bar{x}_2^n), P_1(\bar{x}_1^n, \bar{x}_2^n))]\}\Big\} + a^n e_1^n\Big] \\
&- \Big[(1-a^n)x_1 + a^n\Big\{x_1 - g_1(x_1) + R^{\partial\phi_1}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1) \\
&- [N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1,x_2), P_1(x_1,x_2))]\}\Big\}\Big]\Big\|_1 \\
\le\ & \epsilon_1^n + (1-a^n)\|\bar{x}_1^n - x_1\|_1 + a^n\|(\bar{x}_1^n - x_1) - (g_1(\bar{x}_1^n) - g_1(x_1))\|_1 \\
&+ a^n\|R^{\partial\phi_1^n}_{\rho_1,\eta_1^n}\{(M_1(A_1,B_1)\circ g_1)(\bar{x}_1^n) \\
&- [N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n, \bar{t}_2^n) - \rho_1 Q_1(E_1(\bar{x}_1^n, \bar{x}_2^n), P_1(\bar{x}_1^n, \bar{x}_2^n))]\} \\
&- \Big\{R^{\partial\phi_1}_{\rho_1,\eta_1}\{(M_1(A_1,B_1)\circ g_1)(x_1) \\
&- [N_1(u_1, v_2, w_1, t_2) - \rho_1 Q_1(E_1(x_1,x_2), P_1(x_1,x_2))]\}\Big\}\|_1 + a^n\|e_1^n\|_1.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|\bar{x}_1^{n+1} - x_1\|_1 \le\ & \epsilon_1^n + (1-a^n)\|\bar{x}_1^n - x_1\|_1 \\
&+ a^n\|(\bar{x}_1^n - x_1) - (g_1(\bar{x}_1^n) - g_1(x_1))\|_1 \\
&+ a^n\|R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\bar{\mathcal{Y}}_1^n) - R^{\partial\phi_1, M_1(A_1,B_1)}_{\rho_1,\eta_1}(\mathcal{Y}_1)\|_1 + a^n e_1^n,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\mathcal{Y}}_1^n := \{&(M_1(A_1,B_1)\circ g_1)(\bar{x}_1^n) - [N_1(\bar{u}_1^n, \bar{v}_2^n, \bar{w}_1^n, \bar{t}_2^n \\
&-\rho_1 Q_1(E_1(\bar{x}_1^n, \bar{x}_2^n), P_1(\bar{x}_1^n, \bar{x}_2^n))]\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{Y}_1 := \{&(M_1(A_1,B_1)\circ g_1)(x_1) - [N_1(u_1, v_2, w_1, t_2) \\
&-\rho_1 Q_1(E_1(x_1,x_2), P_1(x_1,x_2))]\}.
\end{aligned}
$$

Using Theorem 4.4, we have

$$
\begin{aligned}
\|\bar{x}_1^{n+1} - x_1\|_1 \le\ & \epsilon_1^n + (1-a^n)\|\bar{x}_1^n - x_1\|_1 + a^n\|\bar{x}_1^n - x_1 - (g_1(\bar{x}_1^n) - g_1(x_1))\|_1 \\
&+ a^n\|R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\bar{\mathcal{Y}}_1^n) - R^{\partial\phi_1^n, M_1^n(A_1^n,B_1^n)}_{\rho_1,\eta_1^n}(\mathcal{Y}_1)\|_1 \\
&+ a^n\Phi_1^n + a^n e_1^n \\
\le\ & \epsilon_1^n + (1-a^n)\|\bar{x}_1^n - x_1\|_1 + a^n\|\bar{x}_1^n - x_1 - (g_1(\bar{x}_1^n) - g_1(x_1))\|_1 \\
&+ a^n L_1^n\|\bar{\mathcal{Y}}_1^n - \mathcal{Y}_1\|_1 + a^n\Phi_1^n + a^n e_1^n. \qquad (4.9)
\end{aligned}
$$

This implies

$$\|\bar{x}_1^{n+1} - x_1\| \leq \epsilon_1^n + (1-a^n)\|\bar{x}_1^n - x_1\|_1 + a^n\{b_1^n\|\bar{x}_1^n - x_1\|_1 + d_1^n\|\bar{x}_2^n - x_2\|_2\} + a^n\Phi_1^n + a^n\|e_1^n\|_1. \tag{4.10}$$

Similarly, we infer that

$$\|\bar{x}_2^{n+1} - x_2\|_2 \leq \epsilon_2^n + (1-a^n)\|\bar{x}_2^n - x_2\|_2 + a^n\{b_2^n\|\bar{x}_2^n - x_2\|_2 + d_2^n\|\bar{x}_1^n - x_1\|_1\} + a^n\Phi_2^n + a^n\|e_2^n\|_2. \tag{4.11}$$

This implies

$$\begin{aligned}&\|\bar{x}_1^{n+1} - x_1\|_1 + \|\bar{x}_2^{n+1} - x_2\|_2 \\ &\leq \epsilon^n + [1 - a^n(1 - \max\{k_1^n, k_2^n\})]\Big(\Big\|\bar{x}_1^n - x_1\Big\|_1 + \Big\|\bar{x}_2^n - x_2\|_2\Big) \\ &\quad + a^n(1-\max\{k_1^n,k_2^n\})\frac{(\Phi_1^n + \Phi_2^n + \|e_1^n\|_1 + \|e_2^n\|_2)}{(1-\max\{k_1^n,k_2^n\})}.\end{aligned} \tag{4.12}$$

Suppose that $\lim_{n\to\infty}\epsilon^n = 0$. If

$$\begin{aligned}\zeta^n &= \Big\|\bar{x}_1^n - x_1\Big\|_1 + \Big\|\bar{x}_2^n - x_2\Big\|_2, \ \hbar^n = \frac{\big\{(\Phi_1^n + \Phi_2^n + \|e_1^n\|_1 + \|e_2^n\|_2\big\}}{(1-\max\{k_1^n,k_2^n\})}, \\ \omega^n &= a^n(1-\max\{k_1^n,k_2^n\}),\end{aligned}$$

then, we have

$$\zeta^{n+1} \leq (1-\omega^n)\zeta^n + \omega^n\hbar^n.$$

Using Lemma 2.6, we have $\zeta^n \to 0$ as $n \to \infty$. This implies $\bar{x}_1^n \to x_1, \bar{x}_2^n \to x_2$ as $n \to \infty$.

Proceeding as in the convergence of the sequence $(u_i^n, v_i^n, w_i^n, t_i^n)$ it follows that $(\bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n, \bar{t}_i^n) \to (u_i, v_i, w_i, t_i)$ as $n \to \infty$.

Conversely, suppose that $(\bar{x}_i^n, \bar{u}_i^n, \bar{v}_i^n, \bar{w}_i^n, t_i^n) \to (x_i, u_i, v_i, w_i, t_i)$ as $n \to \infty$. In view of (4.12), we have

$$\begin{aligned}\epsilon^n &= \epsilon_1^n + \epsilon_2^n \\ &\leq \|\bar{x}_1^{n+1} - x_1\|_1 + \|\bar{x}_2^{n+1} - x_2\|_2 \\ &\quad + [1 - a^n(1-\max\{k_1^n,k_2^n\})]\left(\left\|\bar{x}_1^n - x_1\right\|_1 + \left\|\bar{x}_2^n - x_2\right\|_2\right) \\ &\quad + a^n(1-\max\{k_1^n,k_2^n\})\frac{(\Phi_1^n + \Phi_2^n + \|e_1^n\|_1 + \|e_2^n\|_2)}{(1-\max\{k_1^n,k_2^n\})}.\end{aligned}$$

Therefore, we have $\lim_{n\to\infty} \epsilon^n = 0$. This completes the proof. □

**Remark 4.6.** The problem considered in this paper is more general than the similar problems consider by many researchers in the literature. The results presented in this paper generalize many known results in the literature. The class of $M(.,.)$-$\eta$-proximal mapping is more general than the similar types of operators considered in the literature, see for example [13,19] and the related references cited therein. The stability analysis can further be extended for other classes of variational inclusions and their extensions considered in the literature, see for example $[1, 3–6, 8–11, 13, 17–25, 29–31]$.

**Acknowledgments:** The first author was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the Korea (2018R1D1A1B07045427).

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