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ON OPTIMAL SOLUTIONS OF WELL-POSED PROBLEMS AND VARIATIONAL INEQUALITIES

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Abstract. In this paper, we study well-posed problems and variational inequalities in locally convex Hausdorff topological vector spaces. The necessary and sufficient conditions are obtained for the existence of solutions of variational inequality problems and quasi-variational inequalities even when the underlying set K is not convex. In certain cases, solutions obtained are not unique. Moreover, counter examples are also presented for the authenticity of the main results.

1. Introduction

The theory of variational inequalities has become a rich source of inspiration in both mathematical and engineering sciences, which has begun with the works of Fichera [8] and Hartmann and Stampacchia [11]. Variational inequalities and its various generalizations have been very effective and quite powerful tool in studying the existence of solutions of constrained problems arising in

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mechanics, optimization and control, operations research, general equilibrium problems in economics and transportation, and so on. It also provides us unified and general framework for studying boundary value problems arising out of contact problems in electrostatics, fluid flow through porous media etc.

The study of variational inequality problems has been carried out by several mathematicians, see Anh et al. [1], Baiocchi and Capelo [4], Crank [6], Glowinski, Lions and Tremoliers [9], Kikuchi and Oden [15], Kim [16], Kim et al. [17, 18, 19, 20], Kinderlehrer and Stampacchia [21], Noor [27].

The work is also generalized in several directions like quasi-variational inequality [3], vector variational inequality [2, 19], generalized vector variational inequalities [26, 34], variational-like inequalities for multivalued maps [29], Stampacchia variational inequality with weak convex mappings [5], and so on.

There are several notion of well-posedness related to optimization problems, see for example, [7], [13], [14], [24], [25], [28], [30], [31], [33] and the references therein. These notions of well-posedness can be divided into three classes, namely, Tykhonov type [32], Levitin-Polyak type [23] and Hadamard type [10]. Generally speaking, in the study of Tykhonov well-posedness of a problem one induces the notion of approximate sequence for the solution requires some convergence of such sequences to a solution of the problem, Levintin-Polyak well-posedness of a problem means the convergence of the approximating solution sequence to the problem with some constraints, while Hadamard well-posedness of a problem means the continuous dependence of the solutions on the data or the parameter of the problem, see for example, [7], [13], [14], [24], [25], [28], [30], [31], [33] and the reference therein.

In this paper, we study well-posed problems in locally convex Hausdorff topological vector spaces. The necessary and sufficient conditions are obtained for the existence of optimal solutions of variational and quasi-variational inequality problems by using Cauchy-Schwartz inequality and Theorem 5.1.1 in [22].

The rest of this article is structured as follows: In section 2, we recall some preliminary material on functional analysis and then introduce notion of well-posedness, variational inequality problem (VIP) and quasi-variational inequality problem (QVIP). In section 3, we study well-posed problem for locally convex Hausdorff topological vector space and and gave some examples to illustrate the result. The necessary and sufficient conditions for the variational inequality problem(VIP) and quasi-variational inequality problem (QVIP) to get optimal solution are established in section 4.

2. Formulations and preliminaries

Let (X, τ) be a locally convex topological vector space over field $\Phi(=\mathbb{R})$. A subset B of X is called balanced if $aB \subseteq B$ whenever $|a| \leq 1, a \in \Phi$. A subset B of X is known as absorbing if for every $x \in X$, there exists $a > 0$ such that $x \in aB$.

Let B be a convex, balanced, absorbing set in X. The Minkowski functional (gauge) $p_B : X \longrightarrow R$ is defined as $p_B(x) = inf\,\{a > 0 : x \in aB\}$. Since gauge of a convex balanced absorbing set in topological vector space is a seminorm, it follows that p_B is a seminorm.

Consider $\mathcal{P} = \{p_B : B$ is convex balanced absorbing set$\}$. Then $\mathcal{P}$ is a family of seminorms on X.

For each $p \in \mathcal{P}$ and $U(x, \epsilon, p) = \{y \in X : p(y - x) < \epsilon\}$, let

$$\mathcal{S} = \{U(x, \epsilon, p) : x \in X, \epsilon > 0, p \in \mathcal{P}\}.$$

Then the topology generated by $\mathcal{S}$ as a subbase is denoted by $\tau_{\mathcal{P}}$. It is well known that if τ is a locally convex topology on X then $\tau = \tau_{\mathcal{P}}$.

Let $I : X \longrightarrow (-\infty, \infty]$ be a proper extended real valued function defined on a topological space X. Consider the problem to minimize $I(x)$ subject to $x \in X$ is denoted by (X, I). If $I(x_0) \leq I(x)$, for all $x \in X$, then x_0 is called a global minimizer of (X, I). The set of all global minimizer is denoted by argmin(X, I). For argmin$(X, I) = \{x_0\}$, we say that (X, I) is well-posed if $I(x_n) \longrightarrow I(x_0)$ then $x_n \longrightarrow x_0$. In other words, the problem (X, I) is Tykhonov well-posed if and only if I has a unique global minimum point on X towards which every minimizing sequence converges.

Now we shall work under the following settings, unless otherwise specified:

Let X be a topological vector space with its topological dual X^* and K be a nonempty convex subset of X. The value of $l \in X^*$ at x is denoted by $\langle l, x\rangle$. Let $F : K \longrightarrow X^*$ be a mapping. Then the variational inequality problem (in short,VIP) is to find $x \in K$ such that

$$\langle F(x), y - x\rangle \geq 0, \text{ for all } y \in K \tag{2.1}$$

or equivalently

$$\langle F(x), x\rangle \leq \langle F(x), y\rangle, \text{ for all } y \in K.$$

Suppose $K : X \to 2^X$, where 2^X is the power set of X, be a set-valued map such that $K(x)$ is nonempty. Then the quasi-variational inequality problem (in short, QVIP) consists in finding a vector $x \in K(x)$ such that

$$\langle F(x), y - x\rangle \geq 0, \text{ for all } y \in K(x). \tag{2.2}$$

Now we need following results to prove the main results of the paper.

Theorem 2.1. ([22], Theorem 5.3.1) *Let X be a vector space over Φ, let $L \subset X$ be a real hyperplane, and let $K \subset X$ be convex. Then the following statements are equivalent:*

(i) *K lies strictly on one side of L.*
(ii) *$K \cap L = \phi$.*

Theorem 2.2. ([22], Theorem 13.1.1) *Let $(X, \langle .,. \rangle)$ be an inner product space over Φ. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \ (x, y \in X).$$

Moreover,

$$|\langle x, y\rangle| = \|x\|\|y\|, \ (x, y \in X)$$

if and only if x, y are linear independent.

3. Well-posed problem for locally convex Hausdorff t.v.s

In this section, we shall give a formulation for well-posed problem in a locally convex Hausdorff topological vector space.

Theorem 3.1. *Let (X, τ) be a locally convex Hausdorff topological vector space. Then $I : X \longrightarrow (-\infty, \infty)$ is well-posed if and only if*

(i) *there exists $x_0 \in X$ such that $I(x_0) \leq I(x)$, for all $x \in X$,*
(ii) *for every $\epsilon > 0$, there exists $\delta > 0$ with $|I(x) - I(x_0)| < \delta$ such that $p(x - x_0) < \epsilon$, for all $p \in \mathcal{P}$ and $x \in X$.*

Proof. Since (X, τ) is a locally convex Hausdorff topological vector space, there exists a family of seminorms $\mathcal{P}$ on X such that $\tau = \tau_{\mathcal{P}}$. Suppose that $I : X \longrightarrow (-\infty, \infty)$ is well-posed. We first prove that the condition (ii) is true. If the condition (ii) is false, then there exists $\epsilon > 0$ such that for every $\delta > 0$, $|I(x) - I(x_0)| < \delta$ but $p(x-x_0) \geq \epsilon$, for some $p \in \mathcal{P}$ and $x \in X$. Choose $\delta = \frac{1}{n}$. Then we can find $x_n \in X$ such that $|I(x_n) - I(x_0)| < \frac{1}{n}$ and $p(x_n - x_0) \geq \epsilon$, for some seminorm p. This means that $I(x_n) \longrightarrow I(x_0)$ as $n \longrightarrow \infty$. But $x_n \not\longrightarrow x_0$. Hence the condition (ii) must be true.

Next we prove that the condition (i) is true. If the condition (i) is false, then there exists $y \in X$ such that $I(y) < I(x_0)$ so that $|I(y) - I(x_0)| < \delta$, for all $\delta > 0$. Hence by the condition (ii) $y \in U(x_0, \epsilon, p)$, for every $p \in \mathcal{P}$, $\epsilon > 0$ and hence y belongs to every neighborhood of x_0, which is a contradiction, since (X, τ) is a Hausdorff space. Therefore $I(x_0) \leq I(x)$, for all $x \in X$.

Conversely, suppose that (i) and (ii) are true. From condition (ii), for every $\epsilon > 0$, there exists $\delta > 0$ with $|I(x_n) - I(x_0)| < \delta$ such that $p(x_n - x_0) < \epsilon$. This implies that whenever $\{x_n\}$ is a sequence and $I(x_n) \longrightarrow I(x_0)$, we have $x_n \longrightarrow x_0$ in (X, τ_p). That is, arg min $(X, I) = \{x_0\}$. Hence I is well-posed. $\square$

Example 3.2. Let $X = l^p(N)$, the space of all p^{th} summable sequence of real numbers. Define $I : X \longrightarrow \mathbb{R}$ by $I(x) = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$. For each $n \in \mathbb{N}$, define $p_n : l^p(\mathbb{N}) \longrightarrow \mathbb{R}$ by $p_n(x) = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$. Then, each p_n is a seminorm on X. Let $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$. Then $(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space. Let $x_0(k) = 0$, for every $k \in \mathbb{N}$. Then $x_0 \in l^p$ and $I(x_0) = 0$. Now for each $n \in \mathbb{N}$,

$$p_n\left(x^k\right) \leq I\left(x^k\right). \tag{3.1}$$

Suppose $I\left(x^k\right) \longrightarrow 0$ in $(X, \tau_{\mathcal{P}})$. Then from (3.1) as $p_n\left(x^k\right) \longrightarrow 0$ as $k \longrightarrow \infty$. This is true for every $n \in \mathbb{N}$. Hence $x^k \longrightarrow 0$ as $k \longrightarrow \infty$. Also $I(x_0) \leq I(x)$, for every $x \in l^p(\mathbb{N})$. Thus I is well-posed.

Example 3.3. Let $X = C_b(\mathbb{R})$, the space of all bounded continuous real valued functions defined on $\mathbb{R}$. Let $I : X \longrightarrow \mathbb{R}$ be a function defined as $I(x) = \inf\{|x(t)| : t \in \mathbb{R}\}$. For $m \in \mathbb{N}$, set $p_m(x) = \sup_{|t| \leq m} |x(t)|$. Then p_m is a seminorm. If we take $\mathcal{P} = \{p_m : m \in \mathbb{N}\}$, then $\mathcal{P}$ is a family of seminorms. Then for $n \in \mathbb{N}$, let $x_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$x_n(t) = \begin{cases} |t|, & \text{when } |t| \leq n, \\ n, & \text{when } |t| > n. \end{cases}$$

Then $x_n \in C_b(\mathbb{R})$ for every $n \in \mathbb{N}$. Define $x_0 : \mathbb{R} \longrightarrow \mathbb{R}$ by $x_0(t) = 0$, for every $t \in \mathbb{R}$. Now $I(x_0) = \inf\{|x_o(t)| : t \in \mathbb{R}\} = 0$ and $I(x_n) = 0$, for every $n \in \mathbb{N}$. Consider $|I(x_n) - I(x_0)| = 0$ so that $I(x_n) \to I(x_0)$. But

$$p_1(x_n - x_0) = \sup_{|t| \leq 1} \{|x_n(t) - x_0(t)|\} \geq 1,$$

for every $n \in \mathbb{N}$. This shows that $x_n \not\longrightarrow x_0$ in $(X, \tau_{\mathcal{P}})$. Hence I is not well-posed.

4. Optimal solution of variational inequality problems

In this section, we shall obtain the necessary and sufficient conditions for the variational inequality problem and quasi-variational inequality problem to have optimal solutions.

Theorem 4.1. *Let H be a real Hilbert space and let K be a nonempty convex subset of H. For $x \in K$, let $F : K \longrightarrow H^*$ be defined by $F(x) = f_x$, where $f_x : H \longrightarrow \mathbb{R}$ is given by $f_x(y) = \langle x, y \rangle$, for all $y \in H$. Then*

(i) *if* $0 \in K$, *then for all* $y \in K$, *VIP*-(2.1) *has a solution if and only if* $x = 0$.

(ii) *if* $0 \notin K$, *then for all* $y \in K$, *VIP*-(2.1) *has a solution if and only if* K *contains a vector of smallest norm and* $K \subset H \setminus kerF(x)$.

Proof. (i) Suppose VIP-(2.1) has a solution $x \in K$. Then

$$\langle F(x), x\rangle \leq \langle F(x), y\rangle, \text{ for all } y \in K. \tag{4.1}$$

For $y = 0$, we have

$$\|x\|^2 \leq |\langle x, y\rangle| \leq \|x\|\,\|y\|,$$

which implies that $x = 0$.

Conversely, if $x = 0$, then clearly (4.1) is satisfied. Therefore variational inequality problem has a solution.

(ii) Let $0 \notin K$. Suppose the VIP-(2.1) has a solution for all $y \in K$. We prove that K contains a vector of smallest norm. Now there exists $x \in K$ such that

$$\langle F(x), x\rangle \leq \langle F(x), y\rangle, \text{ for all } y \in K. \tag{4.2}$$

Since $\langle F(x), x\rangle = \|x\|^2$, we can see that $\langle F(x), x\rangle = 0$ if and only if $x = 0$. Therefore from (4.2), we have

$$\|x\|^2 \leq |\langle x, y\rangle| \leq \|x\|\,\|y\|, \text{ for all } y \in K,$$

which implies that $\|x\| \leq \|y\|$ for all $y \in K$, because $x \neq 0$ by hypothesis. Thus $x \in K$ is a vector of smallest norm.

Next we prove that $K \cap \ker F(x) = \phi$. In this case, there exists $z \in K \cap \ker F(x)$. Thus $\langle F(x), z\rangle = 0$ so that $\langle F(x), x\rangle \leq \langle F(x), z\rangle$, which is a contradiction that VIP-(2.1) has a solution. Hence $K \cap \ker F(x) = \phi$.

Conversely, suppose K contains a vector $x \in K$ of smallest norm and $K \cap \ker F(x) = \phi$, that is, $K \subset H \setminus \ker F(x)$. But $\ker F(x)$ is a maximal subspace of H, it is easy to see that $y_1, y_2 \in K$ are linearly dependent. For, if y_1 and y_2 are linearly independent, then $\text{span}\{y_1 \cup \ker F(x)\}$ is a maximal subspace of H which contradicts the maximality of $\ker F(x)$. This proves that all vectors of K are linearly dependent. In view of Theorem 2.2, if x, y are linearly dependent then the equality holds in Cauchy-Schwartz inequality, that is,

$$|\langle x, y\rangle| = \|x\|\,\|y\|, \text{ for all } y \in K.$$

Since $0 \notin K$ and K is convex, so $F(K)$ is convex. Now $\ker F(x)$ is a hyperplane and K is a convex set such that $K \cap \ker F(x) = \phi$. In view of Theorem 2.2, K lies strictly on one side of the $\ker F(x)$. Since $0 \notin K$, so $\langle F(x), x\rangle > 0$. Therefore $\langle F(x), y\rangle > 0$, for all $y \in K$.

Now $\|x\| \leq \|y\|$ implies that

$$\|x\|^2 \leq \|x\| \, \|y\| = |\langle x, y\rangle| = \langle x, y\rangle \,,$$

that is, $\langle F(x), x\rangle \leq \langle F(x), y\rangle,$ for all $y \in K$. Thus x is a solution of VIP(2.1). This completes the proof. □

Example 4.2. Let $X = \mathbb{R}^3$ and K be the region bounded by the lines $x = 1, z = 0$; $y = 0, z = 0$; $x = 3, z = 0$; $2x - y = 2, z = 0$. Then K is a convex set and $\|K\| = [1, 5]$. Define $F : K \longrightarrow (\mathbb{R}^3)^*$ by

$$\langle F(x), y\rangle = x \cdot y.$$

Let $x = (1, 0, 0) \in K$. Then

$$\langle F(x), y\rangle = x \cdot y = (1, 0, 0) \cdot (a, b, c) = a.$$

Clearly $\ker F(x) = YZ$ plane and $\ker F(x) \cap K = \phi$. Now $\langle F(x), x\rangle = x.x = \|x\|^2 = 1$ and $\langle F(x), y\rangle = a$, but $1 \leq a \leq 3$. Thus $\langle F(x), x\rangle \leq \langle F(x), y\rangle$, for all $y \in K$. Therefore x is a solution of VIP-(2.1).

In the next theorem, we shall make use of the following notations:

Let H be a Hilbert space. For $x^* \in H^*$ and $K \subset H$, let

$$K^- = \{y \in K : \langle x^*, y\rangle < 0\} \,,$$

$$K^+ = \{y \in K : \langle x^*, y\rangle > 0\}$$

and

$$K^0 = \{y \in K : \langle x^*, y\rangle = 0\} \,.$$

Clearly that $K = K^- \cup K^0 \cup K^+$.

Now we shall consider the main result of this paper.

Theorem 4.3. *Suppose K is a subset of a Hilbert space H. Let $F : K \longrightarrow H^*$ be a mapping. Consider the variational inequality problem to find $x \in K$ such that*

$$\langle F(x), y - x\rangle \geq 0, \text{ for all } y \in K. \tag{4.3}$$

Then we have the following statements:

(a) *Suppose $K^- \neq \phi$. Then VIP-(4.3) has a solution $x \in K$ if and only if x is a vector of greatest norm in K^-.*
(b) *Suppose $K^- = \phi$ and $K^0 = \phi$. Then VIP-(4.3) has a solution $x \in K$ if and only if x is a vector of smallest norm in K.*
(c) *Suppose $K^- = \phi$ and $K^0 \neq \phi$. Then VIP-(4.3) has a solution x if and only if $x \in K^0$.*

Proof. If K is a finite set or $F(x) = 0$, for $x \in K$, then the proof is trivial. Suppose $F(x) \neq 0$ and the cardinality of K is an infinite set. For the sake of convenience, we denote $F(x)$ by x^* then the inequality (4.3) can be written as

$$\langle x^*, x\rangle \leq \langle x^*, y\rangle, \text{ for all } y \in K. \tag{4.4}$$

Now by using Riesz-Representation theorem [22], there exists $z \in H$ such that $\langle x^*, y\rangle = \langle y, z\rangle$, for every $y \in H$. We claim that $z \neq 0$. If $z = 0$, then we get $x^* = 0$, which is not true. Next we see that $\langle x^*, z\rangle \neq 0$. This can be seen from the relation $\langle x^*, z\rangle = \|z\|^2$. Let $u, v \in H \setminus \ker x^*$. We prove that u, v are linearly dependent. For $M = \text{span}(\{v\} \cup \ker x^*)$ is a proper subspace of H which properly contains $\ker x^*$. This contradicts the fact that $\ker x^*$ is a maximal subspace of H. Hence u, v are linearly dependent.

Now we shall begin with the proof of the main theorem.

(a) In this case, there exists $x \in K$ such that $\langle x^*, x\rangle < 0$. If x_0 is a solution of variational inequality problem, then $\langle x^*, x_0\rangle \leq \langle x^*, y\rangle$, for all $y \in K$ and hence

$$\langle x^*, x_0\rangle \leq \langle x^*, y'\rangle, \text{ for all } y' \in K^-. \tag{4.5}$$

As proved above, $z, x_0, y' \in H \setminus \ker x^*$ are linearly dependent for every $y' \in K^-$. In view of Theorem 2.2, equality holds in Cauchy-Schwartz inequality $|\langle x, y\rangle| \leq \|x\| \|y\|$ for $x, y \in H$ if and only if x and y are linearly dependent. Hence from (4.5), we have

$$\langle x_0, z\rangle \leq \langle y', z\rangle, \text{ for all } y' \in K^-,$$

$$-\langle y', z\rangle \leq -\langle x_0, z\rangle, \text{ for all } y' \in K^-,$$

$$|\langle y', z\rangle| \leq |\langle x_0, z\rangle|, \text{ for all } y' \in K^-$$

or

$$\|y'\| \|z\| \leq \|x_0\| \|z\|, \text{ for all } y' \in K^-.$$

This yields that x_0 is a vector of greatest norm in K^-.

Conversely, suppose that $x_0 \in K$ is an element of greatest norm in K^-. Then we have

$$\|y'\| \|z\| \leq \|x_0\| \|z\|, \text{ for all } y' \in K^-.$$

By reversing the arguments, we have $\langle x^*, x_0\rangle \leq \langle x^*, y'\rangle$, for all $y' \in K^-$. Since for $y \in K^0 \cup K^+$ we have $\langle x^*, y\rangle \geq 0$. Thus $\langle x^*, x_0\rangle \leq \langle x^*, y\rangle$, for all $y \in K$. Thus x_0 is a solution of VIP-(4.3).

(b) In this case $\langle x^*, y\rangle > 0$, for all $y \in K$. Let $x \in K$ be a solution of VIP-(4.3). Then $\langle x^*, x\rangle \leq \langle x^*, y\rangle$, for all $y \in K$ which implies that $|\langle x, z\rangle| \leq |\langle y, z\rangle|$, for all $y \in K$. Hence $\|x\| \|z\| \leq \|y\| \|z\|$, for all $z \in K$. Thus $\|x\| \leq \|y\|$, for all $y \in K$ so that x is a vector of smallest norm in K.

The converse follows by reversing the arguments.

(c) Suppose $K^- = \phi$ and $K^0 \neq \phi$. Then $\langle x^*, y\rangle \geq 0$, for all $y \in K$ and $\langle x^*, y_0\rangle = 0$, for some $y_0 \in K$. Now if $x \in K$ is a solution of VIP-(4.3) then $\langle x^*, x\rangle \leq \langle x^*, y\rangle$, for all $y \in K$. This implies that $0 \leq \langle x^*, x\rangle \leq \langle x^*, y_0\rangle = 0$. Hence $\langle x^*, x\rangle = 0$. Thus $x \in K^0$.

Conversely, suppose $x \in K^0$. Then we show that x is a solution of VIP-(4.3). Now $\langle x^*, x\rangle = 0$ and $\langle x^*, y\rangle \geq 0$, for all $y \in K$. Therefore we can conclude that $\langle x^*, x\rangle \leq \langle x^*, y\rangle$, for all $y \in K$. This proves that x is a solution of VIP-(4.3). □

Corollary 4.4. *Suppose K is a closed convex subset of H and $x^*(K) \subseteq \mathbb{R}^+$. Then VIP-*(4.1) *has a unique solution.*

Proof. By Theorem 4.3-(b), $x \in K$ is a solution of variational inequality problem VIP-(4.3) if and only if x is a vector of smallest norm in K. But K has a unique vector of smallest norm. Thus x is a unique solution of VIP-(4.1). □

Example 4.5. Let $H = \mathbb{R}^2$ and K be the region defined by $-2 \leq x \leq 2;\ 1 \leq y \leq 2$. For $x = (x_1, x_2) \in K$, define $F : K \longrightarrow (\mathbb{R}^2)^*$ by

$$\langle F(x), (y_1, y_2)\rangle = x_1 y_2, \quad \text{for } (y_1, y_2) \in \mathbb{R}^2.$$

Then, clearly $kerF(x) = \left\{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0\right\} = Y-$ axis. For $x = (-2, 2)$, $K^- = [-2, 2] \times [1, 2] \setminus \left\{(x_1, y_1) \in K : x_1 = 0\right\}$, which is nonempty. Now $x = (-2, 2)$ is an element in K^- of greatest norm. In view of Theorem 4.3(a), $x = (-2, 2)$ is a solution of VIP-(4.1).

Example 4.6. Let $H = \mathbb{R}^3$ and and K be the region described by $1 \leq x \leq 2;\ 0 \leq y \leq 1;\ 0 \leq z \leq 1$. For $x = (x_1, x_2, x_3) \in K$, define $F : K \longrightarrow (\mathbb{R}^3)^*$ by

$$\langle F(x), (y_1, y_2, y_3)\rangle = x_1 y_1, \text{ for } (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Then $F(x)$ is a bounded linear functional on $\mathbb{R}^3$. If $(y_1, y_2, y_3) \in \mathbb{R}^3$ and $\langle F(x), y\rangle = x_1 y_1 = 0$, then $y_1 = 0$ because $x_1 \neq 0$ for $(x_1, x_2, x_3) \in K$. That is, $\ker F(x)$ is YZ plane. Clearly $K \cap \ker F(x) = \phi$. Now $K^0 = \phi,\ K^- = \phi$ and $x = (1, 0, 0)$ is an element of smallest norm in K. In view of Theorem 4.3-(b), $x = (1, 0, 0)$ is a solution of VIP-(4.1).

Example 4.7. Let $H = \mathbb{R}^2$ and and K be the region defined by

$$1 \leq x \leq 2;\ 0 \leq y \leq 2.$$

For $x = (x_1, x_2) \in K$, define $F : K \longrightarrow (\mathbb{R}^2)^*$ by

$$\langle F(x), (y_1, y_2)\rangle = x_1 y_2, \text{ for } (y_1, y_2) \in \mathbb{R}^2.$$

Then $F(x) \in (\mathbb{R}^2)^*$ and $\ker F(x) = \{(x,y) : y = 0\} = X-$axis. If we take $x = (2,0) \in K$, then $K^0 = \{(y_1, y_2) : 1 \leq y_1 \leq 2, y_2 = 0\}$.
Also $K^- = \phi$. Therefore in view of Theorem 4.3-(c), every element of K^0 is a solution of VIP-(4.1). Thus VIP-(4.1) can have infinitely many solutions.

In the following theorem, we characterize the existence of the solutions of quasi variational inequality problem.

For the set-valued map $K : H \to 2^H$ and for the mapping $F : H \to H^*$, the following notations will be used:

$$K_x^- = \{y \in K_x : \langle F(x), y\rangle < 0\},$$

$$K_x^+ = \{y \in K_x : \langle F(x), y\rangle > 0\}$$

and

$$K_x^0 = \{y \in K_x : \langle F(x), y\rangle = 0\}.$$

Then, clearly $K_x = K_x^- \cup K_x^+ \cup \ K_x^0$.

Theorem 4.8. *Let H be a Hilbert space and let $K : H \to 2^H$ be a set-valued mapping. Consider the quasi-variational inequality problem (for short, QVIP): Find $x \in K_x$ such that*

$$\langle F(x), y - x\rangle \geq 0, \text{ for all } y \in K_x. \tag{4.6}$$

Then we have the following statements:

(a) *Suppose $K_x^- \neq \phi$. Then QVIP-(4.6) has a solution $x \in K_x$ if and only if x is a vector of greatest norm in K_x^-.*
(b) *Suppose $K_x^- = \phi$ and $K_x^0 = \phi$. Then QVIP-(4.6) has a solution $x \in K_x$ if and only if x is a vector of smallest norm in K_x.*
(c) *Suppose $K_x^- = \phi$ and $K_x^0 \neq \phi$. Then QVIP-(4.6) has a solution x if and only if $x \in K_x^0$.*

Proof. If K_x is a finite for $x \in H$ or $F(x) = 0$, then the proof of this theorem is trivial. Therefore we suppose that K_x is an infinite set and $F(x) \neq 0$. The proof of the theorem is obtained on the same line as in the proof of the Theorem 4.1. □

Conclusion: A criterion for well-posed problem in locally convex Hausdorff topological vector space setting is obtained. The characterizations for the existence of solutions of variational inequality problem and quasi-variational inequality problem are obtained even when the underlying set K is not convex. It has been shown that in some cases there are infinitely many solutions of (VIP) and (QVIP).

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