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L^r INEQUALITIES OF GENERALIZED TURÁN-TYPE INEQUALITIES OF POLYNOMIALS

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Abstract. If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $\rho R \geq k^2$ and $\rho \leq R$, Aziz and Zargar [4] proved that

$$\max_{|z|=1} |p'(z)| \geq n \frac{(R+k)^{n-1}}{(\rho+k)^n} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}.$$

We prove a generalized L^r extension of the above result for a more general class of polynomials $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$. We also obtain another L^r analogue of a result for the above general class of polynomials proved by Chanam and Dewan [6].

1. Introduction

For a polynomial $p(z)$ of degree n having all its zeros in $|z| \leq 1$, Turán [12] proved that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

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The result is sharp and equality holds in (1.1) for polynomials having all their zeros on the unit circle.

By involving $\min_{|z|=1} |p(z)|$, Aziz and Dawood [2] improved (1.1) under the same hypotheses of $p(z)$ that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \left[\max_{|z|=1} |p(z)| + \min_{|z|=1} |p(z)| \right]. \tag{1.2}$$

Equality occurs in (1.2) for the polynomial $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. Malik [8] generalized (1.1) by considering polynomials having all zeros in $|z| \leq k$, $k \leq 1$. He proved

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The result is best possible and the extremal polynomial is $p(z) = (z+k)^n$.

Inequality (1.2) was further generalized by Aziz and Zargar [4].

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for $\rho R \geq k^2$ and $\rho \leq R$*

$$\max_{|z|=1} |p'(z)| \geq n \frac{(R+k)^{n-1}}{(\rho+k)^n} \left[\max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right]. \tag{1.4}$$

Equality holds in (1.4) for $p(z) = (z+k)^n$.

Chanam and Dewan [6] proved the following result which improves Theorem 1.1 by considering the more general class of polynomials

$$p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}, 1 \leq \mu < n$$

and involving certain coefficients of the polynomial.

Theorem 1.2. *If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ and $a_0 \neq 0$, is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$, $k > 0$, then for $\rho R \geq k^2$ and $\rho \leq R$*

$$\begin{aligned}\max_{|z|=R} |p'(z)| \geq \ & n \left\{ \frac{R^\mu n |a_n| k^{\mu-1} + \mu |a_{n-\mu}| R^{\mu-1}}{R^{\mu+1} n |a_n| k^{\mu-1} + n |a_n| k^{2\mu} + \mu |a_{n-\mu}| (R k^{\mu-1} + R^\mu)} \right\} \\ & \times \left(\frac{R+k}{\rho+k} \right)^n \left\{ \max_{|z|=\rho} |p(z)| + \min_{|z|=k} |p(z)| \right\}. \end{aligned} \tag{1.5}$$

Equality holds in (1.5) for $\mu = 1$ and $p(z) = (z+k)^n$.

For a polynomial $p(z)$ of degree n and for every $r > 0$, we know

$$\left\{\int_0^{2\pi} |p'(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.6}$$

Zygmund [13] proved inequality (1.6) for $r \ge 1$ for all trigonometric polynomials of degree n and not only for those which are of the form $p(e^{i\theta})$. The validity of (1.6) for $0 < r < 1$ was proved by Arestov [1].

From a well-known fact of analysis [10, 11], we know that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} = \max_{|z|=1}|p(z)|. \tag{1.7}$$

In view of (1.7), inequality (1.6) is the L^r analogue of the famous Bernstein's inequality [5]. This important fact shows that L^r inequalities of a polynomial generalize ordinary inequalities of polynomials.

2. Lemmas

We need the following lemmas to prove our results.

Lemma 2.1. ([9]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$, *then*

$$|q'(z)| \ge k^{\mu+1}\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}}|p'(z)| \quad \text{on } |z|=1 \tag{2.1}$$

and

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \le 1, \tag{2.2}$$

where

$$q(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Lemma 2.2. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \le 1$, *then for* $|z| = 1$

$$|p'(z)| \ge \frac{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}|q'(z)|, \tag{2.3}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Proof. Since $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, $q(z)$ has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Hence, applying Lemma 2.1 to the polynomial $q(z)$, we have by inequality (2.1)

$$|p'(z)| \geq \left(\frac{1}{k}\right)^{\mu+1} \frac{\frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\left(\frac{1}{k}\right)^{\mu-1} + 1}{1 + \frac{\mu}{n}\frac{|a_{n-\mu}|}{|a_n|}\left(\frac{1}{k}\right)^{\mu+1}} |q'(z)| \quad \text{on } |z| = 1,$$

which simplifies to

$$|p'(z)| \geq \frac{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}} |q'(z)|.$$

□

Lemma 2.3. ([4]) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k > 0$, *then for* $\rho R \geq k^2$ *and* $\rho \leq R$, *we have for* $|z| = 1$

$$|p(Rz)| \geq \left(\frac{R+k}{\rho+k}\right)^n |p(\rho z)|. \tag{2.4}$$

Equality in (2.4) *holds for the polynomial* $p(z) = (z+k)^n$.

Lemma 2.4. ([3]) *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \leq 1$, *then for* $|z| = 1$

$$k^\mu |p'(z)| \geq |q'(z)|. \tag{2.5}$$

3. Main results

In this paper, we first prove a generalized L^r extension of Theorem 1.1. Secondly, we obtain an L^r analogue of Theorem 1.2. We find that our results have significant influences on other well-known inequalities.

The following result is a generalized L^r version of Theorem 1.1.

Theorem 3.1. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k > 0$, *then for* $\rho R \ge k^2$ *and* $\rho \le R$, *and* $s, q \ge 1$ *such that* $\frac{1}{s} + \frac{1}{q} = 1$, *and for each* $r > 0$

$$\left\{ \int_0^{2\pi} \left| p'(Re^{i\theta}) \right|^{qr} d\theta \right\}^{\frac{1}{qr}} \ge n \left(\frac{R+k}{\rho+k} \right)^n \frac{1}{R} \left\{ \int_0^{2\pi} \left| 1 + \left(\frac{k}{R} \right)^{\mu} e^{i\theta} \right|^{sr} d\theta \right\}^{-\frac{1}{sr}} \times \left\{ \int_0^{2\pi} \Big(|p(\rho e^{i\theta})| + m \Big)^r d\theta \right\}^{\frac{1}{r}}, \qquad (3.1)$$

where $m = \min_{|z|=k} |p(z)|$.

Proof. Let α be any real or complex number such that $|\alpha| < 1$. Since $p(z)$ has all its zeros in $|z| \le k$, $k > 0$, by Rouche's theorem, the polynomial $G(z) = p(z) + \alpha m$, where $m = \min_{|z|=k} |p(z)|$, has all its zeros in $|z| \le k$, $k > 0$.

Let $H(z) = G(Rz)$. Then

$$H(z) = a_n R^n z^n + a_{n-\mu} R^{n-\mu} z^{n-\mu} + \cdots + a_1 Rz + (a_0 + \alpha m),$$

where $\rho R \ge k^2$ and $\rho \le R$ (it also implies $R \ge k$). Consequently, $H(z)$ has all its zeros in $|z| \le \frac{k}{R}$, $\frac{k}{R} \le 1$. Applying Lemma 2.4 to $H(z)$, we obtain for $|z| = 1$

$$\left(\frac{k}{R} \right)^{\mu} |H'(z)| \ge |I'(z)|, \qquad (3.2)$$

where $I(z) = z^n \overline{H\left(\frac{1}{\bar{z}} \right)}$. Since $H(z)$ has all its zeros in $|z| \le \frac{k}{R}$, $\frac{k}{R} \le 1$, $H'(z)$ also has all its zeros in $|z| \le \frac{k}{R}$, $\frac{k}{R} \le 1$. Hence by Gauss-Lucas theorem, the polynomial

$$z^{n-1} \overline{H'\left(\frac{1}{\bar{z}} \right)} = nI(z) - zI'(z)$$

has all its zeros in $|z| \ge \frac{R}{k}$, $\frac{R}{k} \ge 1$.

From (3.2), we have for $|z| = 1$

$$|I'(z)| \le \left(\frac{k}{R} \right)^{\mu} \left| H'(z) \right|. \qquad (3.3)$$

We also know that for $|z| = 1$, $|H'(z)| = |nI(z) - zI'(z)|$, and thus, inequality (3.2) gives

$$|I'(z)| \leq \left(\frac{k}{R}\right)^{\mu} \left|nI(z) - zI'(z)\right|. \tag{3.4}$$

Let

$$w(z) = \frac{zI'(z)}{nI(z) - zI'(z)}.$$

Then $w(z)$ is analytic in $|z| \leq 1$, $|w(z)| \leq 1$ for $|z| = 1$ and $w(0) = 0$. Therefore, the function $1 + \left(\frac{k}{R}\right)^{\mu} w(z)$ is subordinate to $1 + \left(\frac{k}{R}\right)^{\mu} z$ for $|z| \leq 1$. Hence, by a well-known property of subordination [7], we have for every $r > 0$

$$\int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^{\mu} w(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^{\mu} e^{i\theta}\right|^r d\theta. \tag{3.5}$$

Now,

$$\begin{aligned} 1 + \left(\frac{k}{R}\right)^{\mu} w(z) =& 1 + \frac{zI'(z)}{nI(z) - zI'(z)} \\ =& \frac{nI(z)}{nI(z) - zI'(z)}. \end{aligned}$$

This implies for $|z| = 1$

$$\begin{aligned} |nI(z)| &= \left|1 + \left(\frac{k}{R}\right)^{\mu} w(z)\right| \left|nI(z) - zI'(z)\right| \\ &= \left|1 + \left(\frac{k}{R}\right)^{\mu} w(z)\right| \left|H'(z)\right|. \end{aligned}$$

Thus, for $r > 0$ and $0 \leq \theta < 2\pi$

$$\left|nI(e^{i\theta})\right|^r \leq \left|1 + \left(\frac{k}{R}\right)^{\mu} w(e^{i\theta})\right|^r \left|H'(e^{i\theta})\right|^r,$$

which implies

$$n^r \int_0^{2\pi} \left|I(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^{\mu} w(e^{i\theta})\right|^r \left|H'(e^{i\theta})\right|^r d\theta.$$

By (3.5), the above inequality becomes

$$n^r \int_0^{2\pi} \left|nI(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^\mu e^{i\theta}\right|^r \left|H'(e^{i\theta})\right|^r d\theta.$$

Applying Holder's inequality, for $q \geq 1$ and $s \geq 1$ with $s^{-1} + q^{-1} = 1$ and $r > 0$, we get

$$n\left\{\int_0^{2\pi} \left|I(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^\mu e^{i\theta}\right|^{rs} d\theta\right\}^{\frac{1}{rs}} \left\{\int_0^{2\pi} \left|H'(e^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}}. \tag{3.6}$$

Since $H(z) = G(Rz) = p(Rz) + \alpha m$, therefore, $H'(z) = Rp'(Rz)$. Then,

$$\left\{\int_0^{2\pi} \left|H'(e^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}} = R\left\{\int_0^{2\pi} \left|p'(Re^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}}.$$

Also, for $|z| = 1$, $|I(z)| = |H(z)| = |G(Rz)|$. Then by Lemma 2.3 for $\rho R \geq k^2$ and $\rho \leq R$

$$|I(z)| \geq |G(Rz)| \geq \left(\frac{R+k}{\rho+k}\right)^n |G(\rho z)|,$$

which implies

$$|I(z)| \geq \left(\frac{R+k}{\rho+k}\right)^n |p(\rho z) + \alpha m|. \tag{3.7}$$

Using (3.6) and (3.7) in (3.5), we obtain

$$\begin{aligned} n\left(\frac{R+k}{\rho+k}\right)^n &\left\{\int_0^{2\pi} \left|p(\rho e^{i\theta}) + \alpha m\right|^r d\theta\right\}^{\frac{1}{r}} \\ &\leq \left\{\int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^\mu e^{i\theta}\right|^{rs} d\theta\right\}^{\frac{1}{rs}} \\ &\times R\left\{\int_0^{2\pi} \left|p'(Re^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}} d\theta. \end{aligned} \tag{3.8}$$

Choosing the argument of α suitably such that

$$\left|p(\rho e^{i\theta}) + \alpha m\right| = \left|p(\rho e^{i\theta})\right| + |\alpha| m,$$

which on letting $|\alpha| \to 1$ gives

$$\left|p(\rho e^{i\theta}) + \alpha m\right| = \left|p(\rho e^{i\theta})\right| + m.$$

Inequality (3.8) thus reduces to

$$n\left(\frac{R+k}{\rho+k}\right)^n \left\{\int_0^{2\pi} \left(\left|p(\rho e^{i\theta})\right| + \alpha m\right)^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq \left\{\int_0^{2\pi} \left|1 + \left(\frac{k}{R}\right)^{\mu} e^{i\theta}\right|^{rs} d\theta\right\}^{\frac{1}{rs}}$$
$$\times R\left\{\int_0^{2\pi} \left|p'(Re^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}} d\theta.$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Taking $\mu = 1$ and letting $r \to \infty$ in Theorem 3.1, we have

$$\max_{|z|=R} |p'(z)| \geq \frac{n(R+k)^n}{R(\rho+k)^n} \left(1 + \frac{k}{R}\right)^{-1} \left\{\max_{|z|=\rho} |p(z)| + \min_{|z|=k} |p(z)|\right\},$$

which simplifies to inequality (1.4) of Theorem 1.1. This verifies that Theorem 3.1 is a generalized L^r version of Theorem 1.1 proved by Aziz and Zargar [4].

Remark 3.3. Again, if we let $r \to \infty$ and taking $\mu = 1$ along with $\rho = R = 1$ in Theorem 3.1 we have the following result which is an improvement of (1.3) due to Malik [8].

Corollary 3.4. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+k} \left\{\max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)|\right\}. \tag{3.9}$$

It is obvious that (3.9) is an improvement of inequality (1.3). Consequently, Theorem 3.1 is an improvement and a generalization of (1.3) due to Malik [8].

Remark 3.5. For $k = 1$, inequality (3.9) of Corollary 3.4 reduces to inequality (1.2) due to Aziz and Dawood [2]. Thus, Theorem 3.1 is an improved and a generalized L^r version of (1.1) due to Turán [12].

Next, we prove the L^r analogue of Theorem 1.2 which further gives a refinement of Theorem 3.1. More precisely, we prove:

Theorem 3.6. *If* $p(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k > 0$, *then for* $\rho R \geq k^2$ *and* $\rho \leq R$, *and* $s \geq 1$, $q \geq 1$ *such that* $\frac{1}{s} + \frac{1}{q} = 1$, *and for each* $r > 0$,

$$\left\{ \int_0^{2\pi} \left| p'(Re^{i\theta}) \right|^{qr} d\theta \right\}^{\frac{1}{qr}} \geq n \left(\frac{R+k}{\rho+k} \right)^n \frac{1}{R} \left\{ \int_0^{2\pi} \left| 1 + Ae^{i\theta} \right|^{sr} d\theta \right\}^{-\frac{1}{sr}} \times \left\{ \int_0^{2\pi} \left(\left| p(\rho e^{i\theta}) \right| + m \right)^r d\theta \right\}^{\frac{1}{r}}, \tag{3.10}$$

where

$$A = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}R^\mu}{n|a_n|k^{\mu-1}R^{\mu+1} + \mu|a_{n-\mu}|R^\mu} \tag{3.11}$$

and $m = \min\limits_{|z|=k} |p(z)|$.

Proof. Since $p(z)$ has all its zeros in $|z| \leq k$, $k > 0$, by Rouche's theorem, for real or complex number α with $|\alpha| < 1$, the polynomial $G(z) = p(z) + \alpha m$, where $m = \min\limits_{|z|=k} |p(z)|$ has all its zeros in $|z| \leq k$, $k > 0$. Therefore,

$$\begin{aligned} H(z) &= G(Rz) \\ &= a_n R^n z^n + a_{n-\mu} R^{n-\mu} z^{n-\mu} + \cdots + a_1 Rz + a_0 + \alpha m, \end{aligned}$$

where $\rho R \geq k^2$ and $\rho \leq R$ (implies $R \geq k$ also), has all its zeros in $|z| \leq \frac{k}{R}$, $\frac{k}{R} \leq 1$. Applying Lemma 2.2 to $H(z)$, it follows from inequality (2.3) that

$$\begin{aligned} |H'(z)| &\geq \frac{n|a_n|R^n \left(\frac{k}{R}\right)^{\mu-1} + \mu|a_{n-\mu}|R^{n-\mu}}{n|a_n|R^n \left(\frac{k}{R}\right)^{2\mu} + \mu|a_{n-\mu}|R^{n-\mu}k^{\mu-1}} |I'(z)| \\ &= \frac{n|a_n|R^{\mu+1}k^{\mu-1} + \mu|a_{n-\mu}|R^\mu}{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}R^\mu} |I'(z)|, \end{aligned} \tag{3.12}$$

where

$$I(z) = z^n \overline{H\left(\frac{1}{\overline{z}}\right)}.$$

Since $H(z)$ has all its zeros in $|z| \leq \frac{k}{R}$, $\frac{k}{R} \leq 1$, $H'(z)$ also has all its zeros in $|z| \leq \frac{k}{R}$, $\frac{k}{R} \leq 1$. Hence by Gauss-Lucas Theorem, the polynomial

$$z^{n-1}\overline{H'\left(\frac{1}{\bar{z}}\right)} = nI(z) - zI'(z)$$

has all its zeros in $|z| \geq \frac{R}{k}$, $\frac{R}{k} \geq 1$.

From (3.12), we have for $|z| = 1$,

$$\begin{aligned} |I'(z)| &\leq \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}R^{\mu}}{n|a_n|k^{\mu-1}R^{\mu+1} + \mu|a_{n-\mu}|R^{\mu}}|H'(z)| \\ &= A|H'(z)|, \end{aligned} \tag{3.13}$$

where

$$A = \frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}R^{\mu}}{n|a_n|k^{\mu-1}R^{\mu+1} + \mu|a_{n-\mu}|R^{\mu}}.$$

Since, for $|z| = 1$, $|H'(z)| = |nI(z) - zI'(z)|$, inequality (3.13) equivalently gives

$$|I'(z)| \leq A|nI(z) - zI'(z)|. \tag{3.14}$$

Using the fact (3.14), we have

$$w(z) = \frac{zI'(z)}{A\left(nI(z) - zI'(z)\right)}$$

is analytic in $|z| \leq 1$, $|w(z)| \leq 1$ for $|z| = 1$ and $w(0) = 0$. Therefore, the function $1+Aw(z)$ is subordinate to $1+Az$ for $|z| \leq 1$. Hence, by a well-known property of subordination [7], we have for $r > 0$

$$\int_0^{2\pi} \left|1 + Aw(e^{i\theta})\right|^r d\theta \leq \int_0^{2\pi} \left|1 + Ae^{i\theta}\right|^r d\theta. \tag{3.15}$$

Now,

$$\begin{aligned} 1 + Aw(z) &= 1 + \frac{zI'(z)}{nI(z) - zI'(z)} \\ &= \frac{nI(z)}{nI(z) - zI'(z)}. \end{aligned}$$

Hence, for $|z| = 1$, it implies from $|H'(z)| = |nI(z) - zI'(z)|$ that

$$\begin{aligned} |nI(z)| &= |1 + Aw(z)|\left|nI(z) - zI'(z)\right| \\ &= |1 + Aw(z)|\,|H'(z)|. \end{aligned}$$

Which gives for $r > 0$ and $0 \le \theta < 2\pi$

$$\left|nI(e^{i\theta})\right|^r \le \left|1 + Aw(e^{i\theta})\right|^r \left|H'(e^{i\theta})\right|^r,$$

which implies

$$n^r \int_0^{2\pi} \left|I(e^{i\theta})\right|^r d\theta \le \int_0^{2\pi} \left|1 + Aw(e^{i\theta})\right|^r \left|H'(e^{i\theta})\right|^r d\theta.$$

Using (3.15), the above inequality gives

$$n^r \int_0^{2\pi} \left|I(e^{i\theta})\right|^r d\theta \le \int_0^{2\pi} \left|1 + Ae^{i\theta}\right|^r \left|H'(e^{i\theta})\right|^r d\theta.$$

Applying Holder's inequality, for $q \ge 1$ and $s \ge 1$ with $\frac{1}{s} + \frac{1}{q} = 1$ and $r > 0$, we get

$$n\left\{\int_0^{2\pi} \left|I(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} \left|1 + Ae^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{sr}} \left\{\int_0^{2\pi} \left|H'(e^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}}. \tag{3.16}$$

Since $H(z) = G(Rz) = p(Rz) + \alpha m$, $H'(z) = Rp'(Rz)$. Then, the factor

$$\left\{\int_0^{2\pi} \left|H'(e^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}} = R\left\{\int_0^{2\pi} \left|p'(Re^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}}. \tag{3.17}$$

Also, since $|I(z)| = |H(z)| = |G(Rz)|$ for $|z| = 1$, by Lemma 2.3 for $\rho R \ge k^2$ and $\rho \le R$

$$|I(z)| \ge |G(Rz)| \ge \left(\frac{R+k}{\rho+k}\right)^n |G(\rho z)|,$$

that is,

$$|I(z)| \ge \left(\frac{R+k}{\rho+k}\right)^n |p(\rho z) + \alpha m|. \tag{3.18}$$

Making use of (3.17) and (3.18) in (3.16), we have

$$n\left(\frac{R+k}{\rho+k}\right)^n\left\{\int\limits_0^{2\pi}\left|p(\rho e^{i\theta})+\alpha m\right|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int\limits_0^{2\pi}\left|1+Ae^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{sr}} \times R\left\{\int\limits_0^{2\pi}\left|p'(Re^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}}. \tag{3.19}$$

Choosing the argument of α suitably such that

$$\left|p(\rho e^{i\theta})+\alpha m\right| = \left|p(\rho e^{i\theta})\right| + |\alpha| m$$

and letting $|\alpha| \to 1$, we have

$$\left|p(\rho e^{i\theta})+\alpha m\right| = \left|p(\rho e^{i\theta})\right| + m.$$

Inequality (3.19) thus reduces to

$$n\left(\frac{R+k}{r+k}\right)^n\left\{\int\limits_0^{2\pi}\left(\left|p(\rho e^{i\theta})\right|+m\right)^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int\limits_0^{2\pi}\left|1+Ae^{i\theta}\right|^{sr} d\theta\right\}^{\frac{1}{sr}} \times R\left\{\int\limits_0^{2\pi}\left|p'(Re^{i\theta})\right|^{qr} d\theta\right\}^{\frac{1}{qr}},$$

from which inequality (3.10) follows. □

Letting $r \to \infty$ in inequality (3.10), we have the following result.

Corollary 3.7. *If $p(z) = a_n z^n + \sum\limits_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$, then for $\rho R \ge k^2$ and $\rho \le R$*

$$\max_{|z|=R}|p'(z)| \ge n\left(\frac{R+k}{\rho+k}\right)^n \frac{1}{R(1+A)}\left\{\max_{|z|=\rho}|p(z)| + \min_{|z|=k}|p(z)|\right\}, \tag{3.20}$$

where A is given by (3.11).

Remark 3.8. Since

$$\frac{1}{R\,(1+A)} = \frac{n|a_n|R^{\mu}k^{\mu-1}+\mu|a_{n-\mu}|R^{\mu-1}}{n|a_n|R^{\mu+1}k^{\mu-1}+n|a_n|k^{2\mu}+\mu|a_{n-\mu}|\left(Rk^{\mu-1}+R^{\mu}\right)},$$

Corollary 3.7 shows that Theorem 3.6 is L^r analogue of Theorem 1.2. Further, as explained by Chanam and Dewan [6], Corollary 3.7 is an improvement of Theorem 1.1 and hence, correspondingly, Theorem 3.6 is a refinement of Theorem 3.1.

Remark 3.9. In view of Corollary 3.7, Theorem 3.6 is L^r version of Theorem 1.2 in a richer form for restrictions concerning the polynomial $p(z)$, namely $a_0 \neq 0$, $\mu \neq n$ and $n \neq 1$ in the hypotheses of Theorem 1.2, have all been dropped in Theorem 3.6 and consequently in Corollary 3.7. In other words, Corollary 3.7 is a better version of Theorem 1.2.

Remark 3.10. Letting $r \to \infty$ in inequality (3.10), and taking $\mu = 1$ along with $\rho = R = k = 1$, it reduces to inequality (1.2) as in Remark 3.5 and hence same consequences of Remark 3.5 follow.

Further, if we take $\mu = 1$ and $\rho = R = 1$ in Corollary 3.7, we have:

Corollary 3.11. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{1+A} \left\{ \max_{|z|=1} |p(z)| + \min_{|z|=k} |p(z)| \right\}, \tag{3.21}$$

where $A = \dfrac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|}$.

Remark 3.12. Inequality (3.21) of Corollary 3.11 is an improvement of (1.3) due to Malik [8]. To see this it is sufficient to show that $\dfrac{n}{1+A} \geq \dfrac{n}{1+k}$, which is equivalent to showing $A \leq k$, where A is defined as in Corollary 3.11.

If $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$, then $q(z) = \sum_{\nu=0}^{n} \overline{a}_\nu z^{n-\nu}$ has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$. Applying Lemma 2.1 to $q(z)$, it follows from (2.2) that for $\mu = 1$

$$\frac{1}{n} \frac{|a_{n-1}|}{|a_n|} \frac{1}{k} \leq 1. \tag{3.22}$$

Now, as $k \leq 1$, in view of (3.22), it is easy to verify that $A \leq k$.

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