Nonlinear Functional Analysis and Applications
Vol. 14, No. 1 (2009), pp. 61–68

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SADDLE POINT CHARACTERIZATION OF SUBHARMONIC SOLUTIONS OF NONAUTONOMOUS SECOND ORDER HAMILTONIAN SYSTEMS

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Abstract. In the present paper, the non-autonomous second order Hamiltonian systems

$$\ddot{u}(t) - \nabla F(t, u(t)) = 0 \qquad a.\ e.\ t \in R$$

are studied and some existence results of subharmonic solutions with saddle point character are obtained by the critical point reduction method.

1. Introduction and Preliminaries

Consider the second order Hamiltonian systems

$$\ddot{u}(t) - \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R \tag{1}$$

where $F : R \times R^N \to R$ is $T-$periodic ($T > 0$) in t for all $x \in R^N$, that is

$$F(t + T, x) = F(t, x) \tag{2}$$

[0]Received July 13, 2007. Revised October 28, 2008.
[0]2000 Mathematics Subject Classification: 34C25,58E20,47H04.
[0]Keywords: Subharmonic solution, Hamiltonian system, saddle point character.
[0]This work was supported by the Science Foundation (G2360247) and Teachers Teaching Group Foundation (E4081708) of Guangzhou Zhong Kai University of Agriculture and Engineering.

for all $x \in R^N$ and a. e. $t \in R$, and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a. e. $t \in [0,T]$, and there exist $a \in L^1(R^+;R^+), b \in L^1(0,T;R^+)$, such that $|F(t,x)| \leq a(|x|)b(t), |\bigtriangledown F(t,x)| \leq a(|x|)b(t)$ for all $x \in R^N$ and a. e. $t \in R$.

A solution of problem (1) is called to be subharmonic if it is $kT-$periodic solution for some positive integer k.

Let

$$H^1_{kT} = \{u : [0,kT] \to R^N |\ u \ is\ absolutely\ continuous,\ u(0) = u(kT)\ and\ \dot{u} \in L^2(0,kT;R^N)\}$$

is a Hilbert space with the norm defined by

$$\|u\| = [\int_0^{kT} |u(t)|^2 dt + \int_0^{kT} |\dot{u}(t)|^2 dt]^{\frac{1}{2}}$$

and $\|u\|_\infty = \max_{0\leq t\leq kT} |u(t)|$ for $u \in H^1_{kT}$.

The corresponding functional φ_k on H^1_{kT} given by

$$\varphi_k(u) = \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt + \int_0^{kT} F(t,u(t))dt$$

is continuously differentiable and weakly lower semi-continuous on H^1_{kT} (see [1]). Moreover one has

$$< \varphi'_k(u), v >= \int_0^{kT} [(\dot{u}(t), \dot{v}(t)) + (\nabla F(t,u(t)), v(t))]dt$$

for all $u, v \in H^1_{kT}$. It is well known that the $kT-$periodic solutions of problem (1) correspond to the critical points of functional φ_k.

For $u \in H^1_{kT}$, let $\overline{u} = (kT)^{-1}\int_0^{kT} u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}$. Then one has Sobolev's inequality

$$\|\widetilde{u}\|^2_\infty \leq \frac{kT}{12}\int_0^{kT} |\dot{u}(t)|^2 dt \tag{3}$$

and Wertinger's inequality

$$\int_0^{kT} |\widetilde{u}(t)|^2 dt \leq \frac{k^2T^2}{4\pi^2}\int_0^{kT} |\dot{u}(t)|^2 dt. \tag{4}$$

It has been proved that problem (1) has infinitely distinct subharmonic solutions under suitable conditions (see [1-4]). Recently, Zhao and Wu [5] consider the existence of $T-$periodic solutions with saddle point character. Inspired and motivated by the results due to Mawhin-Willem [1], F. Giannoni[2], Tang-Wu [4] and Zhao-Wu [5], in this paper, we shall continue to consider the existence of subharmonic solutions with saddle point character, some new existence results are obtained by using the critical point reduction method. The results in this paper develop and generalize the corresponding results .

2. MAIN RESULTS AND PROOF

Now we state and prove our main result.

Theorem 2.1. *Suppose that F satisfies assumption (A), (2) and the following conditions:*
(i). there exists a function $\lambda \in L^1(0,T;R)$ with $\int_0^T \lambda(t)dt > 0$ such that $-\nabla F(t,\cdot)$ is $\lambda(t)-$monotone, that is

$$(-\nabla F(t,x) - (-\nabla F(t,y)), x-y) \geq \lambda(t)|x-y|^2; \tag{5}$$

for all $x, y \in R^N$ and a. e. $t \in [0,T]$;
(ii). there exist $g, h \in L^1(0,T;R), M > 1$ and $\alpha \in [1,2)$ such that

$$F(t,x) \geq g(t)|x|^\alpha + h(t) \tag{6}$$

for all $x \in R^N$ and $|x| \geq M$ and a. e. $t \in [0,T]$;
(iii).

$$F(t,x) \to -\infty \tag{7}$$

as $|x| \to +\infty$ uniformly for a. e. $t \in [0,T]$.
Then problem (1) has $kT-$periodic solutions u_k with saddle point character in H^1_{kT} for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Proof. Without loss of generality, we may assume that functions b in assumption(A) , λ in (5) and g, h in (6) are $T-$ periodic and assumption (A) , (5), (6) and (7) hold for all $t \in R$ by the $T-$ periodicity of $F(t,x)$ in the first variable.

Set $\widetilde{H}^1_{kT} = \{u \in H^1_{kT} | \overline{u} = 0\}$, then $H^1_{kT} = R^N \bigoplus \widetilde{H}^1_{kT}$, obviously. Define the function Ψ as follows:

$$\Psi(u) = \sup_{x \in R^N} \varphi_k(u+x) \qquad \forall u \in \widetilde{H}^1_{kT}.$$

For each fixed $u \in \widetilde{H}^1_{kT}$ and any $x_1, x_2 \in R^N$, one has

$$\int_0^{kT} (-\nabla F(t,u(t)+x_1) - (-\nabla F(t,u(t)+x_2)), x_1 - x_2)dt \geq |x_1 - x_2|^2 \int_0^{kT} \lambda(t)dt$$

Consequently,

$$\langle -\varphi_k'(u(t)+x_1) - (-\varphi_k'(u(t)+x_2)), x_1 - x_2 \rangle \geq |x_1 - x_2|^2 \int_0^{kT} \lambda(t)dt.$$

By virtue of Theorem 2.3 in [6] there exists a continuous mapping $\theta : \widetilde{H}^1_{kT} \to R^N$ such that $\varphi_k(u+\theta(u)) = \Psi(u)$ for all $u \in \widetilde{H}^1_{kT}, \Psi : \widetilde{H}^1_{kT} \to R$ is continuously differentiable, and $\Psi'(u) = \varphi'_k(u+\theta(u))|_{\widetilde{H}^1_{kT}}$ for all $u \in \widetilde{H}^1_{kT}$. Hence, $u \in \widetilde{H}^1_{kT}$ is a critical point of Ψ implies $u+\theta(u)$ is a critical point of φ_k.

Moreover, for each $u \in \widetilde{H}^1_{kT}$, by condition (ii) and Sobolev's inequality one has

$$\begin{aligned}\Psi(u) \geq & \varphi_k(u) = \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt + \int_0^{kT} F(t,u(t))dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt + \int_0^{kT} g(t)|u(t)|^\alpha dt + \int_0^{kT} h(t)dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \|u\|_\infty^\alpha \int_0^{kT} |g(t)|dt + \int_0^{kT} h(t)dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - C_1(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha}{2}} + C_2\end{aligned} \tag{8}$$

for all $u \in \widetilde{H}^1_{kT}$ and some positive constants C_1 and C_2. By Wertinger's inequality, one has

$$\|u\| \to +\infty \Leftrightarrow \|\dot{u}\|_2 \to +\infty$$

on $\widetilde{H}^1_{kT}$, then (8) implies that $\Psi(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, there exists a point $u_0 \in \widetilde{H}^1_{kT}$ such that $\Psi(u_0) = \min_{\widetilde{H}^1_{kT}} \Psi(u)$, and hence $u_k = u_0 + \theta(u_0)$ is a solution with saddle point character of problem (1) in H^1_{kT}.

Since $F(t,x) \to -\infty$ as $|x| \to +\infty$, so for every $\beta > 0$ there exists $M \geq 1$ such that

$$F(t,x) \leq -\beta \tag{9}$$

for all $|x| \geq M$, and by assumption (A), there exists $\gamma \in L^1(0,T)$ such that

$$F(t,x) \leq \gamma(t) \tag{10}$$

for all $x \in R^N$ and a. e. $t \in [0,T]$. Without loss of generality, we may assume that (9) and (10) hold for all $t \in [0,T]$, and that $\gamma(t)$ is $T-$periodic.

Set

$$e_k(t) = k(\cos k^{-1}\omega t)x_0$$

for all $t \in R$ and some $x_0 \in R^N$ with $|x_0| = 1$, where $\omega = 2\pi/T$. Then we have

$$e_k(t) \in \widetilde{H}^1_{kT}$$

obviously and

$$\dot{e}_k(t) = -\omega(\sin k^{-1}\omega t)x_0$$

for all $t \in R$, which implies that

$$\int_0^{kT} |\dot{e}_k(t)|^2 dt = \frac{1}{2} kT\omega^2.$$

By the definition of u_k, we have

$$\varphi_k(u_k) = \min_{u \in \widetilde{H}^1_{kT}} \sup_{x \in R^N} \varphi_k(x+u) \leq \sup_{x \in R^N} \varphi_k(x+e_k) = \sup_{R^N + e_k} \varphi_k. \tag{11}$$

Now we prove that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$. For fixed $x \in R^N$, set

$$A_k = \{t \in [0, kT] | |x + k(\cos k^{-1}\omega t)x_0| \leq M\}.$$

Then we have

$$\text{meas} A_k \leq \frac{kT}{2} \tag{12}$$

for all large k. In fact, if

$$\text{meas} A_k > \frac{kT}{2},$$

then there exists $t_1 \in A_k$ such that

$$\frac{kT}{8} \leq t_1 \leq \frac{3kT}{8} \tag{13}$$

or

$$\frac{5kT}{8} \leq t_1 \leq \frac{7kT}{8}. \tag{14}$$

Moreover, there exists $t_2 \in A_k$ such that $|t_2 - t_1| \geq \frac{kT}{8}$ which implies that

$$|\frac{1}{2} k^{-1}(t_2 - t_1)\omega| \geq \frac{\pi}{8} \tag{15}$$

and $|t_2 - (kT - t_1)| \geq \frac{kT}{8}$ which implies that

$$|\frac{1}{2} k^{-1}(t_2 + t_1)\omega - \pi| \geq \frac{\pi}{8}. \tag{16}$$

By (13) and (14) one has

$$\frac{T}{16} \leq \frac{1}{2} k^{-1}(t_1 + t_2) \leq \frac{15T}{16}$$

which implies that

$$\frac{\pi}{8} \leq \frac{1}{2} k^{-1}(t_1 + t_2) \leq \frac{15\pi}{8}. \tag{17}$$

From (16) and (17) we obtain

$$|\sin(\frac{1}{2} k^{-1}(t_1 + t_2)\omega)| \geq sin\frac{\pi}{8}.$$

From (14), (15) and (16) we obtain

$$|\sin(\frac{1}{2} k^{-1}(t_1 - t_2)\omega)| \geq sin\frac{\pi}{8}.$$

So we have

$$\begin{aligned}&|\cos(k^{-1}\omega t_1)-\cos(k^{-1}\omega t_2)|\\&=2|\sin(\frac{1}{2}k^{-1}(t_1+t_2)\omega)||\sin(\frac{1}{2}k^{-1}(t_1-t_2)\omega)|\\&\geq 2\sin^2\frac{\pi}{8}.\end{aligned}$$

But due to $t_1, t_2 \in A_k$, we have

$$\begin{aligned}&|\cos(k^{-1}\omega t_1)-\cos(k^{-1}\omega t_2)|\\&=\frac{1}{k}|(x+k\cos(k^{-1}\omega t_1)x_0)-(x+k\cos(k^{-1}\omega t_2)x_0)|\\&\leq\frac{2M}{k}\end{aligned}$$

which is a contradiction for large k. Hence (12) holds. Therefore

$$\text{meas}([0,kT]\setminus A_k)\geq\frac{1}{2}kT$$

for large k. By (9) and (10) we have

$$\begin{aligned}k^{-1}\varphi_k(x+e_k)&=\frac{1}{4}T\omega^2-k^{-1}\int_0^{kT}F(t,x+k(\cos k^{-1}\omega t)x_0)dt\\&\leq\frac{1}{4}T\omega^2-k^{-1}\int_{A_k}|\gamma(t)|dt-k^{-1}\int_{[0,kT]\setminus A_k}F(t,x+k(\cos k^{-1}\omega t)x_0)dt\\&\leq\frac{1}{4}T\omega^2+\int_0^T|\gamma(t)|dt-\frac{T}{2}\beta\end{aligned}$$

for all $x\in R^N$ and all large k. Then by the arbitrariness of β, following the same way in [4] we complete our proof. □

Theorem 2.2. *Suppose that F satisfies assumption (A), (2) and the following conditions:*
(i). there exists a function $\mu\in L^1(0,T;R^+)$ with $\int_0^T\mu(t)dt>0$ such that $-F(t,\cdot)$ is $\mu(t)-$ convex for a. e. $t\in[0,T]$. That is, for a . e. $t\in[0,T]$, the function $x\to F(t,x)-\frac{\mu(t)}{2}|x|^2$ is convex;
(ii). there exist $g,h\in L^1(0,T;R^+), M>1$ and $\alpha\in(0,2)$ such that

$$|F(t,x)|\leq g(t)|x|^\alpha+h(t)$$

for all $x\in R^N$ and $|x|\geq M$ and a. e. $t\in[0,T]$;

(iii).

$$F(t,x)dt \to -\infty$$

as $|x| \to +\infty$ *uniformly for a. e.* $t \in [0,T]$. *Then problem (1) has* $kT-$*periodic solutions* u_k *with saddle point character in* H^1_{kT} *for every positive integer* k *such that* $\|u_k\|_\infty \to +\infty$ *as* $k \to +\infty$.

Proof. Similarly, we may assume that functions b, γ, μ, g, h are all $T-$ periodic. Define the function Ψ as follows:

$$\Psi(u) = \sup_{x \in R^N} \varphi_k(u+x) \qquad \forall u \in \widetilde{H}^1_{kT}.$$

For each fixed $u \in \widetilde{H}^1_{kT}$ and any $x_1, x_2 \in R^N$, by condition (i) and Lemma 2.1 in [6] one has

$$\int_0^{kT} (-\nabla F(t,u(t)+x_1)-(-\nabla F(t,u(t)+x_2)), x_1-x_2)dt \geq |x_1-x_2|^2 \int_0^{kT} \mu(t)dt.$$

Similarly as in Theorem 1, $u \in \widetilde{H}^1_{kT}$ is a critical point of Ψ implies $u + \theta(u)$ is a critical point of φ_k.

Moreover, by condition (ii) and Sobolev's inequality one has

$$\begin{aligned} \Psi(u) \geq & \varphi_k(u) = \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt + \int_0^{kT} F(t,u(t))dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} g(t)|u(t)|^\alpha dt - \int_0^{kT} h(t) \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \|u\|_\infty^\alpha \int_0^{kT} g(t)dt - \int_0^{kT} h(t)dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - C_3(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha}{2}} - C_4 \end{aligned} \tag{18}$$

for all $u \in \widetilde{H}^1_{kT}$ and some constants C_3 and C_4. Since $\|u\| \to +\infty \Leftrightarrow \|\dot{u}\|_2 \to +\infty$ on $\widetilde{H}^1_{kT}$, then (18) implies that $\Psi(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, there exists a point $u_0 \in \widetilde{H}^1_{kT}$ such that $\Psi(u_0) = \min_{\widetilde{H}^1_{kT}} \Psi(u)$, and hence $u_k = u_0 + \theta(u_0)$ is a solution with saddle point character of problem (1) in H^1_{kT}. Then the rest continue as same as in Theorem 1, we complete the proof of Theorem 2. □

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