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STRONG CONVERGENCE OF AN ITERATIVE SEQUENCE FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, we consider the problem of finding zeros of m-accretive operators by an iterative process with errors. A strong convergence theorem is established in a real Banach space.

1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a nonlinear mapping. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(T)$ to denote the fixed point set of T.

It is a efficient way to to study nonexpansive mappings by using contractions. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C, \tag{1.1}$$

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where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t \to 0$ even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved that if E is a Hilbert space then x_t converges strongly to a fixed point of T which is nearest to u. Reich [15] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Let E^* be the dual space of a Banach space E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In the sequel, we use j to denote the single-valued normalized duality mapping. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be smooth or said to be have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for all $x \in U$. E is said to be uniformly smooth or said to be have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x, y \in U$. It is known that if E is smooth, then J is single-valued. A Banach space E is uniformly smooth if and only if the duality map J is the single-valued and norm-to-norm uniformly continuous on bounded sets of E.

Recall that a (possibly multivalued) operator A with the domain $D(A)$ and the range $R(A)$ in E is accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, where $i = 1, 2$ there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1) \rangle \geq 0.$$

An accretive operator A is m-accretive if $R(I + rA) = E$ for each $r > 0$. Throughout this article we always assume that A is m-accretive and has a zero (i.e., the inclusion $0 \in A(z)$ is solvable). The set of zeros of A is denoted by F. Hence,

$$F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0).$$

For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from E to $C := \overline{D(A)}$ which will be assumed convex.

The normal Mann iterative process [10] was introduced by Mann in 1976. The normal Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_0 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 0, \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is in the interval $(0,1)$.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [14]). In an infinite-dimensional Hilbert space, the normal Mann's iterative process has only weak convergence. Therefore, many authors try to modify the normal Mann's iterative process to have strong convergence for nonexpansive mappings and its extensions (see [3-8,11-13] and the references therein).

Recently, Qin and Su [13] introduced a modified Mann iterative process for accretive operators in a Banach spaces. To be more precise, they proved the following results.

Theorem QS. *Assume that E is a uniformly smooth Banach space and A is an m-accretive operator in E such that $A^{-1}(0) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $r_n \geq \epsilon$ *for each* $n \geq 0$, $\beta_n \in [0,a)$ *for some* $a \in (0,1)$;

(c) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\sum_{n=0}^{\infty} |\beta_n - \beta_{n+1}| < \infty$ *and* $\sum_{n=0}^{\infty} |r_n - r_{n+1}| < \infty$.

Let $\{x_n\}$ be a sequence defined in the following iterative process:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha u + (1-\alpha_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

Then the sequence $\{x_n\}$ converges strongly to a zero of A.

Inspired and motivated by the above results, we consider the following iterative process

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + \gamma_n J_{r_n} x_n + \delta_n v_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{1.3}$$

where u is an arbitrary (but fixed) element in C, $\{v_n\}$ a bounded sequence in C, $J_{r_n} = (I + r_n A)^{-1}$, $\{\alpha_n\}$ a sequence in $[0,1]$ and $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ such that $\beta_n + \gamma_n + \delta_n \equiv 0$.

If $\delta_n \equiv 0$ in (1.3), then the iterative process (1.3) is reduced to (1.2). If $\delta_n \equiv \beta_n \equiv 0$ in (1.3), then the iterative process (1.3) is reduced to

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0, \tag{1.4}$$

which was considered in Kim and Xu [8] and Xu [18].

The purpose of this paper is to consider the iterative process (1.3) for approximating a zero of accretive operators in the framework of uniformly smooth Banach spaces. The results presented in this paper mainly improve and extend the corresponding results in Kim and Xu [8], Qin and Su [13] and Xu [18].

In order to prove our main results, we also need the following concepts and lemmas.

Let C be a nonempty closed convex subset of a Banach space E and D a nonempty subset of C. Recall that a mapping Q of C onto D is said to be sunny if $Q(Qx + t(x - Q(x))) = Q(x)$ for any $x \in C$ and $t \geq 0$ with $Q(x)+t(x-Q(x)) \in C$; Recall that a subset D of C is said to be a nonexpansive retract of C if there exists a nonexpansive retraction of C onto D.

The following results describe a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1.1 ([16]). *Let E be a smooth Banach space C a nonempty subset of E. Let $Q : E \to C$ be a retraction. Then the following are equivalent:*

(1) *Q is sunny and nonexpansive;*
(2) $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy)\rangle, \quad \forall x, y \in E$;
(3) $\langle x - Qx, j(y - Qx)\rangle \leq 0, \quad \forall x \in E, y \in C$.

Proposition 1.2 ([15]). *Let E be a uniformly smooth Banach space and $T : C \to C$ a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D; that is, Q satisfies the property:*

$$\langle u - Qu, J(y - Qu)\rangle \leq 0, \quad \forall u \in C, y \in D.$$

Lemma 1.3. *In a Banach space E, there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall x, y \in E,$$

where $j(x + y) \in J(x + y)$.

Lemma 1.4 ([17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5 ([2]). *Let* E *be a Banach space* A *an* m*-accretive operator. For* $\lambda > 0$ *and* $\mu > 0$ *and* $x \in E$, *we have*

$$J_\lambda x = J_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x),$$

where $J_\lambda = (I + \lambda A)^{-1}$ *and* $J_\mu = (I + \mu A)^{-1}$.

Lemma 1.6 ([9]). *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{t_n\}$ *is a sequence in* $(0, 1)$. *Assume that the following conditions are satisfied*

(a) $\sum_{n=0}^{\infty} t_n = \infty$ *and* $b_n = o(t_n)$;
(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

Theorem 2.1. *Let* E *be a real smooth Banach space and* A *an* m*-accretive operators in* E. *Assume that* $C := \overline{D(A)}$ *is nonempty and convex. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *be real number sequences in* $[0, 1]$. *Let* Q_C *be the sunny nonexpansive retraction from* E *onto* C *and* $\{v_n\}$ *a bounded sequence in* C. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + \gamma_n J_{r_n} x_n + \delta_n v_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad n \geq 0, \end{cases}$$

where u *is a fixed element in* C, $\{r_n\}$ *is a positive real numbers sequence and* $J_{r_n} = (I + r_n A)^{-1}$. *Assume that the above control sequences satisfy the following restrictions:*

(a) $\beta_n + \gamma_n + \delta_n = 1$, *for each* $n \geq 0$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\sum_{n=0}^{\infty} \delta_n < \infty$;
(e) $r_n \geq \lambda > 0$ *for each* $n \geq 0$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to a zero* $Q(u)$ *of* A.

Proof. First, we prove that $\{x_n\}$ is bounded. Fixing $p \in A^{-1}(0)$, we see that

$$\begin{aligned}\|x_1 - p\| &= \|\alpha_0(u-p) + (1-\alpha_0)(y_0 - p)\| \\ &\leq \alpha_0\|u-p\| + (1-\alpha_0)\|y_0 - p\| \\ &\leq \alpha_0\|u-p\| + (1-\alpha_0)\big(\beta_0\|x_0 - p\| + \gamma_0\|J_{r_0}x_0 - p\| + \delta_0\|v_0 - p\|\big) \\ &\leq \alpha_0\|u-p\| + (1-\alpha_0)\big((1-\delta_0)\|x_0 - p\| + \delta_0\|f_0 - p\|\big) \\ &\leq K,\end{aligned}$$

where $K = \|u-p\| + \|x_0 - p\| + \|v_0 - p\| < \infty$. Next, we prove that

$$\|x_n - p\| \leq M_1, \quad \forall n \geq 1 \tag{2.1}$$

where $M_1 = \max\{K, \sup_{n\geq 0}\|v_n - p\|\}$. It is easy to see that (2.1) holds for $n = 1$. We assume that the result holds for some positive integer m. It follows that

$$\begin{aligned}&\|x_{m+1} - p\| \\ &\leq \alpha_m\|u-p\| + (1-\alpha_m)\|y_m - p\| \\ &= \alpha_m\|u-p\| + (1-\alpha_m)\big(\|\beta_m(x_n - p) + \gamma_m(J_{r_m}x_m - p) + \delta_m(v_m - p)\|\big) \\ &\leq \alpha_m\|u-p\| + (1-\alpha_m)\big(\beta_m\|x_m - p\| + \gamma_m\|J_{r_m}x_m - p\| + \delta_m\|v_m - p\|\big) \\ &\leq \alpha_m\|u-p\| + (1-\alpha_m)\big((1-\delta_m)\|x_m - p\| + \delta_m\|v_m - p\|\big) \\ &\leq \alpha_m M_1 + (1-\alpha_m)\big((1-\delta_m)M_1 + \delta_m M_1\big) \\ &= M_1.\end{aligned}$$

This shows that (2.1) holds for all $n \geq 1$. This claim that $\{x_n\}$ is bounded. If $r_{n+1} \geq r_n$, from Lemma 1.5, we see from the condition (e) that

$$\begin{aligned}\|J_{r_n}x_n - J_{r_{n+1}}x_{n+1}\| &\leq \|\frac{r_n}{r_{n+1}}x_n + (1 - \frac{r_n}{r_{n+1}})J_{r_n}x_n - x_{n+1}\| \\ &= \|\frac{r_n}{r_{n+1}}(x_n - x_{n+1}) + \frac{r_{n+1} - r_n}{r_{n+1}}(J_{r_n}x_n - x_{n+1})\| \\ &\leq \|x_n - x_{n+1}\| + \frac{M_2}{\lambda}(r_{n+1} - r_n),\end{aligned} \tag{2.2}$$

where M_2 is an appropriate constant such that

$$M_2 \geq \sup_{n\geq 0}\{\|J_{r_n}x_n - x_{n+1}\|\}.$$

Put $g_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$ for every $n \geq 0$. This implies that

$$x_{n+1} = (1-\beta_n)g_n + \beta_n x_n, \quad n \geq 0. \tag{2.3}$$

Now, we compute $\|g_{n+1} - g_n\|$. Note that

$$\begin{aligned} & g_{n+1} - g_n \\ &= \frac{\alpha_{n+1}u + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n u + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}(u - y_{n+1}) + (1-\beta_{n+1})J_{r_{n+1}}x_{n+1} + \delta_{n+1}(v_{n+1} - J_{r_{n+1}}x_{n+1})}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n(u-y_n) + (1-\beta_n)J_{r_n}x_n + \delta_n(v_n - J_{r_n}x_n)}{1-\beta_n}. \end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned} \|g_{n+1} - g_n\| \le\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| \\ & + \|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| + \frac{\delta_{n+1}}{1-\beta_{n+1}}\|v_{n+1} - J_{r_{n+1}}x_{n+1}\| \\ & + \frac{\delta_n}{1-\beta_n}\|v_n - J_{r_n}x_n\|. \end{aligned} \tag{2.5}$$

Substituting (2.2) into (2.5), we arrive at

$$\begin{aligned} & \|g_{n+1} - g_n\| - \|x_n - x_{n+1}\| \\ & \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - u\| \\ & \quad + \frac{M_2}{\epsilon}(r_{n+1} - r_n) + \frac{\delta_{n+1}}{1-\beta_{n+1}}\|v_{n+1} - J_{r_{n+1}}x_{n+1}\| + \frac{\delta_n}{1-\beta_n}\|v_n - J_{r_n}x_n\|. \end{aligned}$$

In view of the conditions (b), (c), (d) and (e), we can conclude that

$$\limsup_{n\to\infty} \big(\|g_{n+1} - g_n\| - \|x_n - x_{n+1}\|\big) \le 0. \tag{2.6}$$

In the case of $r_n \ge r_{n+1}$, we can obtain (2.6) by a similar way. It follows from Lemma 1.4 that

$$\lim_{n\to\infty} \|g_n - x_n\| = 0. \tag{2.7}$$

In view of (2.3), we have

$$x_{n+1} - x_n = (1-\beta_n)(g_n - x_n),$$

which combines with (2.7) gives that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.8}$$

On the other hand, we have

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - J_{r_n}x_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\| + \beta_n\|x_n - J_{r_n}x_n\| + \delta_n\|v_n - x_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - J_{r_n}x_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|u - y_n\| + \delta_n\|v_n - x_n\|.$$

By virtue of the the conditions (b)-(d) and (2.8), we conclude that

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{2.9}$$

Take a fixed number r such that $\epsilon > r > 0$. From Lemma 1.5, we obtain that

$$\begin{aligned}\|J_{r_n}x_n - J_r x_n\| &= \|J_r\big(\frac{r}{r_n}x_n + (1-\frac{r}{r_n})J_{r_n}x_n\big) - J_r x_n\| \\ &\leq \|(1-\frac{r}{r_n})(J_{r_n}x_n - x_n)\| \\ &\leq \|J_{r_n}x_n - x_n\|.\end{aligned} \tag{2.10}$$

It follows that

$$\begin{aligned}\|x_n - J_r x_n\| &= \|x_n - J_{r_n}x_n + J_{r_n}x_n - J_r x_n\| \\ &\leq \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\ &\leq 2\|x_n - J_{r_n}x_n\|.\end{aligned}$$

Thanks to (2.9), we obtain that

$$\lim_{n\to\infty}\|x_n - J_r x_n\| = 0. \tag{2.11}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \leq 0, \tag{2.12}$$

where

$$Qu = \lim_{t\to 0} z_t, \quad u \in C$$

and z_t solves the fixed point equation

$$z_t = tu + (1-t)J_r z_t, \quad \forall t \in (0,1),$$

from which it follows that

$$\|z_t - x_n\| = \|(1-t)(J_r z_t - x_n) + t(u - x_n)\|.$$

It follows From Lemma 1.3 that

$$\begin{aligned}\|z_t - x_n\|^2 &\leq (1-t)^2\|J_r z_t - x_n\|^2 + 2t\langle u - x_n, j(z_t - x_n)\rangle \\ &\leq (1-2t+t^2)\|z_t - x_n\|^2 + f_n(t) \\ &+ 2t\langle u - z_t, j(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2,\end{aligned} \tag{2.13}$$

where

$$f_n(t) = (2\|z_t - x_n\| + \|x_n - J_r x_n\|)\|x_n - J_r x_n\| \to 0. \tag{2.14}$$

In view of (2.1), we see that $f_n(t) \to 0$ as $n \to \infty$. It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{2.15}$$

Let $n \to \infty$ in (2.15) yields that

$$\limsup_{n\to\infty}\langle z_t - u, j(z_t - x_n)\rangle \le \frac{t}{2}M_3, \tag{2.16}$$

where $M_3 > 0$ is a constant such that $M_3 \ge \|z_t - x_n\|^2$ for all $t \in (0,1)$ and for all $n \ge 0$. In view of (2.16), we can obtain that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, j(z_t - x_n)\rangle \le 0.$$

So, for any $\epsilon > 0$, there exists a positive number δ_1 such that, for $t \in (0, \delta_1)$ we get that

$$\limsup_{n\to\infty}\langle z_t - u, j(z_t - x_n)\rangle \le \frac{\epsilon}{2}. \tag{2.17}$$

On the other hand, since $z_t \to q$ as $t \to 0$, we see there exists $\delta_2 > 0$ such that, for $t \in (0, \delta_2)$ we have

$$\begin{aligned}
&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, j(z_t - x_n)\rangle| \\
&\le |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| \\
&\quad + |\langle u - q, j(x_n - z_t)\rangle - \langle z_t - u, j(z_t - x_n)\rangle| \\
&\le |\langle u - q, j(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, j(x_n - z_t)\rangle| \\
&\le \|u - q\|\|j(x_n - q) - j(x_n - z_t)\| + \|z_t - q\|\|x_n - z_t\| \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

Choosing $\delta = \min\{\delta_1, \delta_2\}, \forall t \in (0, \delta)$, we have

$$\langle u - Q(u), j(x_n - Q(u))\rangle \le \langle z_t - u, j(z_t - x_n)\rangle + \frac{\epsilon}{2},$$

that is,

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \le \limsup_{n\to\infty}\langle z_t - u, j(z_t - x_n)\rangle + \frac{\epsilon}{2}.$$

It follows from (2.17) that

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \le \epsilon.$$

Since ϵ is chosen arbitrarily, we obtain that (2.12) holds.

Finally, we show that $x_n \to Q(u)$ as $n \to \infty$ and this concludes the proof. Note that

$$\begin{aligned}\|y_n - Q(u)\| &= \|\beta_n x_n + \gamma_n J_{r_n} x_n + \delta_n v_n - Q(u)\| \\ &\leq \beta_n \|x_n - Q(u)\| + \gamma_n \|J_{r_n} x_n - Q(u)\| + \delta_n \|v_n - Q(u)\| \\ &\leq \|x_n - Q(u)\| + \delta_n M_1.\end{aligned}$$

It follows from Lemma 1.3 that

$$\begin{aligned}&\|x_{n+1} - Q(u)\|^2 \\ &= \|(1-\alpha_n)(y_n - Q(u)) + \alpha_n (u - Q(u))\|^2 \\ &\leq (1-\alpha_n)^2 \|y_n - Q(u)\|^2 + 2\alpha_n \langle u - Q(u), j(x_{n+1} - Q(u))\rangle \\ &\leq (1-\alpha_n)(\|x_n - Q(u)\| + \delta_n M_1)^2 + 2\alpha_n \langle u - Q(u), j(x_{n+1} - Q(u))\rangle \quad (2.18) \\ &= (1-\alpha_n)(\|x_n - Q(u)\|^2 + 2\delta_n M_1 \|x_n - Q(u)\| + \delta_n^2 M_1^2) \\ &\quad + 2\alpha_n \langle u - Q(u), j(x_{n+1} - Q(u))\rangle \\ &\leq (1-\alpha_n)\|x_n - Q(u)\|^2 + \eta_n M_3 + 2\alpha_n \langle u - Q(u), j(x_{n+1} - Q(u))\rangle,\end{aligned}$$

where M_3 is an appropriate constant such that

$$M_3 \geq \sup_{n\geq 0}\{2M_1\|x_n - Q(u)\| + \delta_n M_1^2\}.$$

Let $\rho_n = \max\{\langle u - Q(u), j(x_n - Q(u))\rangle, 0\}$. Next, we show that $\lim_{n\to\infty} \rho_n = 0$. Indeed, from (2.12), for any give $\epsilon > 0$, there exists a positive integer n_1 such that

$$\langle u - Q(u), J(x_n - Q(u))\rangle < \epsilon, \quad \forall n \geq n_1.$$

This implies that $0 \leq \rho_n < \epsilon \; \forall n \geq n_1$. Since $\epsilon > 0$ is arbitrary, we see that $\lim_{n\to\infty} \rho_n = 0$. It follows from (2.18) that

$$\|x_{n+1} - Q(u)\|^2 \leq (1-\alpha_n)\|x_n - Q(u)\|^2 + \eta_n M_3 + 2\alpha_n \rho_{n+1}.$$

Put $a_n = \|x_n - Q(u)\|^2$, $t_n = \alpha_n$, $b_n = 2\alpha_n\rho_{n+1}$ and $c_n = \eta_n M_3$ for every $n \geq 0$. In view of Lemma 1.6, we can obtain the desired conclusion. □

Remark 2.2. The proof of Theorem 2.1 is different from Theorem 2.1 of Qin and Su [13]. From computation point of view, Theorem 2.1 can be viewed as an improvement of Theorem 2.1 of Qin and Su [13]. Furthermore, the restrictions imposed on the control sequences are also mild.

As an application, we have the following results.

Corollary 2.3. *Let E be a real smooth Banach space and A an m-accretive operators in E. Assume that $C := \overline{D(A)}$ is nonempty and convex. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $[0,1]$. Let Q_C be the sunny nonexpansive* *retraction from E onto C and $\{x_n\}$ a sequence generated in the following manner:*

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha u + (1-\alpha_n) y_n, \quad \forall n \geq 0, \end{cases}$$

where u is a fixed element in C, $\{r_n\}$ is a positive real numbers sequence and $J_{r_n} = (I + r_n A)^{-1}$. Assume that the above control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*;*
(c) $r_n \geq \lambda > 0$ *for each* $n \geq 0$ *and* $\lim_{n\to\infty} |r_n - r_{n+1}| = 0$.

Then the sequence $\{x_n\}$ converges strongly to a zero $Q(u)$ of A.

References

[1] F.E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Ration. Mech. Anal. **24** (1967), 82–90.

[2] V. Barbu, *Nonlinear Semigroups and differential equations in Banach space*, Noordhoff, 1976.

[3] Y.J. Cho, S.M. Kang and X. Qin, *Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces*, Comput. Math. Appl. **56** (2008), 2058–2064.

[4] J.S. Jung, Y.J. Cho and H. Zhou, *Iterative processes with mixed errors for nonlinear equations with perturbed m-accretive operators in Banach spaces*, Appl. Math. Comput. **133** (2002), 389–406.

[5] J.S. Jung, *Convergence of composite iterative methods for finding zeros of accretive operators*, Nonlinear Anal. **71** (2009) 1736–1746.

[6] J.K. Kim, Y.M. Nam and J.Y. Sim, *Convergence theorems of implicit iterative sequences for a finite family of asymptotically quasi-nonexpansive type mappings*, Nonlinear Anal. TMA, 71(2009), e2839-e2848.

[7] J.K. Kim, G. Li, Y.M. Nam and K.S. Kim, *Nonlinear ergodic theorems for almost-orbits of asymptotically nonexpansive type mappings in Banach spaces*, Bull. Korean Math. Soc. **38** (2001), 587–603.

[8] T.H. Kim and H.K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal. **61** (2005), 51–60.

[9] L.S. Liu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

[10] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[11] G. Marino. V. Colao, X. Qin and S.M. Kang, *Strong convergence of the modified Mann iterative method for strict pseudo-contractions*, Comput. Math. Appl. **57** (2009), 455–465.

[12] X. Qin, Y.J. Cho, J.I. Kang and S.M. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces*, J. Comput. Appl. Math. **230** (2009), 121–127.

[13] X. Qin and Y. Su, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal. Appl. **329** (2007), 415–424.
[14] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.
[15] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.
[16] S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl. **44** (1973), 57–70.
[17] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals*, J. Math. Anal. Appl. **305** (2005), 227–239.
[18] H.K. Xu, *Strong convergence of an iterative method for nonexpansive and accretive operators*, J. Math. Anal. Appl. **314** (2006), 631–643.