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APPLICATION AND FIXED POINT THEOREMS FOR ORTHOGONAL GENERALIZED F-CONTRACTION MAPPINGS ON O-COMPLETE METRIC SPACE

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Abstract. In this paper, we introduce the concepts of an orthogonal generalized F-contraction mapping and prove some fixed point theorems for a self mapping in an orthogonal metric space. The given results are generalization and extension some of the well-known results in the literature. An example to support our result is presented.

1. Introduction

The concept of an orthogonal set has many applications in several branches in mathematics and it has many types of the orthogonality. Gordji et al. [3]

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introduced the new concept of an orthogonality in metric spaces and proved the fixed point result for contraction mappings in metric spaces endowed with the new orthogonality. Furthermore, they gave the application of this results for claming the existence and uniqueness of solution of the first-order ordinary differential equation while the Banach contraction mapping can not be applied in this problem. Gordji and Habibi [4] proved fixed point theorems in a generalized orthogonal metric space. Sawangsup et al. [7] introduced the new concept of an orthogonal F-contraction mapping and proved the fixed point theorems in orthogonal-complete metric spaces. The orthogonal contractive type mappings have been studied by many authors and important results have been obtained by [2, 5, 8, 9, 12].

In this paper, we introduced the new concepts of an orthogonal generalized F-contraction mappings and proved the fixed point theorems in an orthogonal-complete metric space.

2. Preliminaries

Throughout this paper, we denote by V, $\mathbb{R}^+$ and $\mathbb{N}$ the nonempty set, the set of positive real numbers and the set of positive integers, respectively.

Firstly, we recall the concept of an F-contraction function which is introduced by Wardowski [10]. Let $\Im$ denote the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

- (F_1) F is strictly increasing;
- (F_2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We give some examples of functions belonging to $\Im$ as follows:

Example 2.1. Let functions F_1, F_2, F_3, $F_4 : \mathbb{R}^+ \to \mathbb{R}$ be defined by:

(1) $F_1(\alpha) = \ln \alpha$ for $\alpha > 0$;
(2) $F_2(\alpha) = \alpha + \ln \alpha$ for all $\alpha > 0$;
(3) $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ for all $\alpha > 0$;
(4) $F_4(\alpha) = \ln(\alpha^2 + \alpha)$ for all $\alpha > 0$.

Then $F_1, F_2, F_3, F_4 \in \Im$.

Let (V, d) be a metric space. A map $H : V \to V$ is said to be an F-contraction on (V, d) (see [1, 6, 13]) if there exist $F \in \Im$ and $\tau > 0$ such that for all $v, u \in V$ satifying $d(Hv, Hu) > 0$, the following holds:

$$\tau + F(d(Hv, Hu)) \leq F(d(v, u)).$$

In [11], Wardowski and Dung introduced the definition of an F-weak contraction mapping as follows:

Definition 2.2. Let (V, d) be a metric space. A map $H : V \to V$ is said to be an F-weak contraction on (V, d) if there exist $F \in \Im$ and $\tau > 0$ such that for all $v, u \in V$ satisfying $d(Hv, Hu) > 0$, the following holds:

$$\tau + F(d(Hv, Hu)) \leq F(S(v, u)) \tag{2.1}$$

where

$$S(v, u) = \max\left\{d(v, u), d(v, Hv), d(u, Hu), \frac{d(v, Hu) + d(u, Hv)}{2}\right\}.$$

Every F-contraction is an F-weak contraction, but the inverse implication does not hold.

Example 2.3. ([11]) The map $H : [0, 1] \to [0, 1]$ given by

$$Hu = \begin{cases} \frac{1}{2} & \text{if } u \in [0, 1) \\ \frac{1}{4} & \text{if } u = 1 \end{cases}$$

is not continuous, it is an F-contraction, but choosing $F(t) = \ln(t)$, $t > 0$ and $\tau = \ln 3$.

In [11], authors proved the following existence and uniqueness of a fixed point for F-contraction mappings:

Theorem 2.4. ([11]) *Let (V, d) be a complete metric space and $H : V \to V$ be an F-contraction mapping. Then, H has a unique fixed point in V. Moreover, for each $v_0 \in V$, the Picard sequence $\{H^n v_0\}$ converges to the fixed point of H.*

On the other hand, Gordji et al. [3] introduced the concept of an orthogonal set (or O-set). Some examples and properties of the orthogonal sets are as following.

Definition 2.5. ([3]) Let $V \neq \emptyset$ and $\perp \subseteq V \times V$ be a binary relation. V is called an orthogonal set (briefly O-set), if $\perp$ satisfies the following condition:

$$\exists v_0 \in V : (\forall v \in V, v \perp v_0) \quad \text{or} \quad (\forall v \in V, v_0 \perp v),$$

We denote this O-set by $(V, \perp)$.

Example 2.6. ([3]) Let V be the set all people in the world. Define the binary relation $\perp$ on V by $u \perp v$ if u can give blood to v. According to the Table 1, if v_0 is a person such that his(her) blood type is O-, then we have $v_0 \perp v$ for all $v \in V$. This means that $(V, \perp)$ is an O-set. In this O-set, v_0 (in Definition 2.5) is not unique. Note that, in this example, v_0 may be a person with blood type $AB+$. In this case, we have $v \perp v_0$ for all $v \in V$.

Table 1:

Type	You can give blood to	You can receive blood from
A+	A+AB+	A+A-O+O-
O+	O-A+B+AB+	O+O-
B+	B+AB+	B+B-O+O-
AB+	AB+	Everyone
A-	A+A-AB+AB-	A-O-
O-	Everyone	O-
B-	B+B-AB+AB-	B-O-
AB-	AB+AB-	AB-B-O-A-

Example 2.7. ([3]) Let $V = \mathbb{Z}$. Define the binary relation $\perp$ on V by $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence, $(V, \perp)$ is an O-set.

Example 2.8. ([3]) Let (V, d) be a metric space and $H : V \to V$ be a Picard operator, that is, there exists $v^* \in V$ such that $\lim_{n\to\infty} H^n u = v^*$ for all $u \in V$. We define $u \perp v$ if

$$\lim_{n\to\infty} d(v, H^n u) = 0.$$

Then $(V, \perp)$ is an O-set.

Example 2.9. ([3]) Let $V = [0, \infty)$ and define $v \perp u$ if $vu \in \{v, u\}$. Then, by setting $v_0 = 1$, $(V, \perp)$ is an O-set.

Now, we give the concepts of an O-sequence, a $\perp$-continuous mapping, an O-complete orthogonal metric space, a $\perp$-preserving mapping and a weakly $\perp$-preserving mapping.

Definition 2.10. ([3]) Let $(V, \perp)$ be an O-set. A sequence $\{v_n\}$ is called an orthogonal sequence (briefly, O-sequence) if

$$(\forall n \in \mathbb{N}, v_n \perp v_{n+1}) \quad \text{or} \quad (\forall n \in \mathbb{N}, v_{n+1} \perp v_n).$$

Definition 2.11. ([3]) The triplet $(V, \perp, d)$ is called an orthogonal metric space if $(V, \perp)$ is an O-set and (V, d) is a metric space.

Definition 2.12. ([3]) Let $(V, \perp, d)$ be an orthogonal partial metric space. Then, a mapping $H : V \to V$ is said to be orthogonally continuous (or $\perp$-continuous) in $v \in V$ if for each O-sequence $\{v_n\}$ in V with $v_n \to v$ as $n \to \infty$, we have $Hv_n \to Hv$ as $n \to \infty$. Also, H is said to be $\perp$-continuous on V if H is $\perp$-continuous in every $v \in V$.

Remark 2.13. ([3]) Every continuous mapping is $\perp$-continuous and the converse is not true.

Definition 2.14. ([3]) Let $(V, \perp, d)$ be an orthogonal metric space. Then, V is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

Remark 2.15. ([3]) Every complete metric space is O-complete and the converse is not true.

Definition 2.16. ([3]) Let $(V, \perp)$ be an O-set. A mapping $H : V \to V$ is said to be $\perp$-preserving if $Hv \perp Hu$ whenever $v \perp u$. Also $H : V \to V$ is said to be weakly $\perp$-preserving if $Hv \perp Hu$ or $Hu \perp Hv$ whenever $v \perp u$.

In [7], Sawangsup et al. introduced the definition of an orthogonal F-contraction mapping as follows:

Definition 2.17. ([7]) Let $(V, \perp, d)$ be an orthogonal metric space. A map $H : V \to V$ is said to be an orthogonal F-contraction (briefly, $F_\perp$-contraction) if there are $F \in \Im$ and $\tau > 0$ such that the following condition holds:

$$\forall v, u \in V \text{ with } v \perp u \ [d(Hv, Hu) > 0 \Rightarrow \tau + F(d(Hv, Hu)) \leq F(d(v, u))].$$

In [7], authors proved the following existence and uniqueness of a fixed point for F-contraction mappings:

Theorem 2.18. ([7]) *Let $(V, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal element v_0 and H be a self-mapping on V satisfying the following conditions:*

(i) *H is $\perp$- preserving;*
(ii) *H is an $F_\perp$-contraction mapping;*
(iii) *H is $\perp$-continuous.*

Then, H has a unique fixed point in V. Also, the Picard sequence $\{H^n v_0\}$ converges to the fixed point of H.

3. Main results

In this section, inspired by the notions of an F-weak contraction mapping and an orthogonal set, we introduce a new generalized F-contraction mapping and prove some fixed point theorems for a new generalized F-contraction mapping in an orthogonal metric space.

Definition 3.1. Let $(V, \perp, d)$ be a orthogonal metric space. A map $H : V \to V$ is said to be a orthogonal generaized F-contraction (briefly, generalized $F_\perp$-contraction) on $(V, \perp, d)$ if there are $F \in \Im$ and $\tau > 0$ such that the following condition holds:

$$\forall v, u \in V \text{ with } v \perp u \; [d(Hv, Hu) > 0 \Rightarrow \tau + F(d(Hv, Hu)) \leq F(S(v, u))] \tag{3.1}$$

where

$$S(v, u) = \max\left\{d(v, u), d(u, Hu), d(v, Hv), \frac{d(v, Hu) + d(u, Hv)}{2}\right\}.$$

Every orthogonal F-contraction is an orthogonal generalized F-contraction. The inverse implication does not hold.

Now, we give the first fixed point theorem for an orthogonal generalized F-contraction mapping in an O-complete orthogonal metric space $(V, \perp, d)$.

Theorem 3.2. *Let $(V, \perp, d)$ be an O-complete orthogonal metric space with an orthogonal element v_0 and H be a self-mapping on V satisfying the following conditions:*

(i) *H is $\perp$- preserving;*
(ii) *H is generalized $F_\perp$-contraction mapping;*
(iii) *H or F is $\perp$-continuous.*

Then, H has a unique fixed point in V. Also, the Picard sequence $\{H^n v_0\}$ converges to the fixed point of H.

Proof. Since $(V, \perp)$ is an O-set, there exists $v_0 \in V$ such that for all $v \in V$, $v \perp v_0$ or $v_0 \perp v$. It follows that $v_0 \perp H v_0$ or $H v_0 \perp v_0$. Let

$$v_1 := Hv_0, v_2 := Hv_1 = H^2 v_0, ..., v_{n+1} := Hv_n = H^{n+1} v_0$$

for all $n \in \mathbb{N} \cup \{0\}$. If $v_n = v_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then it is clear that v_n is a fixed point of H. Assume that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Denote $t_n := d(v_n, v_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since H is $\perp$-preserving, we have

$$v_n \perp v_{n+1} \quad \text{or} \quad v_{n+1} \perp v_n \tag{3.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{v_n\}$ is an O-sequence. Since H is a generalized $F_\perp$-contraction mapping, we have

$$F(t_n) = F(d(v_n, v_{n+1})) = F(d(Hv_{n-1}, Hv_n)) \leq F(S(v_{n-1}, v_n)) - \tau, \tag{3.3}$$

where

$$\begin{aligned} S(v_n, v_{n-1}) &= \max\left\{ d(v_n, v_{n-1}), d(v_n, Hv_n), d(v_{n-1}, Hv_{n-1}), \right. \\ &\qquad \left. \frac{d(v_n, Hv_{n-1}) + d(v_{n-1}, Hv_n)}{2} \right\} \\ &= \max\left\{ d(v_n, v_{n-1}), d(v_n, v_{n+1}), d(v_{n-1}, v_n), \right. \\ &\qquad \left. \frac{d(v_n, v_n) + d(v_{n-1}, v_{n+1})}{2} \right\} \\ &\leq \max\left\{ d(v_n, v_{n-1}), d(v_n, v_{n+1}), d(v_{n-1}, v_n), \right. \\ &\qquad \left. \frac{d(v_{n-1}, v_n) + d(v_n, v_{n+1})}{2} \right\} \\ &= \max\{t_{n-1}, t_n\}. \end{aligned}$$

If there exists $n \in \mathbb{N}$ such that $S(v_n, v_{n-1}) \leq t_n$. Since F is strictly increasing, (3.3) becomes

$$F(t_n) \leq F(t_n) - \tau < F(t_n).$$

It is a contradiction. Therefore, $S(v_n, v_{n-1}) \leq t_{n-1}$ for all $n \in \mathbb{N}$. Thus, from (3.3) we have

$$F(t_n) \leq F(t_{n-1}) - \tau$$

for all $n \in \mathbb{N}$, which implies that

$$F(t_n) \leq F(t_0) - n\tau, \tag{3.4}$$

where $t_0 = d(v_1, v_0)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ in (3.4), we get

$$\lim_{n\to\infty} F(t_n) = -\infty. \tag{3.5}$$

From the property (F_2), we have

$$\lim_{n\to\infty} t_n = 0. \tag{3.6}$$

From the property (F_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} t_n^k F(t_n) = 0. \tag{3.7}$$

By (3.4), we have

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k n\tau \leq 0 \tag{3.8}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.8) and using (3.6) and (3.7), we obtain

$$\lim_{n\to\infty} n t_n^k = 0. \tag{3.9}$$

From (3.9), there exists $n_1 \in \mathbb{N}$ such that $nt_n^k \leq 1$ for all $n \geq n_1$ and so

$$t_n \leq \frac{1}{n^{\frac{1}{k}}} \tag{3.10}$$

for all $n \geq n_1$.

Now, we claim that $\{v_n\}$ is a Cauchy O-sequence. Using (3.10) and the triangle inequality, it follows that, for all $m > n \geq n_1$,

$$\begin{aligned} d(v_n, v_m) &\leq d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2}) + ... + d(v_{m-1}, v_m) \\ &= t_n + t_{n+1} + ... + t_{m-1} \\ &= \sum_{j=n}^{m-1} t_j \\ &\leq \sum_{j=n}^{m-1} \frac{1}{j^{\frac{1}{k}}}. \end{aligned}$$

Since $\sum_{j=n}^{m-1} \frac{1}{j^{\frac{1}{k}}} < \infty$, it follows that $\{v_n\}$ is a Cauchy O-sequence in V. Since V is O-complete, there exists $h \in V$ such that $\lim_{n\to\infty} v_n = h$. We shall prove that h is a fixed point of H, considering two situations.

First case, we presume that H is $\perp$-continuous. Then, we have

$$Hh = H\big(\lim_{n\to\infty} v_n\big) = \lim_{n\to\infty} v_{n+1} = h$$

and so h is a fixed point of H. In the second case, we presume that F is $\perp$-continuous. Then, for each $n \in \mathbb{N}$, there is $j_n \in \mathbb{N}$ such that $j_n > j_{n-1}$ and $v_{j_n+1} = Hh$ (we take $j_0 = 1$). Then, we have

$$h = \lim_{n\to\infty} v_{j_n+1} = Hh,$$

which proves that h is a fixed point of H. Thus, in both the cases, H has a fixed point h.

In order to prove the uniquenes of fixed point, of H, let $h, g \in V$ be two fixed points of H and suppose that $H^n h = h \neq g = H^n g$ for all $n \in \mathbb{N}$. By choice of v_0, we obtain

$$\Big(v_0 \perp h \quad \text{and} \quad v_0 \perp g\Big) \quad \text{or} \quad \Big(h \perp v_0 \quad \text{and} \quad g \perp v_0\Big).$$

Since H is $\perp$-preserving, we have

$$\Big(H^n v_0 \perp H^n h \quad \text{and} \quad H^n v_0 \perp H^n g\Big) \quad \text{or} \quad \Big(H^n h \perp H^n v_0 \quad \text{and} \quad H^n g \perp H^n v_0\Big)$$

for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} d(h, g) &= d(H^n h, H^n g) \\ &\leq d(H^n h, H^n v_0) + d(H^n v_0, H^n g). \end{aligned}$$

As $n \to \infty$, we obtain $d(h, g) \leq 0$. Thus, $h = g$. Hence H has a unique fixed point in V. $\square$

Next, we present analogous theorem for Theorem 3.2 by withdrawing the continuity of H.

Theorem 3.3. *Theorem* 3.2 *also holds if we replace the hypothesis* (iii) *by the following condition:*

(iii)′ *If* $\{v_n\}$ *is a sequence in* V *such that* $v_n \to h \in V$ *and* $v_n \perp v_{n+1}$ *or* $v_{n+1} \perp v_n$ *for all* $n \in \mathbb{N}$, *then* $v_n \perp v$ *or* $v \perp v_n$ *for all* $n \in \mathbb{N}$.

Proof. From the proof of Theorem 3.2, we have already shown that there exists $h \in V$ such that $v_n \to h$ as $n \to \infty$. Put $\curlyvee = \{n \in \mathbb{N} : Hv_n = Hh\}$.

We consider the following two cases:

Case I. If $\curlyvee$ is not finite, then there is a subsequence $\{v_{n(k)}\}$ of $\{v_n\}$ such that $v_{n(k)+1} = Hv_{n(k)} = Hh$ for all $k \in \mathbb{N}$. Since $v_n \to h$, we have $Hh = h$.

Case II. If $\curlyvee$ is finite, then there is $n_0 \in \mathbb{N}$ such that $v_{n+1} = Hv_n \neq Hh$ for all $n \geq n_0$. In particular, $v_n \neq h$, $d(v_n, h) > 0$ and $d(Hv_n, Hh) > 0$ for all $n \geq n_0$. Since $v_n \perp h$ or $h \perp v_n$ for all $n \in \mathbb{N}$, we have

$$\tau + F(d(v_{n+1}, Hh)) = \tau + F(d(Hv_n, Hh)) \leq F(S(v_n, h)),$$

where

$$S(v_n, h) = \max\left\{d(v_n, h), d(v_n, Hv_n), d(h, Hh), \frac{d(v_n, Hh) + d(h, Hv_n)}{2}\right\}$$

for each $n \geq n_0$. As $n \to \infty$, we have

$$\tau + F(d(h, Hh)) \leq F(d(h, Hh)).$$

Therefore, h is a fixed point of H. This complets the proof. $\square$

Example 3.4. Let $V = [0, 1]$ and $d : V \times V \to [0, \infty)$ be a mapping defined by

$$d(h, g) = |h - g|$$

for all $h, g \in V$. Define the binary relation $\perp$ on V by $h \perp g$ if $hg \leq (h \vee g)$, where $h \vee g = h \quad or \quad g$. Then (V, d) is an O-complete metric space. Define the mapping $H : V \to V$ by

$$Hh = \begin{cases} \frac{1}{4} & \text{if } h \in [0, 1), \\ \frac{1}{8} & \text{if } h = 1. \end{cases}$$

Then, since H is not $\perp$-continuous, H is not an orthogonal F-contraction. Let $h\perp g$. Without loss of generality, we may assume that $hg \leq g$. Then, for $h \in [0,1)$ and $g = 1$,

$$d(Hh, H1) = d\left(\frac{1}{4} - \frac{1}{8}\right) = \left|\frac{1}{4} - \frac{1}{8}\right| = \frac{1}{2} > 0$$

and

$$\max\left\{d(h,1), d(h,Hh), d(1,H1), \frac{d(h,H1)+d(1,Hh)}{2}\right\} = d(1,H1) = \frac{7}{8}.$$

Choosing the function $F(a) = \ln a$, $a \in (0,\infty)$ and $\tau = \ln 3$, we have that H is an orthogonal generalized F-contraction.

4. Application to ordinary differential equations

Recall that, for any $1 \leq p < \infty$, the space $L^p(V,F,\lambda)$ (or $L^p(V)$) consists of all complex-valued measurable functions β on the underlying space V satisfying

$$\int_V |\beta(v)|^p d\lambda(v),$$

where F is the σ-algebra of measurable sets and λ is the measure. When $p = 1$, the space $L^1(V)$ consists of all integrable functions β on V and we define the L^1-norm of β by

$$\|\beta\|_1 = \int_V |\beta(v)| d\lambda(v).$$

In this section, using Theorem 3.2, we show the existence of a solution of the following differential equation:

$$\begin{cases} w'(t) = H(t, w(t)), & \text{a.e. } t \in I := [0,T]; \\ w(0) = b, & b \geq 1, \end{cases} \tag{4.1}$$

where $H : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function satisfying the following conditions:

(i) $H(s,y) \geq 0$ for all $y \geq 0$ and $s \in I$;

(ii) for each $l, m \in L^1(I)$ with $l(s)m(s) \geq l(s)$ or $l(s)m(s) \geq m(s)$ for all $s \in I$, there exist $\beta \in L^1(I)$ and $\tau > 0$ such that

$$|H(s,l(s)) - H(s,m(s))| \leq \frac{\beta(s)}{(1+\tau\sqrt{d(l,m)})^2}|l(s) - m(s)| \tag{4.2}$$

and

$$|l(s) - m(s)| \leq \beta(s)e^{B(s)},$$

for all $s \in I$, where $B(s) := \int_0^s |\beta(r)|dr$.

Theorem 4.1. *If* (i) *and* (ii) *are satisfied, then the differential Eq.* (4.1) *has a unique positive solution.*

Proof. Let $V = \{w \in C(I,\mathbb{R}) : w(t) > 0 \text{ for all } t \in I\}$. Define the orthogonality relation $\perp$ on V by $l \perp m$ if and only if $l(t)m(t) \geq l(t)$ *or* $l(t)m(t) \geq m(t)$ for all $t \in I$. Since $B(t) = \int_0^t |\beta(s)|ds$, we have $B'(t) = |\beta(t)|$ for almost everywhere $t \in I$. Define a mapping $d : V \times V \to [0,\infty)$ by

$$d(l,m) = \|l-m\|_B = \sup_{t\in I} e^{-B(t)}|l(t)-m(t)|$$

for all $l, m \in V$. Then, (V,d) is a complete metric space (see, [3] for details).

Define a mapping $Q : V \to V$ by

$$(Ql)(t) = b + \int_0^t H(s,l(s))ds.$$

Now, we show that Q is $\perp$-preserving. For each $l, m \in V$ with $l \perp m$ and $t \in I$, we have

$$(Ql)(t) = b + \int_0^t H(s,l(s))ds \geq 1.$$

It follows that $[(Ql)(t)][(Qm)(t)] \geq (Qm)(t)$ and so $(Ql)(t) \perp (Qm)(t)$. Then, Q is $\perp$-preserving.

Next, we claim that Q is an orthogonal generalized F-contraction. Let $l, m \in V$ with $l \perp m$. Suppose that $Q(l) \neq Q(m)$. Then, for each $s \in I$, we have

$$|l(s) - m(s)| \leq \beta(s)e^{B(s)}.$$

From (ii), for each $t \in I$, we obtain

$$\begin{aligned}
|(Ql)(t) - (Qm)(t)| &\leq \int_0^t |H(s,l(s)) - H(s,m(s))|ds \\
&\leq \int_0^t \frac{|\beta(s)|}{(1+\tau\sqrt{d(l,m)})^2}|l(s)-m(s)|ds \\
&\leq \int_0^t \frac{|\beta(s)|}{(1+\tau\sqrt{d(l,m)})^2}|l(s)-m(s)|e^{-B(s)}e^{B(s)}ds \\
&\leq \frac{d(l,m)}{(1+\tau\sqrt{d(l,m)})^2}\int_0^t |\beta(s)|e^{B(s)}ds \\
&\leq \frac{d(l,m)}{(1+\tau\sqrt{d(l,m)})^2}\left(e^{B(t)} - 1\right)
\end{aligned}$$

and so

$$
\begin{aligned}
e^{-B(t)}|(Ql)(t)-(Qm)(t)| &\le e^{-B(t)}\left(e^{B(t)}-1\right)\frac{d(l,m)}{(1+\tau\sqrt{d(l,m)})^2}\\
&=\left(1-e^{-B(t)}\right)\frac{d(l,m)}{(1+\tau\sqrt{d(l,m)})^2}\\
&\le\left(1-e^{-\|\beta\|_1}\right)\frac{d(l,m)}{(1+\tau\sqrt{d(l,m)})^2}.
\end{aligned}
$$

Thus, it follows that

$$d(Ql,Qm)\le\frac{d(l,m)}{(1+\tau\sqrt{d(l,m)})^2},$$

it implies that

$$\frac{(1+\tau\sqrt{d(l,m)})^2}{d(l,m)}\le\frac{1}{d(Ql,Qm)},$$

and so, we have

$$\left(\tau+\frac{1}{\sqrt{d(l,m)}}\right)^2\le\frac{1}{d(Ql,Qm)}.$$

Hence, we have

$$\tau+\frac{1}{\sqrt{d(l,m)}}\le\frac{1}{\sqrt{d(Ql,Qm)}},$$

this means that

$$\tau-\frac{1}{\sqrt{d(Ql,Qm)}}\le-\frac{1}{\sqrt{d(l,m)}}.$$

Taking a function $F:\mathbb{R}^+\to\mathbb{R}$ by $F(c)=-\frac{1}{\sqrt{c}}$ for all $c>0$, it follows that Q is an orthogonal generalized F-contraction.

Finally, we claim that Q is $\perp$-continuous. Let $\{l_n\}\subseteq V$ be an O-sequence converging to a point $l\in V$. Applying condition (ii), we get

$$
\begin{aligned}
e^{-B(t)}|(Ql_n)(t)-(Ql)(t)| &\le e^{-B(t)}\int_0^t|H(s,l_n(s))-H(s,l(s))|ds\\
&\le\left(1-e^{-\|\beta\|_1}\right)\frac{d(l_n,l)}{(1+\tau\sqrt{d(l_n,l)})^2}
\end{aligned}
$$

for all $n\in\mathbb{N}$ and $t\in I$. Hence we have

$$||Ql_n-Ql||_B\le\left(1-e^{-\|\beta\|_1}\right)\frac{||l_n-l||_B}{(1+\tau\sqrt{||l_n-l||_B})^2}$$

for all $n\in\mathbb{N}$. Since $1-e^{-\|\beta\|_1}<1$, Q is $\perp$-continuous. By Theorem 3.2, Q has a unique fixed point and hence the differential Eq. (4.1) has a unique positive solution. This comples the proof. □

5. Conclusion

In this paper, we proved fixed point theorems for orthogonal generalized F-contraction mappings on O- complete metric spaces.

References

[1] K. Afassinou and O. K. Narain, *Existence of solutions for boundary value problems via F-contraction mappings in metric spaces*, Nonlinear Funct. Anal. Appl., **25**(2) (2020), 303-319, https://doi.org/10.22771/nfaa.2020.25.02.07

[2] M. Eshaghi and H. Habibi, *Fixed point theory in ε-connected orthogonal metric space*, Shand Commu. Math. Anal., **16**(1) (2019), 35–46.

[3] M.E. Gordji, M. Ramezani, M. De La Sen and Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory Appl.,, **18**(2) (2017), 569–578.

[4] M.E. Gordji and H. Habibi, *Fixed point theory in generalized orthogonal metric space*, J. Linear and Topo. Algebra, **6**(3) (2017), 251–260.

[5] N.B. Gungor and D. Turkoglu, *Fixed point theorems on orthogonal metric spaces via altering distance functions*, AIP Conference Proceedings, **2183**, 040011 (2019).

[6] D. Kitkuan and J. Janwised, *α-admissible Prešić type F-contraction*, Nonlinear Funct. Anal. Appl., **25**(2) (2020), 345-354, https://doi.org/10.22771/nfaa.2020.25.02.10

[7] K. Sawangsup, W. Sintunavarat and Y.J. Cho, *Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces*, J. Fixed Point Theory Appl., **22:10** (2020).

[8] K. Sawangsup and W. Sintunavarat, *Fixed point results for orthogonal Z-contraction mappings in O-complete metric spaces*, Int. J. Appl. Physics Math., **10**(1) (2020), 33-40.

[9] T. Senapati, L.K. Dey, B. Damjanović and A. Chanda, *New fixed results in orthogonal metric spaces with an application*, Kragujevac J. Math., **42**(4) (2018), 505–516.

[10] D. Wardowski, *Fixed points of new type of contractive mappings in complete metric space* Fixed Point Theory Appl., **2012**:94 (2012).

[11] D. Wardowski and N. Van Dung, *Fixed points of F-weak contractions on complete metric space*, Demonstratio Math., **47** (2014), 146–155.

[12] O. Yamaod and W. Sintunavarat, *On new orthogonal contractions in b-metric spaces*, Int. J. Pure Math., **5** (2018), 37-40.

[13] M. Younis, D. Singh, D. Gopal, A. Goyal and M. S. Rathore, *On applications of generalized F-contraction to differential equations*, Nonlinear Funct. Anal. Appl., **24**(1) (2019), 155-174, https://doi.org/10.22771/nfaa.2019.24.01.10.