PARAMETRIC GENERALIZED MULTI-VALUED NONLINEAR QUASI-VARIATIONAL INCLUSION PROBLEM

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Abstract. In this paper, we investigate the behavior and sensitivity analysis of a solution set for a parametric generalized multi-valued nonlinear quasi-variational inclusion problem in a real Hilbert space. For this study, we utilize the technique of resolvent operator and the property of a fixed-point set of a multi-valued contractive mapping. We also examine Lipschitz continuity of the solution set with respect to the parameter under some appropriate conditions.

1. Introduction

In last 30 years, more attention has been given to grow techniques for the sensitivity analysis of solutions for various classes of variational inequalities. The quasi-variational inclusion is a useful and important generalization of a variational inequality. In view of mathematical and engineering problems,

sensitivity properties of different classes of variational inequalities can offer new insight concerning the problem being studied and can stimulate ideas for solving problems, see for the details and applications [1,4,5,10,15-21,28-30]. The sensitivity analysis of solutions for variational inequalities have been studied broadly by many researchers using different methods. By utilizing the projection technique, Dafermos [5], Ding and Luo [8], Mukherjee and Verma [22], Park and Jeong [26] and Yen [30] dealt with the sensitivity analysis for some classes of variational inequalities for single-valued mappings. By using the resolvent operator technique, Adly [1], Agarwal *et al.* [2], Ding [6], Fang and Huang [9], Hassouni and Moudafi [10] and Noor [24] studied the sensitivity analysis of solutions for some classes of parametric variational inclusions for single-valued mappings.

Lately, by using techniques of projection and resolvent operator, Agarwal *et al.* [3], Ding [7], Huang [11], Kazmi and Khan [12,14], Liu *et al.* [19], Noor [25], Peng and Long [27] and Ram [28] examined the behavior and sensitivity of solutions for some important classes of parametric generalized variational inclusions for single and multi-valued mappings.

Motivated by the recent research in this direction, here, we consider a parametric generalized multi-valued nonlinear quasi-variational inclusion problem (in short, PGMNQVIP) involving maximal monotone mappings in a real Hilbert space. Further, using the technique of resolvent operator and the property of a fixed point set of a multi-valued contractive mapping, we examine the behavior and sensitivity analysis of a solution set for PGMNQVIP. Furthermore, under some suitable conditions, Lipschitz continuity of the solution set with respect to the parameter is proved. The technique utilized in this paper can be used to extend and advance the theorems given by many researchers (see [1-9,12-14,21,23-27]).

2. Preliminaries

Suppose that H is a real Hilbert space with a norm $\| \cdot \|$ and an inner product $\langle \cdot, \cdot \rangle$. $C(H)$ denotes the collection of all nonempty compact subsets of H and 2^H denotes the power set of H. The Pompeiu-Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ on $C(H)$ is defined by

$$\mathcal{H}(A, B) \ = \ \max \left\{ \sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A) \right\}, \ A, B \in C(H).$$

First, we review the following concepts and known results.

Definition 2.1. ([9,11-14]) A self-mapping T on H is said to be

(i) monotone, if

$$\langle T(x) - T(y), x - y \rangle \ \geq \ 0, \text{ for all } x, y \in H;$$

(ii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y),\, x - y \rangle \geq \alpha \|x - y\|^2, \text{ for all } x, y \in H;$$

(iii) β-Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta \|x - y\|, \text{ for all } x, y \in H.$$

Definition 2.2. ([9,11-14]) A multi-valued mapping $M : H \to 2^H$ is said to be

(i) monotone, if

$$\langle u - v,\, x - y \rangle \geq 0, \text{ for all } x, y \in H,\ u \in M(x),\ v \in M(y);$$

(ii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle u - v,\, x - y \rangle \geq \alpha \|x - y\|^2, \text{ for all } x, y \in H,\ u \in M(x),\ v \in M(y);$$

(iii) maximal monotone, if M is monotone and $(I + \rho M)(H) = H$ for any $\rho > 0$, where I is the identity mapping on H.

Definition 2.3. ([28]) Let $W : H \to 2^H$ be a maximal monotone mapping. For any fixed $\rho > 0$, the mapping $J_\rho^W : H \to H$, defined by

$$J_\rho^W(x) = (I + \rho W)^{-1}(x), \text{ for all } x \in H,$$

is said to be the resolvent operator of W.

Lemma 2.4. ([28]) *If $W : H \to 2^H$ is a maximal monotone mapping, then the resolvent operator $J_\rho^W : H \to H$ of W is nonexpansive, that is,*

$$\|J_\rho^W(x) - J_\rho^W(y)\| \leq \|x - y\|, \text{ for all } x, y \in H.$$

Lemma 2.5. ([23]) *Let (X, d) be a complete metric space. Suppose that $T : X \to C(X)$ satisfies*

$$\mathcal{H}(T(x),\, T(y)) \leq \theta\, d(x,\, y), \text{ for all } x, y \in X, \tag{2.1}$$

where $\theta \in (0, 1)$ is a constant. Then the mapping T has fixed point in X.

A mapping $T : X \to C(X)$ is said to be θ-$\mathcal{H}$-contraction mapping if it satisfies above inequality (2.1).

Lemma 2.6. ([18]) *Let $T_1, T_2 : X \to C(X)$ be θ-$\mathcal{H}$-contraction mappings on a complete metric space (X, d). Then*

$$\mathcal{H}(F(T_1),\, F(T_2)) \leq (1 - \theta)^{-1} \sup_{x \in X} \mathcal{H}(T_1(x),\, T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are the sets of fixed points of T_1 and T_2, respectively.

3. Formulation of problem

Let Ω be a nonempty open subset of H. Let $N, M : H \times H \times H \times \Omega \to H$ and $m, f : H \times \Omega \to H$ be single-valued mappings, and let $A, B, C, P, Q, R, T, D, G, S : H \times \Omega \to C(H)$ be multi-valued mappings. Suppose that $W : H \times H \times \Omega \to 2^H$ is a multi-valued mapping such that for each given $(z, \lambda) \in H \times \Omega$, $W(\cdot, z, \lambda) : H \to 2^H$ is a maximal monotone mapping with

$$(S(H, \lambda) - m(H, \lambda)) \cap \operatorname{dom} W(\cdot, z, \lambda) \neq \emptyset.$$

In the entire paper, unless otherwise stated, we will study the following parametric generalized multi-valued nonlinear quasi-variational inclusion problem (PGMNQVIP):

For each fixed parameter $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $p(\lambda) \in P(x(\lambda), \lambda)$, $q(\lambda) \in Q(x(\lambda), \lambda)$, $r(\lambda) \in R(x(\lambda), \lambda)$, $t(\lambda) \in T(x(\lambda), \lambda)$, $n(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$ and $s(\lambda) \in S(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), w(\lambda), \lambda) - M(p(\lambda), q(\lambda), r(\lambda), \lambda)$$
$$+ f(t(\lambda), \lambda) + W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda). \tag{3.1}$$

Some special cases:

(i) If $M(p(\lambda), q(\lambda), r(\lambda), \lambda) \equiv 0$, then PGMNQVIP (3.1) reduces to the following parametric quasi-variational inclusion problem: for each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $t(\lambda) \in T(x(\lambda), \lambda)$, $n(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $s(\lambda) \in S(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), w(\lambda), \lambda) + f(t(\lambda), \lambda)$$
$$+ W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda), \tag{3.2}$$

similar type problem has been studied by Ram [28].

(ii) If $N(u(\lambda), v(\lambda), w(\lambda), \lambda) \equiv N(u(\lambda), v(\lambda), \lambda)$ and $f(t(\lambda), \lambda) \equiv 0$, then problem (3.2) reduces to the following parametric quasi-variational inclusion problem: for each fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $n(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $s(\lambda) \in S(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), \lambda) + W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda), \tag{3.3}$$

which has been introduced and studied by Ding [6].

(iii) If $S \equiv g : H \times \Omega \to H$ is a single-valued mapping; $D(x, \lambda) \equiv x$ and $m(x, \lambda) \equiv 0$, for all $(x, \lambda) \in H \times \Omega$, then problem (3.3) reduces to the following parametric quasi-variational inclusion problem: for each

fixed $\lambda \in \Omega$, find $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$ such that

$$0 \in N(u(\lambda), v(\lambda), \lambda) + W(g(x(\lambda), \lambda), z(\lambda), \lambda), \tag{3.4}$$

which has been introduced and studied by Noor [24,25].

For a suitable choice of the mappings $A, B, C, P, Q, R, T, D, G, S, W$, m, f and the space H, it is easy to check that PGMNQVIP (3.1) contains a number of known classes of parametric variational inclusions (inequalities) studied by many researchers as special cases (see [1-3,5-8,11-14,21,23-27]).

Now, for each fixed $\lambda \in \Omega$, the solution set $S(\lambda)$ of PGMNQVIP (3.1) is denoted as

$$S(\lambda) = \Big\{ x(\lambda) \in H : \exists\, u(\lambda) \in A(x(\lambda), \lambda), v(\lambda) \in B(x(\lambda), \lambda), w(\lambda) \in C(x(\lambda), \lambda),$$

$$p(\lambda) \in P(x(\lambda), \lambda), q(\lambda) \in Q(x(\lambda), \lambda), r(\lambda) \in R(x(\lambda), \lambda),$$

$$t(\lambda) \in T(x(\lambda), \lambda), n(\lambda) \in D(x(\lambda), \lambda), z(\lambda) \in G(x(\lambda), \lambda),$$

$$s(\lambda) \in S(x(\lambda), \lambda) \text{ such that}$$

$$0 \in N(u(\lambda), v(\lambda), w(\lambda), \lambda) - M(p(\lambda), q(\lambda), r(\lambda), \lambda) + f(t(\lambda), \lambda)$$

$$+ W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda) \Big\}. \tag{3.5}$$

In this paper, our main aim is to study the behavior and sensitivity analysis of the solution set $S(\lambda)$, and the conditions on mappings $A, B, C, P, Q, R, T,$ D, G, S, W, m, f, under which the solution set $S(\lambda)$ of PGMNQVIP (3.1) is nonempty and Lipschitz continuous with respect to the parameter $\lambda \in \Omega$.

4. SENSITIVITY ANALYSIS OF SOLUTION SET $S(\lambda)$

First, we recall the following useful definitions.

Definition 4.1. ([6,12-14]) A multi-valued mapping $S : H \times \Omega \to C(H)$ is said to be

(i) δ-strongly monotone, if there exists a constant $\delta > 0$ such that

$$\langle s_1 - s_2, x - y \rangle \geq \delta \|x - y\|^2,$$

for all $(x, y, \lambda) \in H \times H \times \Omega$, $s_1 \in S(x, \lambda)$, $s_2 \in S(y, \lambda)$;

(ii) L_S-$\mathcal{H}$-Lipschitz continuous, if there exists a constant $L_S > 0$ such that

$$\mathcal{H}(S(x, \lambda), S(y, \lambda)) \leq L_S \|x - y\|,$$

for all $(x, y, \lambda) \in H \times H \times \Omega$.

Definition 4.2. ([12-14]) A multi-valued mapping $A : H \times \Omega \to C(H)$ is said to be (L_A, l_A)-$\mathcal{H}$-mixed Lipschitz continuous, if there exist constants $L_A, l_A > 0$ such that

$$\mathcal{H}(A(x_1, \lambda_1), A(x_2, \lambda_2)) \leq L_A \|x_1 - x_2\| + l_A \|\lambda_1 - \lambda_2\|,$$

for all $(x_1, \lambda_1), (x_2, \lambda_2) \in H \times \Omega$.

Definition 4.3. Let $A, B, C : H \times \Omega \to C(H)$ be multi-valued mappings. A single-valued mapping $N : H \times H \times H \times \Omega \to H$ is said to be

(i) α-strongly mixed monotone with respect to A, B and C, if there exists a constant $\alpha > 0$ such that

$$\langle N(u_1, v_1, w_1, \lambda) - N(u_2, v_2, w_2, \lambda), x - y \rangle \geq \alpha \|x - y\|^2,$$

for all $(x, y, \lambda) \in H \times H \times \Omega$, $u_1 \in A(x, \lambda)$, $u_2 \in A(y, \lambda)$, $v_1 \in B(x, \lambda)$, $v_2 \in B(y, \lambda)$, $w_1 \in C(x, \lambda)$, $w_2 \in C(y, \lambda)$;

(ii) σ-generalized mixed pseudocontractive with respect to A, B and C, if there exists a constant $\sigma > 0$ such that

$$\langle N(u_1, v_1, w_1, \lambda) - N(u_2, v_2, w_2, \lambda), x - y \rangle \leq \sigma \|x - y\|^2,$$

for all $(x, y, \lambda) \in H \times H \times \Omega$, $u_1 \in A(x, \lambda)$, $u_2 \in A(y, \lambda)$, $v_1 \in B(x, \lambda)$, $v_2 \in B(y, \lambda)$, $w_1 \in C(x, \lambda)$, $w_2 \in C(y, \lambda)$;

(iii) $(L_{(N,1)}, L_{(N,2)}, L_{(N,3)}, l_N)$-mixed Lipschitz continuous, if there exist constants $L_{(N,1)}, L_{(N,2)}, L_{(N,3)}, l_N > 0$ such that

$$\begin{aligned}
\|N(x_1, y_1, z_1, \lambda_1) - N(x_2, y_2, z_2, \lambda_2)\| \leq\ & L_{(N,1)} \|x_1 - x_2\| \\
& + L_{(N,2)} \|y_1 - y_2\| \\
& + L_{(N,3)} \|z_1 - z_2\| \\
& + l_N \|\lambda_1 - \lambda_2\|,
\end{aligned}$$

for all $(x_1, y_1, z_1, \lambda_1)$, $(x_2, y_2, z_2, \lambda_2) \in H \times H \times H \times \Omega$.

Now, we transfer the PGMNQVIP (3.1) into a parametric fixed point problem.

Theorem 4.4. *For each fixed $\lambda \in \Omega$, $x(\lambda) \in S(\lambda)$ is a solution of PGMNQVIP (3.1) if and only if there exist $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $p(\lambda) \in P(x(\lambda), \lambda)$, $q(\lambda) \in Q(x(\lambda), \lambda)$, $r(\lambda) \in R(x(\lambda), \lambda)$, $t(\lambda) \in T(x(\lambda), \lambda)$, $n(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$, $s(\lambda) \in S(x(\lambda), \lambda)$ such that the following relation holds:*

$$s(\lambda) = m(n(\lambda), \lambda) + J_\rho^{W(\cdot, z(\lambda), \lambda)}(s(\lambda) - m(n(\lambda), \lambda) - \rho N(u(\lambda), v(\lambda), w(\lambda), \lambda)$$

$$+ \rho M(p(\lambda), q(\lambda), r(\lambda), \lambda) + f(t(\lambda), \lambda)), \qquad (4.1)$$

where $\rho > 0$ is a constant.

Proof. For each fixed $\lambda \in \Omega$, by the definition of the resolvent operator $J_\rho^{W(\cdot,z(\lambda),\lambda)}$ of $W(\cdot, z(\lambda), \lambda)$, there exist $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $p(\lambda) \in P(x(\lambda), \lambda)$, $q(\lambda) \in Q(x(\lambda), \lambda)$, $r(\lambda) \in R(x(\lambda), \lambda)$, $t(\lambda) \in T(x(\lambda), \lambda)$, $n(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$ and $s(\lambda) \in S(x(\lambda), \lambda)$ such that (4.1) holds if and only if

$$s(\lambda) - m(n(\lambda), \lambda) - \rho N(u(\lambda), v(\lambda), w(\lambda), \lambda) + \rho M(p(\lambda), q(\lambda), r(\lambda), \lambda) + f(t(\lambda), \lambda)$$

$$\in s(\lambda) - m(n(\lambda), \lambda) + \rho W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda). \quad (4.2)$$

The above inclusion holds if and only if

$$0 \in N(u(\lambda), v(\lambda), w(\lambda), \lambda) - M(p(\lambda), q(\lambda), r(\lambda), \lambda)$$

$$+ f(t(\lambda), \lambda) + W(s(\lambda) - m(n(\lambda), \lambda), z(\lambda), \lambda).$$

By the definition of $S(\lambda)$, we obtain that $x(\lambda) \in S(\lambda)$ is a solution of PGM-NQVIP (3.1) if and only if there exist $x(\lambda) \in H$, $u(\lambda) \in A(x(\lambda), \lambda)$, $v(\lambda) \in B(x(\lambda), \lambda)$, $w(\lambda) \in C(x(\lambda), \lambda)$, $p(\lambda) \in P(x(\lambda), \lambda)$, $q(\lambda) \in Q(x(\lambda), \lambda)$, $r(\lambda) \in R(x(\lambda), \lambda)$, $t(\lambda) \in T(x(\lambda), \lambda)$, $n(\lambda) \in D(x(\lambda), \lambda)$, $z(\lambda) \in G(x(\lambda), \lambda)$ and $s(\lambda) \in S(x(\lambda), \lambda)$ such that (4.1) holds. $\square$

Remark 4.5. Theorem 4.4 is a generalized variant of Lemma 3.1 of Adly [1], Lemma 2.1 of Agarwal *et al.* [3], Theorem 3.1 of Ding [6], Lemma 3.1 of Ding and Luo [8], Lemma 2.1 of Peng and Long [27], and Theorem 3.1 of Ram [28].

Now, we prove that the following theorem which ensures that the solution set $S(\lambda)$ of PGMNQVIP (3.1) is nonempty and closed for each $\lambda \in \Omega$.

Theorem 4.6. *Let $A, B, C, P, Q, R, T, D, G, S : H \times \Omega \to C(H)$ be multivalued mappings such that $A, B, C, P, Q, R, T, D, G$ and S are $\mathcal{H}$-Lipschitz continuous in the first argument with constants $L_A, L_B, L_C, L_P, L_Q, L_R, L_T, L_D, L_G$ and L_S, respectively, and let $S : H \times \Omega \to C(H)$ be δ-strongly monotone. Let $m : H \times \Omega \to H$ be (L_m, l_m)-mixed Lipschitz continuous and $f : H \times \Omega \to H$ be (L_f, l_f)-mixed Lipschitz continuous. Let $N : H \times H \times H \times \Omega \to H$ be α-strongly mixed monotone with respect to A, B and C, and $(L_{(N,1)}, L_{(N,2)}, L_{(N,3)})$-mixed Lipschitz continuous and let $M : H \times H \times H \times \Omega \to H$ be σ-generalized mixed pseudocontractive with respect to P, Q and R, and $(L_{(M,1)}, L_{(M,2)}, L_{(M,3)})$-mixed Lipschitz continuous. Suppose that the multi-valued mapping $W : H \times H \times \Omega \to 2^H$ is such that for each fixed $(z, \lambda) \in H \times \Omega$, $W(\cdot, z, \lambda) : H \to 2^H$ is a maximal monotone mapping satisfying*

$$(S(H, \lambda) - m(H, \lambda)) \cap \operatorname{dom} W(\cdot, z, \lambda) \neq \emptyset.$$

Suppose that there exist constants $k_1, k_2 > 0$ such that

$$\|J_\rho^{W(\cdot,x_1,\lambda_1)}(t) - J_\rho^{W(\cdot,x_2,\lambda_2)}(t)\| \leq k_1 \|x_1 - x_2\| + k_2 \|\lambda_1 - \lambda_2\|, \quad (4.3)$$

for all $x_1, x_2, t \in H$; $\lambda_1, \lambda_2 \in \Omega$, and suppose that there is a constant $\rho > 0$ such that

$$\theta = k + t(\rho) < 1, \tag{4.4}$$

where

$$k := 2\sqrt{1 - 2\delta + L_S^2} + 2L_m L_D + L_f L_T + k_1 L_G ;$$

$$t(\rho) := \sqrt{1 - 2\rho(\alpha - \sigma) + 2\rho^2(L_N^2 + L_M^2)};$$

$$L_N := (L_A L_{(N,1)} + L_B L_{(N,2)} + L_C L_{(N,3)})$$

and

$$L_M := (L_P L_{(M,1)} + L_Q L_{(M,2)} + L_R L_{(M,3)}).$$

Then, for each $\lambda \in \Omega$, the solution set $S(\lambda)$ of PGMNQVIP (3.1) is nonempty and closed.

Proof. For all $(x, \lambda) \in H \times \Omega$, define a multi-valued mapping $F : H \times \Omega \to 2^H$ by

$$F(x, \lambda) = \bigcup_K G, \tag{4.5}$$

where

$$G := \Big[x - s + m(n, \lambda) + J_\rho^{W(\cdot, z, \lambda)}(s - m(n, \lambda) - \rho N(u, v, w, \lambda)$$

$$+\rho M(p, q, r, \lambda) + f(t, \lambda))\Big]$$

and

$$K := \{u \in A(x, \lambda), v \in B(x, \lambda), w \in C(x, \lambda), p \in P(x, \lambda), q \in Q(x, \lambda),$$

$$r \in R(x, \lambda), t \in T(x, \lambda), n \in D(x, \lambda), z \in G(x, \lambda), s \in S(x, \lambda)\}.$$

For any $(x, \lambda) \in H \times \Omega$, since $A(x, \lambda)$, $B(x, \lambda)$, $C(x, \lambda)$, $P(x, \lambda)$, $Q(x, \lambda)$, $R(x, \lambda)$, $T(x, \lambda)$, $D(x, \lambda)$, $G(x, \lambda)$, $S(x, \lambda) \in C(H)$, and $m, f, J_\rho^{W(\cdot, z, \lambda)}$ are continuous, we know that $F(x, \lambda) \in C(H)$.

Now for each fixed $\lambda \in \Omega$, we prove that $F(x, \lambda)$ is a multi-valued contractive mapping. For any $(x, y, \lambda) \in H \times H \times \Omega$ and any $a \in F(x, \lambda)$, there exist $u_1 \in A(x, \lambda)$, $v_1 \in B(x, \lambda)$, $w_1 \in C(x, \lambda)$, $p_1 \in P(x, \lambda)$, $q_1 \in Q(x, \lambda)$, $r_1 \in R(x, \lambda)$, $t_1 \in T(x, \lambda)$, $n_1 \in D(x, \lambda)$, $z_1 \in G(x, \lambda)$ and $s_1 \in S(x, \lambda)$ such that

$$a = x - s_1 + m(n_1, \lambda) + J_\rho^{W(\cdot, z_1, \lambda)}(s_1 - m(n_1, \lambda) - \rho N(u_1, v_1, w_1, \lambda)$$

$$+\rho M(p_1, q_1, r_1, \lambda) + f(t_1, \lambda)). \tag{4.6}$$

Since $A(y, \lambda)$, $B(y, \lambda)$, $C(y, \lambda)$, $P(y, \lambda)$, $Q(y, \lambda)$, $R(y, \lambda)$, $T(y, \lambda)$, $D(y, \lambda)$, $G(y, \lambda)$, $S(y, \lambda) \in C(H)$, there exist $u_2 \in A(y, \lambda)$, $v_2 \in B(y, \lambda)$, $w_2 \in C(y, \lambda)$,

$p_2 \in P(y, \lambda)$, $q_2 \in Q(y, \lambda)$, $r_2 \in R(y, \lambda)$, $t_2 \in T(y, \lambda)$, $n_2 \in D(y, \lambda)$, $z_2 \in G(y, \lambda)$ and $s_2 \in S(y, \lambda)$ such that

$$\|u_1 - u_2\| \leq \mathcal{H}(A(x, \lambda), A(y, \lambda)) \leq L_A \|x - y\|,$$

$$\|v_1 - v_2\| \leq \mathcal{H}(B(x, \lambda), B(y, \lambda)) \leq L_B \|x - y\|,$$

$$\|w_1 - w_2\| \leq \mathcal{H}(C(x, \lambda), C(y, \lambda)) \leq L_C \|x - y\|,$$

$$\|p_1 - p_2\| \leq \mathcal{H}(P(x, \lambda), P(y, \lambda)) \leq L_P \|x - y\|,$$

$$\|q_1 - q_2\| \leq \mathcal{H}(Q(x, \lambda), Q(y, \lambda)) \leq L_Q \|x - y\|,$$

$$\|r_1 - r_2\| \leq \mathcal{H}(R(x, \lambda), R(y, \lambda)) \leq L_R \|x - y\|,$$

$$\|t_1 - t_2\| \leq \mathcal{H}(T(x, \lambda), T(y, \lambda)) \leq L_T \|x - y\|,$$

$$\|n_1 - n_2\| \leq \mathcal{H}(D(x, \lambda), D(y, \lambda)) \leq L_D \|x - y\|,$$

$$\|z_1 - z_2\| \leq \mathcal{H}(G(x, \lambda), G(y, \lambda)) \leq L_G \|x - y\|,$$

$$\|s_1 - s_2\| \leq \mathcal{H}(S(x, \lambda), S(y, \lambda)) \leq L_S \|x - y\|. \qquad (4.7)$$

Let

$$b = y - s_2 + m(n_2, \lambda) + J_\rho^{W(\cdot, z_2, \lambda)}(s_2 - m(n_2, \lambda) - \rho N(u_2, v_2, w_2, \lambda)$$

$$+ \rho M(p_2, q_2, r_2, \lambda) + f(t_2, \lambda)). \qquad (4.8)$$

Then we have $b \in F(y, \lambda)$. It follows that

$$\begin{aligned}
\|a - b\| \leq\ & \|x - y - (s_1 - s_2)\| + \|m(n_1, \lambda) - m(n_2, \lambda)\| \\
& + \|J_\rho^{W(\cdot, z_1, \lambda)}(s_1 - m(n_1, \lambda) - \rho N(u_1, v_1, w_1, \lambda) \\
& + \rho M(p_1, q_1, r_1, \lambda) + f(t_1, \lambda)) - J_\rho^{W(\cdot, z_2, \lambda)}(s_2 - m(n_2, \lambda) \\
& - \rho N(u_2, v_2, w_2, \lambda) + \rho M(p_2, q_2, r_2, \lambda) + f(t_2, \lambda))\| \\
\leq\ & \|x - y - (s_1 - s_2)\| + \|m(n_1, \lambda) - m(n_2, \lambda)\| \\
& + \|J_\rho^{W(\cdot, z_1, \lambda)}(s_1 - m(n_1, \lambda) - \rho N(u_1, v_1, w_1, \lambda) \\
& + \rho M(p_1, q_1, r_1, \lambda) + f(t_1, \lambda)) \\
& - [J_\rho^{W(\cdot, z_1, \lambda)}(s_2 - m(n_2, \lambda) - \rho N(u_2, v_2, w_2, \lambda) \\
& + \rho M(p_2, q_2, r_2, \lambda) + f(t_2, \lambda))]\| \\
& + \|J_\rho^{W(\cdot, z_1, \lambda)}(s_2 - m(n_2, \lambda) - \rho N(u_2, v_2, w_2, \lambda) \\
& + \rho M(p_2, q_2, r_2, \lambda) + f(t_2, \lambda)) \\
& - [J_\rho^{W(\cdot, z_2, \lambda)}(s_2 - m(n_2, \lambda) - \rho N(u_2, v_2, w_2, \lambda) \\
& + \rho M(p_2, q_2, r_2, \lambda) + f(t_2, \lambda))]\|
\end{aligned}$$

$$\leq \;\|x - y - (s_1 - s_2)\| + \|m(n_1, \lambda) - m(n_2, \lambda)\|$$
$$+ \|s_1 - m(n_1, \lambda) - \rho N(u_1, v_1, w_1, \lambda) + \rho M(p_1, q_1, r_1, \lambda) + f(t_1, \lambda)$$
$$- [s_2 - m(n_2, \lambda) - \rho N(u_2, v_2, w_2, \lambda) + \rho M(p_2, q_2, r_2, \lambda) + f(t_2, \lambda)]\|$$
$$+ k_1 \|z_1 - z_2\|$$
$$\leq 2\|x - y - (s_1 - s_2)\| + 2\|m(n_1, \lambda) - m(n_2, \lambda)\| + k_1 \|z_1 - z_2\|$$
$$+ \|f(t_1, \lambda) - f(t_2, \lambda)\| + \|x - y - \rho(N(u_1, v_1, w_1, \lambda)$$
$$- N(u_2, v_2, w_2, \lambda) - M(p_1, q_1, r_1, \lambda) + M(p_2, q_2, r_2, \lambda))\|. \tag{4.9}$$

Since N is α-strongly mixed monotone and mixed Lipschitz continuous, M is σ-generalized mixed pseudocontractive and mixed Lipschitz continuous. Also, A, B, C, P, Q, R are $\mathcal{H}$-Lipschitz continuous, then by using $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$, we have

$$\|x - y - \rho(N(u_1, v_1, w_1, \lambda) - N(u_2, v_2, w_2, \lambda) - M(p_1, q_1, r_1, \lambda) + M(p_2, q_2, r_2, \lambda))\|^2$$
$$\leq \|x - y\|^2 - 2\rho \Big[\langle N(u_1, v_1, w_1, \lambda) - N(u_2, v_2, w_2, \lambda)$$
$$- M(p_1, q_1, r_1, \lambda) + M(p_2, q_2, r_2, \lambda), x - y \rangle \Big]$$
$$+ 2\rho^2 \Big[\|N(u_1, v_1, w_1, \lambda) - N(u_2, v_2, w_2, \lambda)\|^2$$
$$+ \|M(p_1, q_1, r_1, \lambda) - M(p_2, q_2, r_2, \lambda)\|^2 \Big]$$
$$\leq \|x - y\|^2 - 2\rho(\alpha - \sigma)\|x - y\|^2 + 2\rho^2 \Big[(L_A L_{(N,1)} + L_B L_{(N,2)} + L_C L_{(N,3)})^2$$
$$+ (L_P L_{(M,1)} + L_Q L_{(M,2)} + L_R L_{(M,3)})^2 \Big] \|x - y\|^2$$
$$\leq \Big(1 - 2\rho(\alpha - \sigma) + 2\rho^2 [(L_A L_{(N,1)} + L_B L_{(N,2)} + L_C L_{(N,3)})^2$$
$$+ (L_P L_{(M,1)} + L_Q L_{(M,2)} + L_R L_{(M,3)})^2] \Big) \|x - y\|^2. \tag{4.10}$$

Since S is δ-strongly monotone and L_S-$\mathcal{H}$-Lipschitz continuous, we have

$$\|x - y - (s_1 - s_2)\|^2 = \|x - y\|^2 - 2\langle x - y, s_1 - s_2 \rangle + \|s_1 - s_2\|^2$$
$$\leq \|x - y\|^2 - 2\delta\|x - y\|^2 + [\mathcal{H}(S(x, \lambda), S(y, \lambda))]^2$$
$$\leq \|x - y\|^2 - 2\delta\|x - y\|^2 + L_S^2 \|x - y\|^2$$

and hence,

$$\|x - y - (s_1 - s_2)\| \leq \sqrt{1 - 2\delta + L_S^2}\, \|x - y\|. \tag{4.11}$$

By the mixed Lipschitz continuity of m and the $\mathcal{H}$-Lipschitz continuity of D, we have

$$\|m(n_1, \lambda) - m(n_2, \lambda)\| \leq L_m \|n_1 - n_2\| \leq L_m \mathcal{H}(D(x, \lambda), D(y, \lambda))$$
$$\leq L_m L_D \|x - y\|. \tag{4.12}$$

By the $\mathcal{H}$-Lipschitz continuity of G, we have

$$\|z_1 - z_2\| \leq \mathcal{H}(G(x,\lambda), G(y,\lambda)) \leq L_G\|x-y\|. \qquad (4.13)$$

By the mixed Lipschitz continuity of f and the $\mathcal{H}$-Lipschitz continuity of T, we have

$$\|f(t_1,\lambda) - f(t_2,\lambda)\| \leq L_f\|t_1 - t_2\| \leq L_f\,\mathcal{H}(T(x,\lambda), T(y,\lambda)) \leq L_f L_T\|x-y\|. \qquad (4.14)$$

Combining (4.9)-(4.14), we obtain

$$\|a - b\| \leq \theta\,\|x-y\|, \qquad (4.15)$$

where $\theta := k + t(\rho)$; $k := 2\sqrt{1 - 2\delta + L_S^2} + 2L_m L_D + L_f L_T + k_1 L_G$;

$$t(\rho) := \sqrt{1 - 2\rho(\alpha - \sigma) + 2\rho^2(L_N^2 + L_M^2)} \ ;$$

$$L_N := (L_A L_{(N,1)} + L_B L_{(N,2)} + L_C L_{(N,3)})$$

and

$$L_M := (L_P L_{(M,1)} + L_Q L_{(M,2)} + L_R L_{(M,3)}).$$

It follows from condition (4.4) that $\theta < 1$. Hence, we have

$$d(a, F(y,\lambda)) = \inf_{b \in F(y,\lambda)} \|a - b\| \leq \theta\|x-y\|.$$

Since $a \in F(x,\lambda)$ is arbitrary, we obtain

$$\sup_{a \in F(x,\lambda)} d(a, F(y,\lambda)) \leq \theta\|x-y\|.$$

By using same argument, we can prove that

$$\sup_{b \in F(y,\lambda)} d(F(x,\lambda), b) \leq \theta\|x-y\|.$$

By the definition of the Pompeiu-Hausdorff metric $\mathcal{H}$ on $C(H)$, and for all $(x,y,\lambda) \in H \times H \times \Omega$, we obtain that

$$\mathcal{H}\Big(F(x,\lambda), F(y,\lambda)\Big) \leq \theta\|x-y\|, \qquad (4.16)$$

which shows that $F(x,\lambda)$ is a uniform θ-$\mathcal{H}$-contraction mapping with respect to $\lambda \in \Omega$. By Lemma 2.5, for each $\lambda \in \Omega$, $F(x,\lambda)$ has a fixed point $x(\lambda) \in H$, that is, $x(\lambda) \in F(x(\lambda),\lambda)$ and hence Theorem 4.4 ensures that $x(\lambda) \in S(\lambda)$ is a solution of PGMNQVIP (3.1) and so $S(\lambda) \neq \emptyset$.

Further, for each $\lambda \in \Omega$, let $\{x_n\} \subset S(\lambda)$ with $\lim_{n \to \infty} x_n = x_0$, we have $x_n \in F(x_n,\lambda)$ for all $n \geq 1$. By virtue of (4.16), we have

$$d(x_0, F(x_0,\lambda)) \leq \|x_0 - x_n\| + \mathcal{H}(F(x_n,\lambda), F(x_0,\lambda))$$

$$\leq (1+\theta)\|x_n - x_0\| \to 0 \text{ as } n \to \infty,$$

that is, $x_0 \in F(x_0,\lambda)$ and hence $x_0 \in S(\lambda)$. Thus $S(\lambda)$ is a closed set in H. $\quad\square$

Next, we show that the solution set $S(\lambda)$ of PGMNQVIP (3.1) is $\mathcal{H}$-Lipschitz continuous for each $\lambda \in \Omega$.

Theorem 4.7. *Let $A, B, C, P, Q, R, T, D, G$ and S be $\mathcal{H}$-mixed Lipschitz continuous with pairs of constants (L_A, l_A), (L_B, l_B), (L_C, l_C), (L_P, l_P), (L_Q, l_Q), (L_R, l_R), (L_T, l_T), (L_D, l_D), (L_G, l_G) and (L_S, l_S), respectively. Let N be α-strongly mixed monotone with respect to A, B and C, and $(L_{(N,1)}, L_{(N,2)}, L_{(N,3)}, l_N)$-mixed Lipschitz continuous, let M be σ-generalized mixed pseudocontractive with respect to P, Q and R, and $(L_{(M,1)}, L_{(M,2)}, L_{(M,3)}, l_M)$-mixed Lipschitz continuous. Let m, f, W be same as in Theorem 4.6 and condition (4.4) holds. Then for each $\lambda \in \Omega$, the solution set $S(\lambda)$ of PGMNQVIP (3.1) is a $\mathcal{H}$-Lipschitz continuous mapping.*

Proof. In view of Theorem 4.6, for each $\lambda, \bar{\lambda} \in \Omega$, $S(\lambda)$ and $S(\bar{\lambda})$ are both nonempty and closed subsets of H. Again in view of Theorem 4.6, we conclude that $F(x, \lambda)$ and $F(x, \bar{\lambda})$ are multi-valued θ-$\mathcal{H}$-contraction mappings with same contractive constant $\theta \in (0, 1)$. From Lemma 2.6, we get

$$\mathcal{H}(S(\lambda), S(\bar{\lambda})) \leq \left(\frac{1}{1-\theta}\right) \sup_{x \in H} \mathcal{H}(F(x, \lambda), F(x, \bar{\lambda})). \qquad (4.17)$$

Taking any $a \in F(x, \lambda)$, there exist $u(\lambda) \in A(x, \lambda)$, $v(\lambda) \in B(x, \lambda)$, $w(\lambda) \in C(x, \lambda)$, $p(\lambda) \in P(x, \lambda)$, $q(\lambda) \in Q(x, \lambda)$, $r(\lambda) \in R(x, \lambda)$, $t(\lambda) \in T(x, \lambda)$, $n(\lambda) \in D(x, \lambda)$, $z(\lambda) \in G(x, \lambda)$, $s(\lambda) \in S(x, \lambda)$ such that

$$a = x - s(\lambda) + m(n(\lambda), \lambda) + J_\rho^{W(\cdot, z(\lambda), \lambda)}(s(\lambda) - m(n(\lambda), \lambda)$$

$$- \rho N(u(\lambda), v(\lambda), w(\lambda), \lambda) + \rho M(p(\lambda), q(\lambda), r(\lambda), \lambda) + f(t(\lambda), \lambda)). \quad (4.18)$$

It is easy to see that there exist $u(\bar{\lambda}) \in A(x, \bar{\lambda})$, $v(\bar{\lambda}) \in B(x, \bar{\lambda})$, $w(\bar{\lambda}) \in C(x, \bar{\lambda})$, $p(\bar{\lambda}) \in P(x, \bar{\lambda})$, $q(\bar{\lambda}) \in Q(x, \bar{\lambda})$, $r(\bar{\lambda}) \in R(x, \bar{\lambda})$, $t(\bar{\lambda}) \in Q(x, \bar{\lambda})$, $n(\bar{\lambda}) \in D(x, \bar{\lambda})$, $z(\bar{\lambda}) \in G(x, \bar{\lambda})$ and $s(\bar{\lambda}) \in S(x, \bar{\lambda})$ such that

$$\|u(\lambda) - u(\bar{\lambda})\| \leq \mathcal{H}(A(x, \lambda), A(x, \bar{\lambda})) \leq l_A \|\lambda - \bar{\lambda}\|,$$

$$\|v(\lambda) - v(\bar{\lambda})\| \leq \mathcal{H}(B(x, \lambda), B(x, \bar{\lambda})) \leq l_B \|\lambda - \bar{\lambda}\|,$$

$$\|w(\lambda) - w(\bar{\lambda})\| \leq \mathcal{H}(C(x, \lambda), C(x, \bar{\lambda})) \leq l_C \|\lambda - \bar{\lambda}\|,$$

$$\|p(\lambda) - p(\bar{\lambda})\| \leq \mathcal{H}(P(x, \lambda), P(x, \bar{\lambda})) \leq l_P \|\lambda - \bar{\lambda}\|,$$

$$\|q(\lambda) - q(\bar{\lambda})\| \leq \mathcal{H}(Q(x, \lambda), Q(x, \bar{\lambda})) \leq l_Q \|\lambda - \bar{\lambda}\|,$$

$$\|r(\lambda) - r(\bar{\lambda})\| \leq \mathcal{H}(R(x, \lambda), R(x, \bar{\lambda})) \leq l_R \|\lambda - \bar{\lambda}\|,$$

$$\|t(\lambda) - t(\bar{\lambda})\| \leq \mathcal{H}(T(x, \lambda), T(x, \bar{\lambda})) \leq l_T \|\lambda - \bar{\lambda}\|,$$

$$\|n(\lambda) - n(\bar{\lambda})\| \leq \mathcal{H}(D(x, \lambda), D(x, \bar{\lambda})) \leq l_D \|\lambda - \bar{\lambda}\|,$$

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \mathcal{H}(G(x, \lambda), G(x, \bar{\lambda})) \leq l_G \|\lambda - \bar{\lambda}\|,$$

$$\|s(\lambda) - s(\bar{\lambda})\| \leq \mathcal{H}(S(x, \lambda), S(x, \bar{\lambda})) \leq l_S \|\lambda - \bar{\lambda}\|. \qquad (4.19)$$

Let

$$b = x - s(\bar{\lambda}) + m(n(\bar{\lambda}), \bar{\lambda}) + J_\rho^{W(\cdot, z(\bar{\lambda}), \bar{\lambda})}(s(\bar{\lambda}) - m(n(\bar{\lambda}), \bar{\lambda})$$
$$- \rho N(u(\bar{\lambda}), v(\bar{\lambda}), w(\bar{\lambda}), \bar{\lambda}) + \rho M(p(\bar{\lambda}), q(\bar{\lambda}), r(\bar{\lambda}), \bar{\lambda}) + f(t(\bar{\lambda}), \bar{\lambda})). \tag{4.20}$$

Then $b \in F(x, \bar{\lambda})$.

Since N and M are mixed Lipschitz continuous and in view of (4.8), (4.18)-(4.20) and with $l = s(\bar{\lambda}) - m(n(\bar{\lambda}), \bar{\lambda}) - \rho N(u(\bar{\lambda}), v(\bar{\lambda}), w(\bar{\lambda}), \bar{\lambda}) + \rho M(p(\bar{\lambda}), q(\bar{\lambda}), r(\bar{\lambda}), \bar{\lambda}) + f(t(\bar{\lambda}), \bar{\lambda})$, we have

$$
\begin{aligned}
\|a - b\| \leq{}& \|s(\lambda) - s(\bar{\lambda})\| + \|m(n(\lambda), \lambda) - m(n(\bar{\lambda}), \bar{\lambda})\| \\
&+ \|J_\rho^{W(\cdot, z(\lambda), \lambda)}(s(\lambda) - m(n(\lambda), \lambda) - \rho N(u(\lambda), v(\lambda), w(\lambda), \lambda) \\
&+ \rho M(p(\lambda), q(\lambda), r(\lambda), \lambda) + f(t(\lambda), \lambda)) - J_\rho^{W(\cdot, z(\lambda), \lambda)}(l)\| \\
&+ \|J_\rho^{W(\cdot, z(\lambda), \lambda)}(l) - J_\rho^{W(\cdot, z(\bar{\lambda}), \lambda)}(l)\| \\
&+ \|J_\rho^{W(\cdot, z(\bar{\lambda}), \lambda)}(l) - J_\rho^{W(\cdot, z(\bar{\lambda}), \bar{\lambda})}(l)\| \\
\leq{}& 2\|s(\lambda) - s(\bar{\lambda})\| + 2\|m(n(\lambda), \lambda) - m(n(\bar{\lambda}), \bar{\lambda})\| \\
&+ \|f(t(\lambda), \lambda) - f(t(\bar{\lambda}), \bar{\lambda})\| \\
&+ \rho\|N(u(\lambda), v(\lambda), w(\lambda), \lambda) - N(u(\bar{\lambda}), v(\bar{\lambda}), w(\bar{\lambda}), \bar{\lambda})\| \\
&+ \rho\|M(p(\lambda), q(\lambda), r(\lambda), \lambda) - M(p(\bar{\lambda}), q(\bar{\lambda}), r(\bar{\lambda}), \bar{\lambda})\| \\
&+ k_1\|z(\lambda) - z(\bar{\lambda})\| + k_2\|\lambda - \bar{\lambda}\|.
\end{aligned}
\tag{4.21}
$$

By the $\mathcal{H}$-mixed Lipschitz continuity of S in $\lambda \in \Omega$, we have

$$\|s(\lambda) - s(\bar{\lambda})\| \ \leq \ \mathcal{H}(S(x, \lambda), S(x, \bar{\lambda})) \ \leq \ l_S\|\lambda - \bar{\lambda}\|. \tag{4.22}$$

By the mixed Lipschitz continuity of m and $\mathcal{H}$-mixed Lipschitz continuity of D, we have

$$
\begin{aligned}
\|m(n(\lambda), \lambda) - m(n(\bar{\lambda}), \bar{\lambda})\| \leq{}& \|m(n(\lambda), \lambda) - m(n(\bar{\lambda}), \lambda)\| \\
&+ \|m(n(\bar{\lambda}), \lambda) - m(n(\bar{\lambda}), \bar{\lambda})\| \\
\leq{}& L_m\|n(\lambda) - n(\bar{\lambda})\| + l_m\|\lambda - \bar{\lambda}\| \\
\leq{}& L_m \mathcal{H}(D(x, \lambda), D(x, \bar{\lambda})) + l_m\|\lambda - \bar{\lambda}\| \\
\leq{}& (L_m l_D + l_m) \ \|\lambda - \bar{\lambda}\|.
\end{aligned}
\tag{4.23}
$$

By the mixed Lipschitz continuity of f and $\mathcal{H}$-mixed Lipschitz continuity of T, we have

$$
\begin{aligned}
\|f(t(\lambda), \lambda) - f(t(\bar{\lambda}), \bar{\lambda})\| \leq \ & \|f(t(\lambda), \lambda) - f(t(\bar{\lambda}), \lambda)\| \\
& + \|f(t(\bar{\lambda}), \lambda) - f(t(\bar{\lambda}), \bar{\lambda})\| \\
\leq \ & L_f \|t(\lambda) - t(\bar{\lambda})\| + l_f \|\lambda - \bar{\lambda}\| \\
\leq \ & L_f \, \mathcal{H}(T(x, \lambda), T(x, \bar{\lambda})) + l_f \|\lambda - \bar{\lambda}\| \\
\leq \ & (L_f l_T + l_f) \, \|\lambda - \bar{\lambda}\|.
\end{aligned} \tag{4.24}
$$

By the mixed Lipschitz continuity of N, we have

$$
\begin{aligned}
\|N(u(\lambda), & v(\lambda), w(\lambda), \lambda) - N(u(\bar{\lambda}), v(\bar{\lambda}), w(\bar{\lambda}), \bar{\lambda})\| \\
& \leq L_{(N,1)} \|u(\lambda) - u(\bar{\lambda})\| + L_{(N,2)} \|v(\lambda) - v(\bar{\lambda})\| \\
& \quad + L_{(N,3)} \|w(\lambda) - w(\bar{\lambda})\| + l_N \|\lambda - \bar{\lambda}\| \\
& \leq (l_A L_{(N,1)} + l_B L_{(N,2)} + l_C L_{(N,3)} + l_N) \, \|\lambda - \bar{\lambda}\|.
\end{aligned} \tag{4.25}
$$

Also, by the mixed Lipschitz continuity of M, we have

$$
\begin{aligned}
\|M(p(\lambda), & q(\lambda), r(\lambda), \lambda) - M(p(\bar{\lambda}), q(\bar{\lambda}), r(\bar{\lambda}), \bar{\lambda})\| \\
& \leq \ L_{(M,1)} \|p(\lambda) - p(\bar{\lambda})\| + L_{(M,2)} \|q(\lambda) - q(\bar{\lambda})\| \\
& \quad + L_{(M,3)} \|r(\lambda) - r(\bar{\lambda})\| + l_M \|\lambda - \bar{\lambda}\| \\
& \leq (l_P L_{(M,1)} + l_Q L_{(M,2)} + l_R L_{(M,3)} + l_M) \, \|\lambda - \bar{\lambda}\|.
\end{aligned} \tag{4.26}
$$

By the $\mathcal{H}$-mixed Lipschitz continuity of G, we have

$$
\|z(\lambda) - z(\bar{\lambda})\| \ \leq \ \mathcal{H}(G(x, \lambda), G(x, \bar{\lambda})) \ \leq \ l_G \, \|\lambda - \bar{\lambda}\|. \tag{4.27}
$$

In view of (4.21)-(4.27), we obtain that

$$
\|a - b\| \ \leq \ \theta_1 \, \|\lambda - \bar{\lambda}\|, \tag{4.28}
$$

where $\theta_1 := 2(l_S + L_m l_D + l_m) + L_f l_T + l_f + k_1 l_G + k_2 + \rho(l_A L_{(N,1)} + l_B L_{(N,2)} + l_C L_{(N,3)} + l_N + l_P L_{(M,1)} + l_Q L_{(M,2)} + l_R L_{(M,3)} + l_M)$. Hence, we obtain

$$
\sup_{a \in F(x, \lambda)} d(a, F(x, \bar{\lambda})) \ \leq \ \theta_1 \|\lambda - \bar{\lambda}\|.
$$

By using an analogous argument as above, we can have

$$
\sup_{b \in F(x, \bar{\lambda})} d(F(x, \lambda), b) \ \leq \ \theta_1 \|\lambda - \bar{\lambda}\|.
$$

Hence, it follows that

$$
\mathcal{H}(F(x, \lambda), F(x, \bar{\lambda})) \leq \theta_1 \|\lambda - \bar{\lambda}\|.
$$

By Lemma 2.6, we obtain

$$
\mathcal{H}\Big(S(\lambda), S(\bar{\lambda})\Big) \ \leq \ \Big(\frac{\theta_1}{1 - \theta}\Big) \|\lambda - \bar{\lambda}\|.
$$

This shows that $S(\lambda)$ is $\mathcal{H}$-Lipschitz continuous at $\lambda \in \Omega$. $\square$

5. Discussion

For $k_1, k_2, \rho > 0$, it is clear that $\alpha > \sigma$; $L_S > \sqrt{2\delta - 1}$; $k \in (0,1)$ and

$$\left| \rho - \frac{(\alpha - \sigma)}{2(L_N^2 + L_M^2)} \right| > \frac{\sqrt{(\alpha - \sigma)^2 - 2(L_N^2 + L_M^2)}}{2(L_N^2 + L_M^2)} \; ; \; \alpha > \sigma + \sqrt{2(L_N^2 + L_M^2)},$$

where

$$L_N := (L_A L_{(N,1)} + L_B L_{(N,2)} + L_C L_{(N,3)})$$

and

$$L_M := (L_P L_{(M,1)} + L_Q L_{(M,2)} + L_R L_{(M,3)}).$$

Further, $\theta \in (0,1)$ and condition (4.4) of Theorem 4.6 holds for some suitable values of constants, for example, $\alpha = 4$, $\sigma = \delta = L_S = 2.5$, $k_1 = 0.2$, $k_2 = 0.3$.

6. Conclusion

This paper is based on the study of the behavior and sensitivity analysis of a solution set for PGMNQVIP (3.1). For this, we used the technique of resolvent operator of maximal monotone mappings along with the property of a fixed point set of a multi-valued contractive mapping in a real Hilbert space. Since the PGMNQVIP (3.1) includes many known classes of parametric generalized variational inclusions as special cases, Theorems 4.6-4.7 improve and generalize the known results (see [1-3,5-8,11-14,21,23-27]). The concepts and results presented in this paper will be very helpful in the theoretical study of behavior of solutions for some classes of nonlinear operator equation problems.

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