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A TRIPLE MIXED QUADRATURE BASED ADAPTIVE SCHEME FOR ANALYTIC FUNCTIONS

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Abstract. An efficient adaptive scheme based on a triple mixed quadrature rule of precision nine for approximate evaluation of line integral of analytic functions has been constructed. At first, a mixed quadrature rule $SM_1(f)$ has been formed using Gauss-Legendre three point transformed rule and five point Booles transformed rule. A suitable linear combination of the resulting rule and Clenshaw-Curtis seven point rule gives a new mixed quadrature rule $SM_{10}(f)$. This mixed rule is termed as triple mixed quadrature rule. An adaptive quadrature scheme is designed. Some test integrals having analytic function integrands have been evaluated using the triple mixed rule and its constituent rules in non-adaptive mode. The same set of test integrals have been evaluated using those rules as base rules in the adaptive scheme. The triple mixed rule based adaptive scheme is found to be the most effective.

1. Introduction

Despite the simple nature of the problem and the practical value of its method, numerical integration has been of great interest to both pure and applied mathematicians like Archimedes, Kepler, Huygens, Newton, Euler, Gauss, Jacobi, Chebyshev, Markhoff, Fejer, Polyya, Szego, Schoenberg and Sobolov. There are several rules [3,4,11] for the approximate evaluation of real definite integral

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$$I(f) = \int_a^b f(x)dx \text{ and } \int_{-1}^{1} f(z)dz. \quad (1.1)$$

However there are only few quadrature rules for evaluating an integral of type

$$I(f) = \int_L f(z)dz, \quad (1.2)$$

where L is a directed line segment from the point $(z_0 - h)$ to $(z_0 + h)$ in the domain of f. Using the transformation $z = z_0 + ht, t \in [-1, 1]$ (due to [6]), we transformed the integral (1.2) to the form

$$h \int_{-1}^{1} f(z_0 + ht)dt \quad (1.3)$$

and made the approximation of the integral by applying standard quadrature rule meant for approximate evaluation of real definite integral (1.1). The rules so formed are termed as transformed rules for numerical integration of (1.2).

The integral (1.1) has been successfully approximated by several authors [7,8,9] by applying the mixed quadrature rule in the complex plane. In literature, precision of a quadrature rule has been enhanced through Richardson extrapolation and Kronrod extension [8,9]. These methods of precision enhancement are very much cumbersome and each having single base rule. But the enhancement of precision by mixed quadrature approach is very much simple with the aid of two rules and easy to handle.

In 1996, Das and Pradhan [3] breed the concept of mixed quadrature, after that Dash and his research team, Archarya have been developing mixed quadrature rules of different combinations.

In this paper, a new mixed quadrature rule of precision nine has been designed by a convex combination of three rules,
(i) Gauss-Legendre three point transformed rule $GL(f)$,
(ii) Bools transformed rule $BL(f)$,
(iii) Clenshaw-Curtis 7-point rule $CC_7(f)$.

This new mixed rule is termed as Triple Mixed Rule $SM_{10}(f)$.

This paper consists of seven sections. Section 1 is introductory one. Section 2 speaks about the constituent rules $GL(f)$, $BL(f)$ and the formation of the mixed rule $SM_1(f)$ as well as their truncation errors. Section 3 describes about Clenshaw-Curtis 7-point rule $CC_7(f)$ and its truncation error. Section 4 explains how the new rule named Triple mixed rule $SM_{10}(f)$ is constructed. Section 5 gives an account of error analysis of the Triple mixed rule. In Section 6 numerical verification of the new rule and its constituent rules is done evaluating test integrals in non-adaptive environment. The effectiveness of the Triple mixed rule $SM_{10}(f)$ is presented through Tables and Figures. Section 7 consists of an adaptive integration scheme and tabulated results of the test integrals in this adaptive scheme taking the rule $SM_{10}(f)$ and its constituents as base rules. A conclusion is drawn highlighting the role of $SM_{10}(f)$ in the last section, Section 8.

2. Construction of the constituent mixed rule $SM_1(f)$

For construction of the constituent mixed rule $SM_1(f)$ let us consider following two quadrature rules of precision five.

2.1. **Gauss-Legendre 3-point transformed rule** $GL(f)$**.** The Gauss-Legendre 3-point transformed rule [1,2,11,12] is given by

$$I(f) \approx GL(f) = \frac{h}{9}\left[5f\left(z_0 - h\sqrt{\frac{3}{5}}\right) + 8f(z_0) + 5f\left(z_0 + h\sqrt{\frac{3}{5}}\right)\right]. \quad (2.1)$$

Appling Taylor's theorem, (2.1) becomes

$$GL(f) = 2h\Big[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{3}{5^2}\frac{h^6}{5!}f^{vi}(z_0) + \frac{3^2}{5^3}\frac{h^8}{8!}f^{viii}(z_0) + \frac{3^3}{5^4}\frac{h^{10}}{10!}f^{x}(z_0) + \frac{3^4}{5^5}\frac{h^{12}}{12!}f^{xii}(z_0) + \cdots\Big]. \quad (2.2)$$

The exact value of the integral due to Taylor [11]

$$I(f) = 2h\Big[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0) + \frac{h^8}{9!}f^{viii}(z_0) + \frac{h^{10}}{11!}f^{x}(z_0) + \frac{h^{12}}{13!}f^{xii}(z_0) + \cdots\Big]. \quad (2.3)$$

Error due to the rule $GL(f)$ is denoted by $E_{GL}(f)$ and given by $E_{GL}(f) = I(f) - GL(f)$. Using (2.2) and (2.3), we get

$$E_{GL}(f) = \frac{8}{5^2}\frac{h^7}{7!}f^{vi}(z_0) + \frac{88}{5^3}\frac{h^9}{9!}f^{viii}(z_0) + \frac{656}{5^4}\frac{h^{11}}{11!}f^{x}(z_0) + \frac{4144}{5^5}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots \quad (2.4)$$

The error term establishes that the degree of precision of rule $GL(f)$ is five.

2.2. **Boole's Quadrature transformed rule** $BL(f)$**.** The Boole's transformed rule [1,7,11] is given by

$$I(f) \approx BL(f) = \frac{h}{45}\Big[7f(z_0 - h) + 32f(z_0 - \frac{h}{2}) + 12f(z_0) + 32f(z_0 + \frac{h}{2}) + 7f(z_0 + h)\Big]. \quad (2.5)$$

$$BL(f) = 2h\Big[f(z_0)+\frac{h^2}{3!}f^{ii}(z_0)+\frac{h^4}{5!}f^{iv}(z_0)+\frac{h^6}{6\times 6!}f^{vi}(z_0)+\frac{57}{45\times 8}\frac{h^8}{8!}f^{viii}(z_0)$$
$$+\frac{5}{32}\frac{h^{10}}{10!}f^{x}(z_0)+\frac{897}{45\times 128}\frac{h^{12}}{12!}f^{xii}(z_0)+\cdots\Big]. \tag{2.6}$$

Error due to the rule $BL(f)$ is denoted by $E_{BL}(f)$, so $E_{BL}(f) = I(f) - BL(f)$.

$$E_{BL}(f) = \frac{-1}{3}\frac{h^7}{7!}f^{vi}(z_0)+\frac{-17}{20}\frac{h^9}{9!}f^{viii}(z_0)$$
$$+\frac{-23}{16}\frac{h^{11}}{11!}f^{x}(z_0)+\frac{1967}{15\times 128}\frac{h^{13}}{13!}f^{xii}(z_0)+\cdots. \tag{2.7}$$

The error term establishes that the degree of precision of rule $BL(f)$ is five.

2.3. **The mixed rule $SM_1(f)$.** The following theorem gives the construction of the mixed rule $SM_1(f)$.

Theorem 2.1. *If f(z) is analytic in the given domain $\Omega \supset [z_0-h, z_0+h]$, then the mixed $SM_1(f)$ and error due to the rule $ESM_1(f)$ given by*

$$SM_1(f) = \frac{1}{49}\left[25GL(f)+24BL(f)\right]$$

and

$$ESM_1(f) \equiv \frac{-14}{245}\frac{h^9}{9!}f^{viii}(z_0).$$

Proof. We have

$$I(f) = GL(f) + E_{GL}(f) \tag{2.8}$$

and

$$I(f) = BL(f) + E_{BL}(f). \tag{2.9}$$

Adding 24 times of (2.9) with 25 times of (2.8) we have

$$49I(f) = \left[25GL(f)+24BL(f)\right]+\left[25E_{GL}(f)+24E_{BL}(f)\right],$$

this implies that

$$I(f) = \frac{1}{49}\left[25GL(f)+24BL(f)\right]+\frac{1}{49}\left[25E_{GL}(f)+24E_{BL}(f)\right].$$

Therefore, we have

$$I(f) = SM_1(f) + ESM_1(f),$$

where

$$SM_1(f) = \frac{1}{49}\left[25GL(f)+24BL(f)\right] \tag{2.10}$$

is a mixed rule and

$$ESM_1(f) = \frac{1}{49}\left[25E_{GL}(f)+24E_{BL}(f)\right]$$

is the truncation error due to the mixed rule. Using (2.4) and (2.7) after simplification, we get

$$ESM_1(f) = \frac{-14}{245}\frac{h^9}{9!}f^{viii}(z_0) + \frac{413}{2450}\frac{h^{11}}{11!}f^{x}(z_0) + \frac{-48069}{147000}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots . \tag{2.11}$$

Hence we have (neglecting the higher order terms)

$$ESM_1(f) = \frac{(-14)}{245}\frac{h^9}{9!}f^{viii}(z_0).$$

This completes the proof. □

Note: Using (2.1) and (2.5) on (2.10) , we get

$$\begin{aligned} SM_1(f) = {} & \frac{125h}{441}\left\{f\left(z_0 - h\sqrt{\frac{3}{5}}\right) + f\left(z_0 + h\sqrt{\frac{3}{5}}\right)\right\} \\ & + \frac{24h}{315}\left\{f(z_0-h) + f(z_0+h)\right\} \\ & + \frac{256h}{735}\left\{f(z_0-h/2) + f(z_0+h/2)\right\} + \frac{184h}{315}f(z_0). \end{aligned} \tag{2.12}$$

(2.12) is known as expansion form of the rule $SM_1(f)$.

3. Clenshaw-Curtis 7-point transformed rule $CC_7(f)$

The Clenshaw-Curtis 7-point transformed rule [4,5,8] is given by

$$\begin{aligned} I(f) = \int_{z_0-h}^{z_0+h} f(z)dz \equiv CC_7(f) = \frac{h}{315}\Bigg[& 9f(z_0-h) + 80f\left(z_0 - \frac{\sqrt{3}}{2}h\right) \\ & + 144f\left(z_0 - \frac{h}{2}\right) + 164f(z_0) + 144f\left(z_0 + \frac{h}{2}\right) \\ & + 80f\left(z_0 + \frac{\sqrt{3}}{2}h\right) + 9f(z_0+h)\Bigg] \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} CC_7(f) = 2h & \left[f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{7!}f^{vi}(z_0)\right] \\ & + \left[\frac{31}{140}\frac{h^9}{8!}f^{viii}(z_0) + \frac{5}{25}\frac{h^{11}}{10!}f^{x}(z_0) + \cdots\right]. \end{aligned} \tag{3.2}$$

Corollary 3.1. *If $f(z)$ is analytic in the given domain $\Omega \supset [z_0-h, z_0+h]$, then the rule $CC_7(f)$ is of precision-7 and the truncation error due to the rule is $ECC_7(f) = o(h^9)$.*

Proof. From $I(f) = CC_7(f) + ECC_7(f)$, we have

$$ECC_7(f) = I(f) - CC_7(f). \tag{3.3}$$

Using (2.3) and (3.2) on (3.3), the truncation error due to the rule $CC_7(f)$ is

$$ECC_7(f) = \frac{1}{140}\frac{h^9}{9!}f^{viii}(z_0) + \frac{1}{28}\frac{h^{11}}{11!}f^{x}(z_0) + \cdots . \tag{3.4}$$

(3.4) indicate that the degree of precision of the rule $CC_7(f)$ is seven and $ECC_7(f) = o(h^9)$. □

4. Formulation of the triple mixed quadrature rule $SM_{10}(f)$

The following theorem gives the formulation of the proposed Triple mixed quadrature rule.

Theorem 4.1. *If f(z) is analytic in the given domain $\Omega \supset [z_0 - h, z_0 + h]$, then the triple mixed quadrature $SM_{10}(f)$ and truncation error due to the rule $ESM_{10}(f)$ are given by*

$$SM_{10}(f) = \frac{1}{9}[8CC_7(f) + SM_1(f)]$$

and

$$ESM_{10}(f) = \frac{1}{9}[8ECC_7(f) + ESM_1(f)].$$

Proof. Resuming

$$I(f) = CC_7(f) + ECC_7(f) \tag{4.1}$$

and

$$I(f) = SM_1(f) + ESM_1(f). \tag{4.2}$$

Adding 8 times of (4.1) to the equation (4.2), we get

$$9I(f) = SM_1(f) + 8CC_7(f) + ESM_1(f) + 8ECC_7(f).$$

Hence

$$I(f) = \frac{1}{9}[SM_1(f) + 8CC_7(f)] + \frac{1}{9}[ESM_1(f) + 8ECC_7(f)].$$

Therefore, we have

$$I(f) = SM_{10}(f) + ESM_{10}(f),$$

where

$$SM_{10}(f) = \frac{1}{9}[SM_1(f) + 8CC_7(f)] \tag{4.3}$$

and

$$ESM_{10}(f) = \frac{1}{9}[ESM_1(f) + 8ECC_7(f)]. \tag{4.4}$$

(4.3) is the required triple mixed quadrature rule and (4.4) is the truncation error associated due to the rule. □

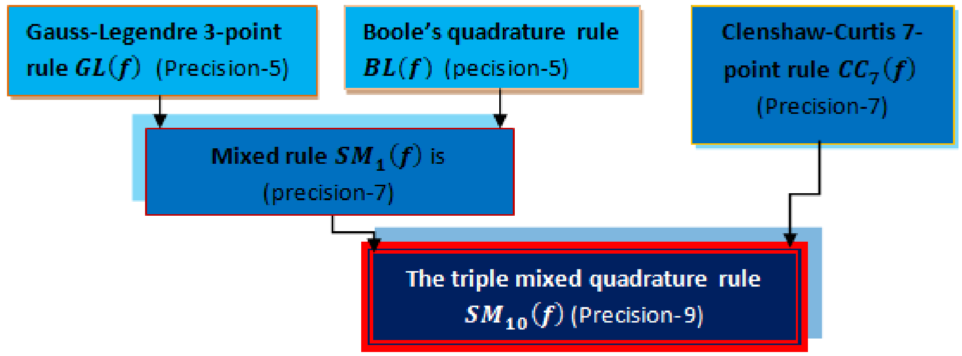

FIGURE 1. Representation of construction of the rule $SM_{10}(f)$

5. Error analysis

An error analysis of the constructed Triple mixed rule has been obtained by the following theorems.

Theorem 5.1. *If f(z) is analytic in the given domain* $\Omega \supset [z_0 - h, z_0 + h]$, *then the truncation error due to the rule* $SM_{10}(f)$ *is denoted by* $ESM_{10}(f)$ *and* $\mid ESM_{10}(f) \mid \cong \frac{53}{1050} \frac{h^{11}}{11!} f^{x}(z_0)$.

Proof. Using (2.11) and (3.4) on (4.4), we get

$$ESM_{10}(f) = \frac{53}{1050} \frac{h^{11}}{11!} f^{x}(z_0) + \cdots .$$

Hence,

$$ESM_{10}(f) \cong \frac{53}{1050} \frac{h^{11}}{11!} f^{x}(z_0)$$

and

$$\mid ESM_{10}(f) \mid \cong \frac{53}{1050} \frac{h^{11}}{11!} f^{x}(z_0).$$

□

Lemma 5.2. *The Error bound of the constructed quadrature rule is*

$$\mid ESM_{10}(f) \mid \leq \frac{2M}{315} \frac{h^9}{9!} |\xi_2 - \xi_1|, \ \xi_1, \xi_2 \in [-1, 1]$$

where $M = \max_{-1 \leq z \leq 1} |f^{ix}(z)|$.

Proof. From (3.4), we get $ECC_7(f) \cong \frac{1}{140} \frac{h^9}{9!} f^{viii}(\xi_1)$, $\xi_1 \in [-1, 1]$ and from (2.11), we get $ESM_1(f) \cong \frac{-14}{245} \frac{h^9}{9!} f^{viii}(\xi_2)$, $\xi_2 \in [-1, 1]$. Using these two values on (4.4), we can write

$$\begin{aligned} ESM_{10}(f) &\cong \frac{1}{9}\left[\left\{\frac{2}{35}\frac{h^9}{9!}f^{viii}(\xi_1)\right\} - \left\{\frac{2}{35}\frac{h^9}{9!}f^{viii}(\xi_2)\right\}\right] \\ &= \frac{2}{315}\frac{h^9}{9!}\left\{f^{viii}(\xi_1) - f^{viii}(\xi_2)\right\} \\ &= \frac{-2}{315}\frac{h^9}{9!}\left\{f^{viii}(\xi_2) - f^{viii}(\xi_1)\right\} \\ &= \frac{-2}{315}\frac{h^9}{9!}\int_{\xi_1}^{\xi_2} f^{ix}(z)dz. \end{aligned}$$

It implies that

$$\begin{aligned} \mid ESM_{10} \mid &\cong \frac{2}{315}\frac{h^9}{9!} \mid \int_{\xi_1}^{\xi_2} f^{ix}(z)dz \mid \leq \frac{2}{315}\frac{h^9}{9!}\int_{\xi_1}^{\xi_2} \mid f^{ix}(z) \mid dz \\ &\leq \frac{2}{315}\frac{h^9}{9!}\int_{\xi_1}^{\xi_2} Mdz, \end{aligned}$$

where $M = \max_{-1\leq z\leq 1} \mid f^{ix}(z) \mid$. Hence, we have

$$\mid ESM_{10}(f) \mid \leq \frac{2M}{315}\frac{h^9}{9!} \mid \xi_2 - \xi_1 \mid . \tag{5.1}$$

Since ξ_1 and ξ_2 are arbitrarily chosen points in the interval $[-1, 1]$, (5.1) shows that the absolute value of the truncation error will be less if the points ξ_1 and ξ_2 are closure to each other. □

Corollary 5.3. *The error bound for the truncation error is*

$$\mid ESM_{10}(f) \mid \leq \frac{4M}{315}\frac{h^9}{9!},$$

where $M = \max_{-1\leq z\leq 1} \mid f^{ix}(z) \mid$.

Proof. From the Lemma 5.2,

$$\mid ESM_{10}(f) \mid \leq \frac{2M}{315}\frac{h^9}{9!}|\xi_2 - \xi_1|,\ \xi_1, \xi_2 \in [-1, 1]$$

where $M = \max_{-1\leq z\leq 1} |f^{ix}(z)|$.

Using the relation $\mid \xi_2 - \xi_1 \mid \leq 2$ [9], we have

$$\mid ESM_{10}(f) \mid \leq \frac{4M}{315}\frac{h^9}{9!}.$$

□

Theorem 5.4. *If f(z) is analytic in the given domain* $\Omega \supset [z_0 - h, z_0 + h]$, *then the error committed due to the mixed quadrature rule* $SM_{10}(f)$ *is less than its constituent rules.*

Proof. From (2.4) and Theorem 5.1 $\mid ESM_{10}(f) \mid \leq \mid EGL(f) \mid$. From (2.7) and Theorem 5.1 $\mid ESM_{10}(f) \mid \leq \mid EBL(f) \mid$. From (2.11) and Theorem 5.1 $\mid ESM_{10}(f) \mid \leq \mid ESM_1(f) \mid$. From (3.4) and Theorem 5.1 $\mid ESM_{10}(f) \mid \leq \mid ECC_7(f) \mid$. □

6. Numerical verification

The effectiveness of the rule $SM_{10}(f)$ is verified by applying it and its constituents on test integrals in Non-adaptive mode (see remarks below on Tables 1,2 and Figures 2,3,4).

Table-1.

Integrals	Values obtained by different quadrature rules				
	$GL(f)$	$BL(f)$	$SM_1(f)$	$CC_7(f)$	$SM_{10}(f)$
$I_1 = \int_0^i e^{-z^2} dz$	1.462409711 47732195i	1.462909438 97296967i	1.462654475 96498614i	1.462651370 23528938i	1.4626517153 163668i
$I_2 = \int_{-i}^i coszdz$	2.350336928 6800113i	2.3504709035 69372i	2.350402549 03398i	2.350402366 6962997i	2.3504023869 5604246i
$I_3 = \int_{-\sqrt{3}i}^{\sqrt{3}i} z^8 dz$	20.20264061 94833i	44.427103214 1417025i	32.06768352 29895i	31.06556841 28960673i	31.176914536 2397823i
$I_4 = \int_{-i/3}^{i/3} coszdz$	0.654389422 5254678i	0.6543893634 69878i	0.654389393 600281i	0.654389393 591309492i	0.6543893935 92306327i

Table-2.

Integrals	Exact value	$\lvert Error \rvert$ due to quadrature rules				
		$\lvert EGL(f)\rvert$	$\lvert EBL(f)\rvert$	$\lvert ESM_1(f)\rvert$	$\lvert ECC_7(f)\rvert$	$\lvert ESM_{10}(f)\rvert$
I_1	1.4626517459 07182i	0.0002420 3442986	0.0002576 930658	0.0000027 30058	0.0000003 75672	0.000000030 590815
I_2	2.3504023872 87602913i	0.00006545 86075916	0.0000685 16281769	0.0000001 61746377	0.0000000 205913032	0.000000000 331560453
I_3	31.176914536 23979128349 4i	10.9742739 16756491	13.250188 67790191	0.8907689 86749708	0.1113461 233437239	0.000000000 000008983494
I_3	0.6543893935 9230448i	0.00000002 89331633	0.0000000 30122426	0.0000000 00007976	0.0000000 00000995	0.000000000 000001847

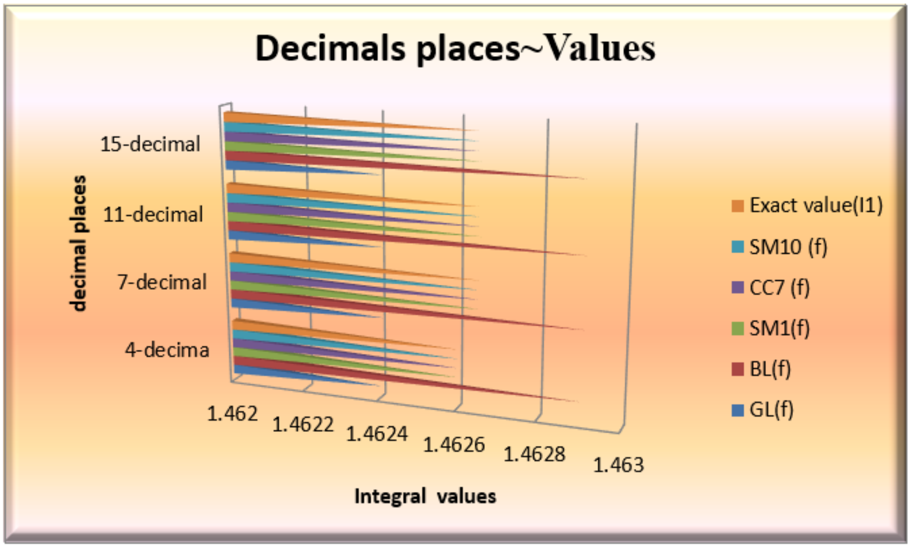

FIGURE 2. Values of I_1 obtained by different quadrature rules.

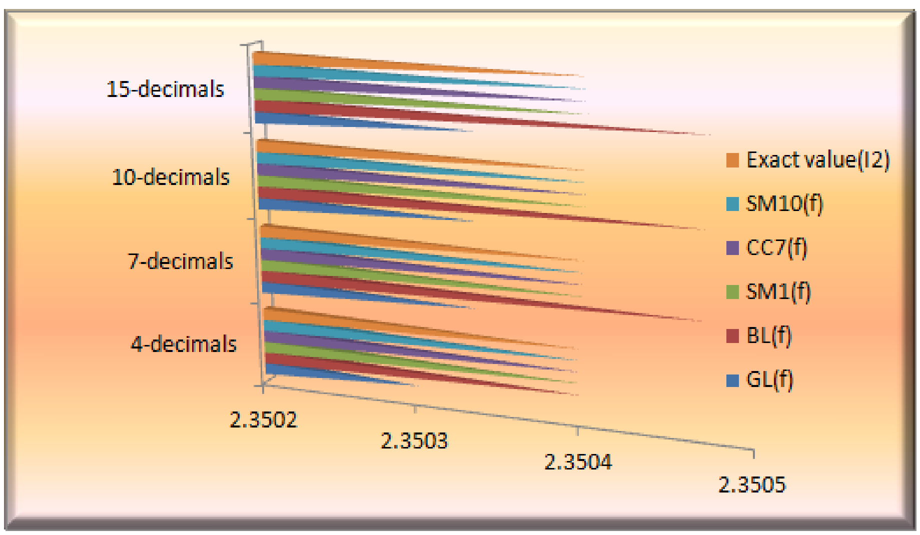

FIGURE 3. Values of I_2 obtained by different quadrature rules.

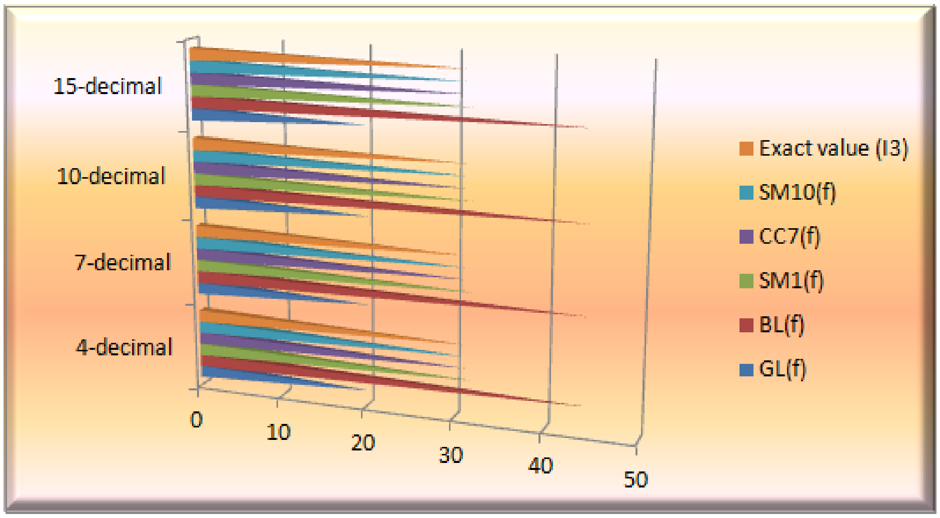

FIGURE 4. Values of I_3 obtained by different quadrature rules.

Remark 6.1. From the figures and Table-2 we mark as below:

(i) In the figure-2, the values obtained from the triple mixed rule $SM_{10}(f)$ covers the exact value $I_1(f)$ upto seven decimal places but the constituent rules fail after 3- 6 decimal places.

(ii) In the figure-3, the values obtained from the rule $SM_{10}(f)$ covers the exact value $I_2(f)$ upto nine decimal places but the constituent rules fail after 3-7 decimal places.

(iii) In the figure-4, the values obtained from the rule $SM_{10}(f)$ covers the exact value $I_3(f)$ upto 14 decimal places but the constituent rules fail to a single decimal place.

7. Application of the quadrature rule in adaptive quadrature routines

An effective adaptive strategy is given in following algorithm [4,10,13].

Algorithm 7.1. The input to this scheme is $a, b, \in, n, f$. The output is

$$P \cong \int_a^b f(x)dx$$

with error less than $\in$, n is the number of interval initially chosen. The adaptive strategy is outlined in the following four steps.

Step-1 An approximation I_1 to $I = \int_a^b f(x)dx$ is computed.

Step-2 The interval is divided into pieces, $[a, c]$ and $[c, b]$ where $c = \frac{(a+b)}{2}$, and then $I_2 \approx \int_a^c f(x)dx$ and $I_3 \approx \int_c^b f(x)dx$ are computed.

Step-3 $I_2 + I_3$ is compared with I_1, to estimate error in $I_2 + I_3$.

Step-4 If | estimated error $| \leq \in$ (termination criterion), then I_2+I_3 is accepted as an approximation to $\int_a^b f(x)dx$. Otherwise the same procedure is applied to [a, c] and [c, b], allowing each piece to a tolerance of $\frac{\in}{2}$.

Applying quadrature routines to the proposed quadrature rule to each of the sub intervals covering [a, b] until the termination criterion is satisfied.

If the termination criterion is not satisfied in one or more of the sub intervals, then those subintervals must be further subdivided and entire process repeated.

Table-3. Approximation of the test integrals by the constructed rule $SM_{10}(f)$ and the constituent rules using the adaptive quadrature routines.

Let us consider the prescribed tolerance $\epsilon = 1.0 \times 10^{-8}$.

Integrals	For the Tripple Mixed rule $SM_{10}(f)$		
I	Approximate value(P)	No of steps required	$\mid Error \mid = \mid P - I \mid$
$I_a = \int_{-i}^{i} coszdz$	2.35040238728724233i	01	3.605 $\times 10^{-13}$
$I_b = \int_{-i}^{i} e^z dz$	1.682941969615179328i	01	2.665 $\times 10^{-16}$
$I_c = \int_{-i/3}^{i/3} coshzdz$	0.654389393592304491	01	1.025 $\times 10^{-17}$

Integrals	For the constituent Mixed rule $SM_1(f)$			For the constituent rule $CC_7(f)$		
I	Approximate value(P)	No of steps required	$\mid Error \mid = \mid P - I \mid$	Approximate value(P)	No of steps required	$\mid Error \mid = \mid P - I \mid$
I_a	2.35040238729040218i	03	2.799 $\times 10^{-12}$	2.3504023872872526i	03	1.923 $\times 10^{-13}$
I_b	1.6829419696178327i	01	2.039 $\times 10^{-12}$	1.68294196961553835i	01	2.546 $\times 10^{-13}$
I_c	0.654389393592335284i	01	3.08 $\times 10^{-14}$	0.654389393592300641i	01	3.838 $\times 10^{-15}$

8. Conclusions

From the tables it is evident that the mixed quadrature rule when applied on each of the test integrals gives better results than that of constituent rules in non-adaptive mode. This mixed quadrature rule $SM_{10}(f)$ also dominates its constituents in adaptive environment. Though in some cases the number of steps required to achieve the desired accuracy is reduced but in all cases the absolute error due to the triple mixed rule is significantly less in comparison to other rules.

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