



NON-INVARIANT HYPERSURFACES OF A (ϵ, δ) -TRANS SASAKIAN MANIFOLDS

Toukeer Khan¹ and Sheeba Rizvi²

¹School of Liberal Arts and Science
Ear University, Lucknow-226003, India
e-mail: toukeerkhan@gmail.com

²School of Liberal Arts and Science
Era University, Lucknow-226003, India
e-mail: drsheeba@erauniversity.in

Abstract. The object of this paper is to study non-invariant hypersurface of a (ϵ, δ) -trans Sasakian manifolds equipped with (f, g, u, v, λ) -structure. Some properties obeyed by this structure are obtained. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurface with (f, g, u, v, λ) -structure of a (ϵ, δ) -trans Sasakian manifolds to be totally geodesic. The second fundamental form of a non-invariant hypersurface of a (ϵ, δ) -trans Sasakian manifolds with (f, g, u, v, λ) -structure has been traced under the condition when f is parallel.

1. INTRODUCTION

The study of (ϵ) -Sasakian manifolds have been studies by Bejancu and Duggal [2], and Xufeng and Xiaoli [10] studied that these manifolds are real hypersurface of indefinite Kahlerian manifolds. Tripathi et al. [9] introduced and studied (ϵ) -almost para contact manifolds. De and Sarkar [4] also introduced (ϵ) -Kenmotsu manifolds and studied conformally flat, Weyl semisymmetric, ϕ -recurrent (ϵ) -Kenmotsu manifolds. Nagaraja et al. [7] studied (ϵ, δ) -trans Sasakian structure.

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⁰Corresponding author: T. Khan(toukeerkhan@gmail.com).

In 1970, Goldberg et al. [5] introduced the notion of a non-invariant hypersurface of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the (1,1)-structure tensor field f defining the almost contact structure is never tangent to the hypersurface. The notion of (f, g, u, v, λ) -structure was given by Yano and Okumura [11]. It is well known ([12] and [3]) that hypersurface of an almost contact metric manifold always admits a (f, g, u, v, λ) -structure. In [5], author proved that there always exists a (f, g, u, v, λ) -structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. Prasad [8] studied the non-invariant hypersurface of trans Sasakian manifolds. Khan [6] studied the non-invariant hypersurface of Nearly Kenmotsu manifold. Ahmed et al. [1] studied the non-invariant hypersurface of nearly hyperbolic Sasakian manifold. In the present paper, we study the non-invariant hypersurface of (ϵ, δ) -trans Sasakian manifolds.

This paper is organized as follows. In section 2, we give a brief description of (ϵ, δ) -trans Sasakian manifolds. In section 3, introduce the non-invariant hypersurface and induced (f, g, u, v, λ) -structure on non-invariant hypersurface M getting some equation. Some results of non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifolds. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifolds to be totally geodesic.

2. PRELIMINARIES

Let \widetilde{M} be a n -dimensional almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g) where a tensor ϕ of type (1,1), a vector field ξ , called structure vector field and η , the dual 1-form and a Riemannian metric g satisfying the following,

$$\phi^2 X = -X + \eta(X)\xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi\xi = 0. \quad (2.2)$$

An almost contact metric manifold \widetilde{M} is called an (ϵ) -almost contact metric manifold if

$$\eta(X) = \epsilon g(X, \xi), \quad g(\xi, \xi) = \epsilon, \quad (2.3)$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad (2.4)$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad (2.5)$$

for all $X, Y \in TM$ [10], where $\epsilon = g(\xi, \xi) = \pm 1$.

An (ϵ) -almost contact metric manifold is called an (ϵ, δ) -trans Sasakian manifold [9] if

$$(\tilde{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \epsilon\eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \delta\eta(Y)\phi X\}, \tag{2.6}$$

$$\tilde{\nabla}_X \xi = -\epsilon\alpha(\phi X) - \delta\beta\phi^2 X, \tag{2.7}$$

hold for some smooth function α and β on \tilde{M} and $\epsilon = \pm 1, \delta = \pm 1$. For $\beta = 0, \alpha = 1$ an (ϵ, δ) -trans Sasakian manifold reduces to an (ϵ) -Sasakian manifold and for $\alpha = 0, \beta = 1$, it is reduced to a (δ) -Kenmotsu manifold.

A hypersurface of an almost contact metric manifold \tilde{M} is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of $(1, 1)$ tensor field ϕ defining the contact structure is never tangent to the hypersurface. Let X be tangent vector on non-invariant hypersurface of an almost contact metric manifold \tilde{M} . Then ϕX is never to tangent of the hypersurface. Let \tilde{M} be a non-invariant hypersurface of an almost contact metric manifold. Now, we define the following:

$$\phi X = fX + u(X)\tilde{N}, \tag{2.8}$$

$$\phi\tilde{N} = -U, \tag{2.9}$$

$$\xi = V + \lambda\tilde{N}, \lambda = \eta(\tilde{N}), \tag{2.10}$$

$$\eta(X) = v(X), \tag{2.11}$$

where f is $(1,1)$ tensor field, u and v are 1-form, \tilde{N} is a unit normal to the hypersurface, $X \in TM$ and $u(X) \neq 0$. Then we get an induced (f, g, u, v, λ) -structure on \tilde{M} satisfying the conditions

$$\left. \begin{aligned} f^2 &= -I + u \otimes U + v \otimes V, \\ uof &= \lambda v, vof = -\lambda u, \\ v(V) &= 1 - \lambda^2, u(V) = v(U) = 0, u(U) = 1 - \lambda^2, \\ fV &= \lambda U, fU = \lambda V, \\ u(X) &= \epsilon g(X, U), v(X) = \epsilon g(X, V), \\ g(fX, fY) &= g(X, Y) - u(X)u(Y) - \epsilon v(X)v(Y), \\ g(fX, Y) &= -g(X, fY), \end{aligned} \right\} \tag{2.12}$$

for all $X, Y \in TM$ and $\lambda = \eta(\tilde{N})$.

The Gauss and Weingarten formula are given by

$$\tilde{\nabla}_X Y = \tilde{\nabla}_X Y + h(X, Y)\tilde{N}, \tag{2.13}$$

$$\tilde{\nabla}_X \tilde{N} = -A_{\tilde{N}}X, \tag{2.14}$$

for all $X, Y \in TM$, where $\tilde{\nabla}$ and ∇ are the Riemannian and induced connection on \tilde{M} and M respectively and \tilde{N} is the unit normal vector in the normal bundle $T^\perp M$. In this formula h is the second fundamental form on M related to $A_{\tilde{N}}$ by

$$h(X, Y) = g(A_{\tilde{N}}X, Y), \tag{2.15}$$

for all $X, Y \in TM$.

3. SOME PROPERTIES OF NON-INVARIANT HYPERSURFACES

Lemma 3.1. *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \tilde{M} . Then*

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X f)Y - u(Y)A_{\tilde{N}}X + h(X, Y)U + ((\nabla_X u)Y + h(X, fY))\tilde{N}, \tag{3.1}$$

$$(\tilde{\nabla}_X \eta)Y = (\nabla_X v)Y - \lambda h(X, Y), \tag{3.2}$$

$$\tilde{\nabla}_X \xi = \nabla_X V - \lambda A_{\tilde{N}}X + (h(X, V) + X\lambda)\tilde{N}, \tag{3.3}$$

for all $X, Y \in TM$.

Proof. Consider:

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y).$$

Using (2.8) and (2.13), we have

$$\begin{aligned} (\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X(fX + u(Y)\tilde{N}) - \phi(\tilde{\nabla}_X Y + h(X, Y)\tilde{N}) \\ (\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X fX + \tilde{\nabla}_X(u(Y)\tilde{N}) - \phi\tilde{\nabla}_X Y - h(X, Y)\phi\tilde{N} \\ (\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X fX + h(X, fY)\tilde{N} + u(Y)\tilde{\nabla}_X \tilde{N} + (\tilde{\nabla}_X u(Y))\tilde{N} - f(\tilde{\nabla}_X Y) \\ &\quad - u(\tilde{\nabla}_X Y)\tilde{N} + h(X, Y)U \\ (\tilde{\nabla}_X \phi)Y &= (\tilde{\nabla}_X f)X + f(\tilde{\nabla}_X X) - u(Y)A_{\tilde{N}}X + h(X, Y)U + h(X, fY)\tilde{N} \\ &\quad - f(\tilde{\nabla}_X Y) + (\tilde{\nabla}_X u(Y))\tilde{N} - u(\tilde{\nabla}_X Y)\tilde{N} \\ (\tilde{\nabla}_X \phi)Y &= (\tilde{\nabla}_X f)X - u(Y)A_{\tilde{N}}X + h(X, Y)U + h(X, fY)\tilde{N} + (\tilde{\nabla}_X u(Y))\tilde{N} \\ &\quad - u(\tilde{\nabla}_X Y)\tilde{N} \\ (\tilde{\nabla}_X \phi)Y &= (\tilde{\nabla}_X f)X - u(Y)A_{\tilde{N}}X + h(X, Y)U + h(X, fY)\tilde{N} + (\tilde{\nabla}_X u(Y)) \\ &\quad + h(X, u(Y))\tilde{N} - u(\tilde{\nabla}_X Y)\tilde{N} \\ (\tilde{\nabla}_X \phi)Y &= (\tilde{\nabla}_X f)X - u(Y)A_{\tilde{N}}X + h(X, Y)U + ((\tilde{\nabla}_X u)Y + h(X, fY))\tilde{N}. \end{aligned}$$

Also, we have

$$(\tilde{\nabla}_X \eta)Y = \tilde{\nabla}_X \eta(Y) - \eta(\tilde{\nabla}_X Y).$$

Using (2.8), (2.11) and (2.13), we have

$$\begin{aligned} (\tilde{\nabla}_X \eta)Y &= \tilde{\nabla}_X(v(Y)) - \eta(\tilde{\nabla}_X Y), \\ (\tilde{\nabla}_X \eta)Y &= \tilde{\nabla}_X(v(Y)) + h(X, v(Y))\tilde{N} - \eta(\tilde{\nabla}_X Y + h(X, Y)\tilde{N}), \\ (\tilde{\nabla}_X \eta)Y &= \tilde{\nabla}_X(v(Y) - \eta(\tilde{\nabla}_X Y) - h(X, Y)\eta(\tilde{N})), \\ (\tilde{\nabla}_X \eta)Y &= \tilde{\nabla}_X v(Y) - v(\tilde{\nabla}_X Y) - h(X, Y)\eta(\tilde{N}), \\ (\tilde{\nabla}_X \eta)Y &= (\tilde{\nabla}_X v)Y - \lambda h(X, Y). \end{aligned}$$

Further, consider using (2.13) and using (2.10), we have

$$\begin{aligned} \tilde{\nabla}_X \xi &= \tilde{\nabla}_X \xi + h(X, \xi)\tilde{N}, \\ \tilde{\nabla}_X \xi &= \tilde{\nabla}_X(V + \lambda\tilde{N}) + h(X, V + \lambda\tilde{N})\tilde{N}, \\ \tilde{\nabla}_X \xi &= \tilde{\nabla}_X V + \tilde{\nabla}_X(\lambda\tilde{N}) + h(X, V)\tilde{N} + \lambda h(X, \tilde{N})\tilde{N}, \\ \tilde{\nabla}_X \xi &= \tilde{\nabla}_X V + \lambda(\tilde{\nabla}_X \tilde{N}) + (X\lambda)\tilde{N} + h(X, V)\tilde{N}, \\ \tilde{\nabla}_X \xi &= \tilde{\nabla}_X V - \lambda A_{\tilde{N}}X + (h(X, V) + X\lambda)\tilde{N}, \end{aligned}$$

for all $X, Y \in TM$. □

Theorem 3.2. *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \tilde{M} . Then we have*

$$h(X, \xi) = \epsilon\alpha f^2 X - \epsilon\alpha u(X)U - \delta\beta fX + f(\tilde{\nabla}_X \xi), \tag{3.4}$$

$$u(\tilde{\nabla}_X \xi) = -\epsilon\alpha u(fX) + \delta\beta u(X), \tag{3.5}$$

for all $X, Y \in TM$.

Proof. Consider

$$\begin{aligned} (\tilde{\nabla}_X \phi)\xi &= \tilde{\nabla}_X \phi\xi - \phi(\tilde{\nabla}_X \xi), \\ (\tilde{\nabla}_X \phi)\xi &= -\phi(\tilde{\nabla}_X \xi). \end{aligned} \tag{3.6}$$

Using equations (2.2), (2.7), (2.8) and (2.9) in above, we have

$$\begin{aligned} (\tilde{\nabla}_X \phi)\xi &= -\phi(-\epsilon\alpha(\phi X) - \delta\beta\phi^2 X), \\ (\tilde{\nabla}_X \phi)\xi &= \phi(\epsilon\alpha(fX + u(X)\tilde{N})) + \delta\beta\phi(-X + \eta(X)\xi), \\ (\tilde{\nabla}_X \phi)\xi &= \epsilon\alpha f^2 X + \epsilon\alpha u(Xf)\tilde{N} - \epsilon\alpha u(X)U - \delta\beta fX - \delta\beta u(X)\tilde{N}. \end{aligned} \tag{3.7}$$

Using equation (2.13) in (3.6), we get

$$(\tilde{\nabla}_X \phi)\xi = -\phi(\tilde{\nabla}_X \xi) - h(X, \xi)\phi\tilde{N}.$$

Using equation (2.8) and (2.9) in above, we get

$$(\tilde{\nabla}_X \phi)\xi = -f(\tilde{\nabla}_X \xi) - u(\tilde{\nabla}_X \xi)\tilde{N} + h(X, \xi)U. \tag{3.8}$$

Comparing equation (3.7) and (3.8), we have

$$\begin{aligned}
 & -f(\tilde{\nabla}_X \xi) - u(\tilde{\nabla}_X \xi)\tilde{N} + h(X, \xi)U \\
 & = \epsilon\alpha f^2 X + \epsilon\alpha u(Xf)\tilde{N} - \epsilon\alpha u(X)U - \delta\beta fX - \delta\beta u(X)\tilde{N}.
 \end{aligned}$$

Equating tangential and normal parts on both sides, we have

$$h(X, \xi)U = \epsilon\alpha f^2 X - \epsilon\alpha u(X)U - \delta\beta fX + f(\tilde{\nabla}_X \xi)$$

and

$$u(\tilde{\nabla}_X \xi) = -\epsilon\alpha u(Xf) + \delta\beta u(X),$$

for all $X, Y \in TM$. □

Theorem 3.3. *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \tilde{M} . Then, we have*

$$\begin{aligned}
 (\nabla_X f)Y &= u(Y)A_{\tilde{N}}X - h(X, Y)U + \alpha g(X, Y)V \\
 &\quad - \epsilon\alpha v(Y)X + \beta g(fX, Y)V - \delta\beta v(Y)fX
 \end{aligned} \tag{3.9}$$

and

$$(\tilde{\nabla}_X u)Y = \lambda\alpha g(X, Y) + \lambda\beta g(fX, Y) - \delta\beta v(Y)u(X) - h(X, fY), \tag{3.10}$$

for all $X, Y \in TM$.

Proof. Consider covariant differentiation, then we have

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi(\tilde{\nabla}_X Y). \tag{3.11}$$

Using equation (2.8) in (2.13), we have

$$(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X fY + \tilde{\nabla}_X (u(Y)\tilde{N}) - \phi\tilde{\nabla}_X Y - h(X, Y)\phi\tilde{N}.$$

Using (2.8), (2.9) and (2.13), we have

$$\begin{aligned}
 (\tilde{\nabla}_X \phi)Y &= \tilde{\nabla}_X fY + h(X, fY)\tilde{N} + u(Y)(\tilde{\nabla}_X \tilde{N}) \\
 &\quad + (\tilde{\nabla}_X u(Y))\tilde{N} - f\tilde{\nabla}_X Y - u(\tilde{\nabla}_X Y)\tilde{N} + h(X, Y)U.
 \end{aligned}$$

Using (2.13) and (2.14) in above, we have

$$(\tilde{\nabla}_X \phi)Y = (\tilde{\nabla}_X f)Y - u(Y)A_{\tilde{N}}X + h(X, Y)U + ((\tilde{\nabla}_X u)Y + h(X, fY))\tilde{N}. \tag{3.12}$$

Now, using (2.8), (2.10) and (2.11) in (2.6), we have

$$\begin{aligned}
 (\tilde{\nabla}_X \phi)Y &= \alpha g(X, Y)V + \lambda\alpha g(X, Y)\tilde{N} - \epsilon\alpha v(Y)X + \beta g(fX, Y)V \\
 &\quad + \lambda\beta g(fX, Y)\tilde{N} - \delta\beta v(Y)fX - \delta\beta v(Y)u(X)\tilde{N}.
 \end{aligned} \tag{3.13}$$

Comparing (3.12) and (3.13), we have

$$\begin{aligned} (\nabla_X f)Y - u(Y)A_{\tilde{N}}X + h(X, Y)U + ((\tilde{\nabla}_X u)Y + h(X, fY))\tilde{N} \\ = \alpha g(X, Y)V + \lambda \alpha g(X, Y)\tilde{N} - \epsilon \alpha v(Y)X + \beta g(fX, Y)V \\ + \lambda \beta g(fX, Y)\tilde{N} - \delta \beta v(Y)fX - \delta \beta v(Y)u(X)\tilde{N}. \end{aligned}$$

Equating tangential and normal part, we have

$$\begin{aligned} (\nabla_X f)Y = u(Y)A_{\tilde{N}}X - h(X, Y)U + \alpha g(X, Y)V - \epsilon \alpha v(Y)X \\ + \beta g(fX, Y)V - \delta \beta v(Y)fX \end{aligned}$$

and

$$(\tilde{\nabla}_X u)Y = \lambda \alpha g(X, Y) + \lambda \beta g(fX, Y) - \delta \beta v(Y)u(X) - h(X, fY),$$

for all $X, Y \in TM$. □

Theorem 3.4. *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \tilde{M} . Then we have*

$$\tilde{\nabla}_X V = \lambda A_{\tilde{N}}X - \epsilon \alpha fX + \delta \beta X - \delta \beta v(X)V \tag{3.14}$$

and

$$h(X, V) = -\epsilon \alpha u(X) - \lambda \delta \beta v(X) - X\lambda, \tag{3.15}$$

for all $X, Y \in TM$.

Proof. Using equation (2.1), (2.8) and (2.11) in (2.7), we have

$$\tilde{\nabla}_X \xi = -\epsilon \alpha fX - \epsilon \alpha u(X)\tilde{N} + \delta \beta X - \delta \beta v(X)V - \lambda \delta \beta v(X)\tilde{N}. \tag{3.16}$$

Comparing equation (3.16) and (3.3) we have

$$\begin{aligned} \tilde{\nabla}_X V - \lambda A_{\tilde{N}}X + (h(X, V) + X\lambda)\tilde{N} \\ = -\epsilon \alpha fX - \epsilon \alpha u(X)\tilde{N} + \delta \beta X - \delta \beta v(X)V - \lambda \delta \beta v(X)\tilde{N}. \end{aligned}$$

Equating tangential and normal part, we have

$$\tilde{\nabla}_X V = \lambda A_{\tilde{N}}X - \epsilon \alpha fX + \delta \beta X - \delta \beta v(X)V$$

and

$$h(X, V) = -\epsilon \alpha u(X) - \lambda \delta \beta v(X) - X\lambda,$$

for all $X, Y \in TM$. □

Theorem 3.5. *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \tilde{M} . Then, we have*

$$\begin{aligned}
 (\tilde{\nabla}_X \phi)Y &= \alpha g(X, Y)V - \epsilon v(Y)X + \beta g(fX, Y)V - \delta \beta v(Y)fX \\
 &\quad + (\lambda \alpha g(X, Y) + \lambda \beta g(fX, Y) - \delta \beta v(Y)u(X))\tilde{N} \tag{3.17}
 \end{aligned}$$

for all $X, Y \in TM$.

Proof. Using (3.9) and (3.10) in (3.13), we have

$$\begin{aligned}
 (\tilde{\nabla}_X \phi)Y &= u(Y)A_{\tilde{N}}X - h(X, Y)U + \alpha g(X, Y)V - \epsilon \alpha v(Y)X + \beta g(fX, Y)V \\
 &\quad - \delta \beta v(Y)fX - u(Y)A_{\tilde{N}}X + h(X, Y)U + (\lambda \alpha g(X, Y) + \lambda \beta g(fX, Y) \\
 &\quad - \delta \beta v(Y)u(X) - h(X, fY) + h(X, fY))\tilde{N}, \\
 (\tilde{\nabla}_X \phi)Y &= \alpha g(X, Y)V - \epsilon \alpha v(Y)X + \beta g(fX, Y)V - \delta \beta v(Y)fX + (\lambda \alpha g(X, Y) \\
 &\quad + \lambda \beta g(fX, Y) - \delta \beta v(Y)u(X))\tilde{N}, \\
 (\tilde{\nabla}_X \phi)Y &= \alpha \{g(X, Y)V - \epsilon v(Y)X\} + \beta \{g(fX, Y)V - \delta \beta v(Y)fX\} \\
 &\quad + (\lambda \alpha g(X, Y) + \lambda \beta g(fX, Y) - \delta \beta v(Y)u(X))\tilde{N},
 \end{aligned}$$

for all $X, Y \in TM$. □

Theorem 3.6. *Let M be a totally umbilical non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \tilde{M} . Then it is totally geodesic if and only if*

$$\epsilon \alpha u(X) + \lambda \delta \beta v(X) + X\lambda = 0, \tag{3.18}$$

for all $X, Y \in TM$.

Proof. Using equation (2.1), (2.8) and (2.11) in (2.7), we have

$$\tilde{\nabla}_X \xi = -\epsilon \alpha fX - \epsilon \alpha u(X)\tilde{N} + \delta \beta X - \delta \beta v(X)V - \lambda \delta \beta v(X)\tilde{N}.$$

Using (3.3) in above equation, we have

$$\begin{aligned}
 \tilde{\nabla}_X V - \lambda A_{\tilde{N}}X + (h(X, V) + X\lambda)\tilde{N} &= -\epsilon \alpha fX - \epsilon \alpha u(X)\tilde{N} + \delta \beta X \\
 &\quad - \delta \beta v(X)V - \lambda \delta \beta v(X)\tilde{N}.
 \end{aligned}$$

Equating normal part, we have

$$h(X, V) = -\epsilon \alpha u(X) - \lambda \delta \beta v(X) - X\lambda. \tag{3.19}$$

If M is totally umbilical, then $A_{\tilde{N}} = \zeta I$, where ζ is Kahlerian metric

$$\begin{aligned}
 h(X, Y) &= g(A_{\tilde{N}}X, Y) = g(\zeta X, Y) = \zeta g(X, Y) = \zeta v(X), \\
 h(X, V) &= \zeta g(X, V) = \zeta v(X). \tag{3.20}
 \end{aligned}$$

Then, from (3.19) and (3.20), we, have

$$\epsilon\alpha u(X) + \lambda\delta\beta v(X) + X\lambda + \zeta v(X) = 0. \tag{3.21}$$

If M is totally geodesic, that is, $\zeta = 0$, then from (3.21), we have

$$\epsilon\alpha u(X) + \lambda\delta\beta v(X) + X\lambda = 0,$$

for all $X, Y \in TM$. □

Theorem 3.7. *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of (ϵ, δ) -trans Sasakian manifold \widetilde{M} . If U is parallel, then we have*

$$\epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) = 0, \tag{3.22}$$

for all $X, Y \in TM$.

Proof. Consider covariant differentiation, then we have

$$(\widetilde{\nabla}_X\phi)\widetilde{N} = \widetilde{\nabla}_X\phi\widetilde{N} - \phi(\widetilde{\nabla}_X\widetilde{N}). \tag{3.23}$$

Using equation (2.8), (2.9), (2.13) and (2.14) in above, we have

$$\begin{aligned} (\widetilde{\nabla}_X\phi)\widetilde{N} &= \nabla_X\phi\widetilde{N} + h(X, \phi\widetilde{N})\widetilde{N} - f(\widetilde{\nabla}_X\widetilde{N}) - u(\widetilde{\nabla}_X\widetilde{N})\widetilde{N}, \\ (\widetilde{\nabla}_X\phi)\widetilde{N} &= -\nabla_XU + f(A_{\widetilde{N}}X). \end{aligned} \tag{3.24}$$

From (2.6), we have

$$\begin{aligned} (\widetilde{\nabla}_X\phi)\widetilde{N} &= \alpha\{g(X, \widetilde{N})\xi - \epsilon\lambda X\} + \beta\{g(\phi X, \widetilde{N})\xi - \delta\lambda\phi X\}, \\ (\widetilde{\nabla}_X\phi)\widetilde{N} &= -\epsilon\alpha\lambda X - \beta\delta\lambda(fX) - \beta\delta\lambda u(X)\widetilde{N}. \end{aligned} \tag{3.25}$$

From (3.25) and (3.26), we have

$$\begin{aligned} -\nabla_XU + f(A_{\widetilde{N}}X) &= -\epsilon\alpha\lambda X - \beta\delta\lambda(fX) - \beta\delta\lambda u(X)\widetilde{N}, \\ \nabla_XU &= \epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) + \beta\delta\lambda u(X)\widetilde{N}. \end{aligned}$$

If U is parallel, then $\nabla_XU = 0$, so from above equation, we have

$$\epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) + \beta\delta\lambda u(X)\widetilde{N} = 0.$$

Now, equating tangential part, we have

$$\epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) = 0,$$

for all $X, Y \in TM$. □

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