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INERTIAL PICARD NORMAL S-ITERATION PROCESS

Samir Dashputre[1], Padmavati[2] and Kavita Sakure[3]

[1]Department of Mathematics
Govt. College, Arjunda Balod 491225, India
e-mail: samir231973@gmail.com

[2]Department of Mathematics
Govt. V.Y.T. Autonomous P.G. College, Durg 491001, India
e-mail: padmavati.sudha62@gmail.com

[3]Department of Mathematics
Govt. Digvijay Autonomous P.G. College, Rajnandgaon 491441, India
e-mail: kavitaaage@gmail.com

Abstract. Many iterative algorithms like that Picard, Mann, Ishikawa and S-iteration are very useful to elucidate the fixed point problems of a nonlinear operators in various topological spaces. The recent trend for elucidate the fixed point via inertial iterative algorithm, in which next iterative depends on more than one previous terms. The purpose of the paper is to establish convergence theorems of new inertial Picard normal S-iteration algorithm for nonexpansive mapping in Hilbert spaces. The comparison of convergence of InerNSP and InerPNSP is done with InerSP (introduced by Phon-on *et al.* [25]) and MSP (introduced by Suparatulatorn *et al.* [27]) via numerical example.

1. Introduction

Fixed point theory plays very crucial role in the fields of pure and applied mathematics as well as in many other branches of science (see [14, 15, 19, 29, 30] and references therein). One of the most fundamental problems in the operator theory is to find fixed points of nonlinear operators (see [8–10]). Many

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[0]Corresponding author: Kavita Sakure(kavitaaage@gmail.com).

problems arising in different areas such as image reconstruction and signal processing [6], variational inequalities [22], convex feasibility problems [3] can be modeled in the form of fixed point problems:

Find $x \in C$ such that

$$Tx = x, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is a nonlinear operator. The solution set of the fixed point problem (1.1) is denoted by $Fix(T)$. As we know, many literatures have been published in both direct and iterative technique to find the fixed points of nonexpansive mappings. The iterative technique is used to solve problems in information theory, game theory, optimization etc., by formulating them into fixed point problems. One of the most used iterative techniques was introduced by Mann [20], which is given as follows:

For any initial point $x_1 \in C$,

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n, \ \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{a_n\}$ is a real sequence in $(0, 1)$. If T is a nonexpansive mapping and iteration parameter a_n satisfies the condition $\sum_{n=0}^{\infty} a_n(1 - a_n) = \infty$. Then, sequence $\{x_n\}$ defined by (1.2) converges weakly to a fixed point of T.

It is well known that the Mann iteration method for the approximation of fixed points of pseudocontractive mappings may not well behave (see [7]). To become free of this problem, Ishikawa [16] introduced an iterative technique, which is extensively studied for the approximation of fixed points of pseudocontractive and nonexpansive mappings by many authors in different spaces (see for example Takahashi *et al.* [28] and Dotson [13]). Agarwal *et al.* [1] introduced an iteration method which is called S-iteration method. Its convergence rate is faster than both Mann and Ishikawa iteration method for contraction mappings. The S-iteration algorithm defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \ \forall n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) = \infty$. The algorithmic design of S-iteration method (1.3) is comparatively different and independent of Mann and Ishikawa iteration methods, that is, neither Mann nor Ishikawa iteration can be reduced into S-iteration and vice-versa. In 2011, [26] introduced another form of S-iteration, named as normal S-iteration method which is defined by

$$\begin{aligned} x_{n+1} &= T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \ \forall n \in \mathbb{N}, \end{aligned} \tag{1.4}$$

where $\{\beta_n\}$ is sequence in $(0, 1)$. Normal S-iteration (1.4) is also known as Hybrid-Picard Mann iteration method [18].

The particular interest is an inertial iteration methods, which are used in computing fixed points for nonexpansive mappings from the algorithms mentioned above, we observe that the next iterate of algorithms depends on the previous iterate only, but the defining property of inertial method is that the next iterate depends on more than one previous iterates.

The inertial Mann algorithm, the combination of inertial extrapolation and Mann algorithm, is introduced by Mainge [21] in 2008. Nakajo and Takahashi [23] introduced the CQ-algorithm, the modification of Mann algorithm. Then Dong *et al.* [12] introduced modified inertial Mann algorithm, combination of inertial extrapolation and modified Mann algorithm and inertial CQ-algorithm, combination of inertial extrapolation and CQ-algorithm. In 2019, Phon-on *et al.* [25] introduced the inertial S-iteration process, combination of inertial extrapolation and modified S-iteration process, whereas modified S-iteration process is introduced by Suparatulatorn *et al.* [27].

In Section 2, we present some basic definitions and results. We introduce the inertial Picard normal S-iteration process (InerPNSP), combination of inertial extrapolation and Picard normal S-iteration process, whereas Picard normal S-iteration process is introduced by Kadioglu *et al.* [17] and present the weak and strong convergence of InerPNSP in Section 3. Also, we introduce the inertial normal S-iteration process (InerNSP), combination of inertial extrapolation and normal S-iteration process, whereas normal S-iteration process is introduced by Sahu [26] and present weak convergence of InerNSP in Section 4. The rate of convergence of InerPNSP, InerNSP, MSP and InerSP is discussed via numerical example in Section 5.

2. Preliminaries

In this section, we summarize notations, definitions and lemmas which play significant role in convergence analysis of our algorithm. Let X be a Banach space with the norm $\|\cdot\|$. Throughout this paper, we adopt the following notations:

- $x_n \to x$ stands the strong convergence of sequence $\{x_n\}$ to x.
- $x_n \rightharpoonup x$ stands the weak convergence of sequence $\{x_n\}$ to x.
- A set of fixed points of mapping $T : X \to X$ is denoted by

$$Fix(T) = \{x \in X : Tx = x\}.$$

- Assume that X is a Banch space and C is a nonempty subset of Banach space X. Then a mapping $T : C \to C$ is said to be nonexpansive, if

for all $x, y \in X$
$$\|Tx - Ty\| \leq \|x - y\|.$$

Lemma 2.1. ([4]) *Assume that H is a real Hilbert space. Then following inequality holds:*
$$||cx + (1-c)y||^2 \leq c||x||^2 + (1-c)||y||^2 - c(1-c)||x-y||^2, \tag{2.1}$$
where $c \in \mathbb{R}, x, y \in H$.

Lemma 2.2. ([23]) *Assume that X is a uniformly convex Banach space and $\{s_n\}$ is sequence in $[\delta, 1-\delta]$ for $\delta \in (0,1)$. Assume that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $\liminf_{n\to\infty} \|x_n\| \leq c, \liminf_{n\to\infty} \|y_n\| \leq c$, and $\liminf_{n\to\infty} \|s_n x_n + (1-s_n)y_n\| = c$ for some $c \geq 0$. Then*
$$\liminf_{n\to\infty} \|x_n - y_n\| = 0.$$

Definition 2.3. ([24]) Assume that X is a Banach space and sequence $\{x_n\}$ in X converges weakly to x. Then X is said to have Opial's property, if
$$\liminf_{n\to\infty} \|x_n - x\| \leq \liminf_{n\to\infty} \|x_n - y\|$$
for all $y \in X, y \neq x$.

Lemma 2.4. ([11]) *Assume that X is a Banach space with Opial's property. Assume that $\{x_n\}$ is a sequence in X and $x, y \in H$ such that $\lim_{n\to\infty} \|x_n - x\|$ and $\lim_{n\to\infty} \|x_n - y\|$ exist. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are two subsequences of $\{x_n\}$ converge to x and y, respectively. Then $x = y$.*

Lemma 2.5. ([2]) *Assume that $\{r_n\}$, $\{d_n\}$ and $\{q_n\}$ are sequences in $[0,\infty)$ such that*
$$r_{n+1} \leq r_n + q_n(r_n - r_{n-1}) + d_n$$
for all $n \geq 1$, $\sum_{n=1}^{\infty} d_n < \infty$ and there is real number q with $q_n < q < 1$ for all $n \geq 1$. Then

(1) $\sum_{n\geq 1}[r_n - r_{n-1}]_+ < \infty$ *where* $[a]_+ = \max\{a, 0\}$.
(2) *there is* $r^* \in [0,\infty)$ *such that* $\lim_{n\to\infty} r_n = r^*$.

Lemma 2.6. ([4]) *Assume that C is a nonempty convex closed subset of a Hilbert space H and $T : C \to H$ is a nonexpansive mapping. Assume that $\{x_n\}$ is a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ as $n \to \infty$. Then $x \in Fix(T)$.*

Lemma 2.7. ([4]) *Assume that C is a nonempty subset of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C and $x \in H$ such that the following conditions holds:*

(1) *for all* $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ *exists.*
(2) *every sequential weak cluster point of* $\{x_n\}$ *is in* C.

Then, the sequence $\{x_n\}$ *converges weakly to a point in* C.

We can compare the rate of convergence of two iterative methods by the following result of Berinde [5].

Definition 2.8. Assume that $\{s_n\}$ and $\{t_n\}$ are two sequences of nonnegative numbers which converge to s and t, respectively. Also, assume that

$$\lim_{n\to\infty} \frac{|s_n - s|}{|t_n - t|} = L.$$

(1) If $L = 0$, then the sequence $\{s_n\}$ converges to s is faster than sequence $\{t_n\}$ converges to t.
(2) If $0 < L < \infty$, then sequences $\{s_n\}$ and $\{t_n\}$ have same rate of convergence.

3. Inertial Picard normal S-iteration process and its convergence analysis

In this section, we introduce Inertial Picard normal S-iteration process (InerPNSP), by combining the inertial extrapolation and Picard normal S-iteration process and study convergence analysis for finding fixed points of nonexpansive mapping in the framework of a Hilbert space.

First we introduce our InerPNSP algorithm.

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$.

InerPNSP Algorithm:

(1) **Initialization:** Select x_0, x_1 arbitrarily.
(2) **Iteration Step:** Select $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ as iteration parameters in $[0, 1]$ and compute $(n+1)^{th}$ iterative term as follows:

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}), \\ z_n = (1-\beta_n)w_n + \beta_n T(w_n), \\ y_n = (1-\alpha_n)z_n + \alpha_n T(z_n), \\ x_{n+1} = T y_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ satisfy:

(A1) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\gamma_n \subset [0, \gamma], 0 \leq \gamma < 1, \{\alpha_n\}, \{\beta\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 0.5)$;

(A2) $\{T(w_n) - w_n\}$ is bounded;
(A3) $\{T(w_n) - p\}$ is bounded.

Theorem 3.1. *Let X be a Hilbert space. Let $p \in F = Fix(T)$. Let the sequence $\{x_n\}$ generated by (3.1) satisfying condition $(A_1), (A_2)$ and (A_3). Then*

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*
(2) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$

Proof. Since T is a nonexpansive mapping, by triangular inequality,

$$\begin{aligned} \|z_n - p\| &= \|(1-\beta_n)w_n + \beta_n T(w_n) - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n\|T(w_n) - p\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\| \\ &= \|w_n - p\|, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2),

$$\begin{aligned} \|y_n - p\| &= \|(1-\alpha_n)z_n + \alpha_n T(z_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|T(z_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \\ &\leq \|w_n - p\|. \end{aligned} \tag{3.3}$$

Using (3.1), (3.2) and (3.3),

$$\begin{aligned} \|x_{n+1} - p\| &= \|T(y_n) - p\| \\ &\leq ||w_n - p||. \end{aligned} \tag{3.4}$$

Now we will prove $\{w_n - p\}$ is bounded. By condition (A_1) and (A_2),

$$\begin{aligned} \|w_n - p\| &= \|w_n - T(w_n) + T(w_n) - p\| \\ &\leq \|T(w_n) - w_n\| + \|T(w_n) - p\| \\ &\leq M \end{aligned}$$

for some $M \in (0, \infty)$. Thus $\{w_n - p\}$ is bounded and by (3.4), $\{x_n - p\}$ and $\{x_n - x_{n-1}\}$ are bounded. By (2.1),

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \gamma_n(x_n - x_{n-1} - p)\|^2 \\ &= \|(1+\gamma_n)(x_n - p) - \gamma_n(x_{n-1} - p)\|^2 \\ &= (1+\gamma_n)\|x_n - p\|^2 - \gamma_n\|x_{n-1} - p\|^2 \\ &\quad + \gamma_n(1+\gamma_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 \\ &= (1+\gamma_n)\|x_n - p\|^2 - \gamma_n\|x_{n-1} - p\|^2 \\ &\quad + \gamma_n(1+\gamma_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.6}$$

Let $r_n = \|x_n - p\|^2$. Then by (3.6)

$$r_{n+1} \leq r_n + \gamma_n(r_n - r_{n-1}) + d_n,$$

where $d_n = \gamma_n(1+\gamma_n)\|x_n - x_{n-1}\|^2$.
By condition (A_1),

$$\begin{aligned} \sum_{n=1}^{\infty} d_n &= \sum_{n=1}^{\infty} \gamma_n(1+\gamma_n)\|x_n - x_{n-1}\|^2 \\ &\leq \sum_{n=1}^{\infty} \gamma(1+\gamma)(2M)^2 \\ &< \infty. \end{aligned}$$

From Lemma 2.5, there is $r^* \in [0,\infty)$ such that $\lim_{n\to\infty} r_n = r^*$. Therefore $\lim_{n\to\infty} \|x_n - p\|^2$ exists and hence $\lim_{n\to\infty} \|x_n - p\|$ exists.

Now we will prove $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.
Assume that $c = \lim_{n\to\infty} \|x_n - p\|$. Since T is nonexpansive,

$$\begin{aligned} \|x_n - T(x_n)\| &\leq \|x_n - p\| + \|T(x_n) - p\| \\ &\leq \|x_n - p\| + \|x_n - p\| \\ &= 2\|x_n - p\|. \end{aligned} \tag{3.7}$$

If $c = 0$, then by (3.7), $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.
Assume that $c > 0$. Now $\sum_{n=1}^{\infty} d_n < \infty$ implies that $\lim_{n\to\infty} d_n = 0$. From (3.5), we have

$$\begin{aligned} \lim_{n\to\infty} \|w_n - p\|^2 &= \lim_{n\to\infty} \big((1+\gamma_n)\|x_n - p\|^2 - \gamma_n\|x_{n-1} - p\|^2 \\ &\quad + \gamma_n(1+\gamma_n)\|x_n - x_{n-1}\|^2\big) \\ &= \lim_{n\to\infty} \|x_n - p\|^2 \\ &= c^2, \end{aligned}$$

which implies $\lim_{n\to\infty} \|w_n - p\| = c$. Therefore

$$\begin{aligned} \limsup_{n\to\infty} \|y_n - p\| &\leq \limsup_{n\to\infty} \|w_n - p\| \\ &= c. \end{aligned} \tag{3.8}$$

Now we claim that $\liminf_{n\to\infty} \|y_n - p\| \geq c$. Since T is a nonexpansive mapping, by (3.1)

$$\begin{aligned} \|x_{n+1} - p\| &= \|T(y_n) - p\| \\ &\leq \|y_n - p\|. \end{aligned}$$

On taking limit inferior both sides

$$\liminf_{n\to\infty} \|x_{n+1} - p\| \leq \liminf_{n\to\infty} \|y_n - p\|$$

and hence

$$c \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{3.9}$$

By (3.8) and (3.9),

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Now, by (3.2),

$$\begin{aligned} \limsup_{n\to\infty} \|z_n - p\| &\leq \limsup_{n\to\infty} \|w_n - p\| \\ &= c. \end{aligned} \tag{3.10}$$

Since T is a nonexpansive mapping, by (3.1)

$$\begin{aligned} \|x_{n+1} - p\| &= \|T(y_n) - p\| \\ &\leq \|y_n - p\| \\ &\leq \|(1-\alpha_n)z_n + \alpha_n T(z_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|T(z_n) - p\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\| \\ &= \|z_n - p\|. \end{aligned}$$

On taking limit inferior both sides

$$\liminf_{n\to\infty} \|x_{n+1} - p\| \leq \liminf_{n\to\infty} \|z_n - p\|$$

and hence

$$c \leq \liminf_{n\to\infty} \|z_n - p\|. \tag{3.11}$$

By (3.10) and (3.11),

$$\lim_{n\to\infty} \|z_n - p\| = c.$$

Now

$$\limsup_{n\to\infty} \|T(w_n) - p\| \leq \limsup_{n\to\infty} \|w_n - p\| \leq c,$$

$$\limsup_{n\to\infty} \|(1-\beta_n)(w_n - p) + \beta_n(T(w_n) - p)\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq c,$$

by Lemma 2.2,

$$\lim_{n\to\infty} \|T(w_n) - w_n\| = 0. \tag{3.12}$$

Now

$$\limsup_{n\to\infty} \|T(z_n) - p\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq c,$$

$$\limsup_{n\to\infty} \|(1-\alpha_n)(z_n - p) + \alpha_n(T(z_n) - p)\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq c,$$

by Lemma 2.2,

$$\lim_{n\to\infty} \|T(z_n) - z_n\| = 0.$$

Now, since $z_n - w_n = \beta_n(T(w_n) - w_n)$, by (3.12)

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|z_n - w_n\| \\ &= \lim_{n\to\infty} \beta_n \|T(w_n) - w_n\| \\ &\leq \lim_{n\to\infty} \|(T(w_n) - w_n)\| \\ &= 0. \end{aligned} \tag{3.13}$$

Now, since $y_n - z_n = \alpha_n(T(z_n) - z_n)$, by (3)

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|y_n - z_n\| \\ &= \lim_{n\to\infty} \alpha_n \|T(z_n) - z_n\| \\ &\leq \lim_{n\to\infty} \|(T(z_n) - z_n)\| \\ &= 0. \end{aligned} \tag{3.14}$$

Now, by (3.13) and (3.14),

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|y_n - w_n\| \\ &\leq \lim_{n\to\infty} \|y_n - z_n\| + \lim_{n\to\infty} \|z_n - w_n\| \\ &= 0. \end{aligned} \tag{3.15}$$

Now, since $w_n - x_n = \gamma_n(x_n - x_{n-1})$, by (3.4)

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|w_n - x_n\| \\ &= \lim_{n\to\infty} \gamma_n \|x_n - x_{n-1}\| \\ &= 0. \end{aligned} \tag{3.16}$$

Now, using (3.12) and (3.16),

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|Tw_n - x_n\| \\ &\leq \lim_{n\to\infty} \|Tw_n - w_n\| + \lim_{n\to\infty} \|w_n - x_n\| \\ &= 0. \end{aligned} \tag{3.17}$$

Using (3.15) and (3.16),

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \|x_n - y_n\| \\ &\leq \lim_{n\to\infty} \|x_n - w_n\| + \lim_{n\to\infty} \|w_n - y_n\| \\ &= 0. \end{aligned} \tag{3.18}$$

Now, since T is a nonexpansive and using (3.13), (3.14), (3.17), (3.18), we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \|T(x_n) - x_n\| \\
&= \lim_{n\to\infty} \|T(x_n) - T(y_n)\| + \lim_{n\to\infty} \|T(y_n) - T(z_n)\| \\
&\quad + \lim_{n\to\infty} \|T(z_n) - T(w_n)\| + \lim_{n\to\infty} \|T(w_n) - x_n\| \\
&= \lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} \|y_n - z_n\| + \lim_{n\to\infty} \|z_n - w_n\| \\
&\quad + \lim_{n\to\infty} \|T(w_n) - x_n\| \\
&= 0.
\end{aligned}$$

Therefore, we have $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$. □

Theorem 3.2. *Assume that H is a Hilbert space. Also assume that $T : H \to H$ is a nonexpansive mapping with $F = Fix(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by* (3.1) *weakly converges to fixed point of T.*

Proof. Assume that $p \in F$. Then from Theorem 3.1, $\lim_{n\to\infty} ||x_n - p||$ exists, therefore $\{x_n\}$ is bounded. Assume that $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are two subsequences of the seqeunce $\{x_n\}$ with weak limits p_1 and p_2, respectively. Again by Theorem 3.1, $\lim_{n\to\infty} ||x_{n_i} - T(x_{n_i})|| = 0$ and $\lim_{n\to\infty} ||x_{n_j} - T(x_{n_j})|| = 0$. Since every Hilbert space has Opial's property (see [24]) and by Lemma 2.6, $T(p_1) = p_1$ and $T(p_2) = p_2$, that is, $p_1, p_2 \in F$. From Theorem 3.1, $\lim_{n\to\infty} ||x_n - p_1||$ and $\lim_{n\to\infty} ||x_n - p_2||$ exist and both sequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ weakly converge to p_1 and p_2, respectively. From Lemma 2.4, $p_1 = p_2$. Thus $\{x_n\}$ converges weakly to fixed point of T. □

4. Inertial normal S-iteration process and its convergence analysis

In this section, we introduce Inertial normal S-iteration process (InerNSP), by combining the inertial extrapolation and normal S-iteration process and study convergence analysis for finding fixed points of nonexpansive mapping in the framework of Hilbert space.

First we introduce our InerNSP algorithm.

Let C be a nonempty closed convex subset of real Hilbert space H and $T : C \to C$ be nonexpansive mapping with $Fix(T) \neq \emptyset$.

InerNSP Algorithm:

(1) **Initialization:** Select x_0, x_1 arbitrarily.

(2) **Iteration Step:** Select $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ as iteration parameters in $[0,1]$ and compute $(n+1)^{th}$ iterative term as follows:

$$\begin{cases} w_n = x_n + \gamma_n(x_n - x_{n-1}), \\ y_n = (1-\alpha_n)w_n + \alpha_n T(w_n), \\ x_{n+1} = Ty_n, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\gamma_n\}$ satisfy:

(B1) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\gamma_n \subset [0,\gamma], 0 \leq \gamma < 1, \{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 0.5)$;

(B2) $\{T(w_n) - w_n\}$ is bounded;

(B3) $\{T(w_n) - p\}$ is bounded.

Theorem 4.1. *Let X be a Hilbert space. Let $p \in F = Fix(T)$. Let the sequence $\{x_n\}$ generated by (4.1) satisfying condition $(B_1), (B_2)$ and (B_3). Then*

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists.*
(2) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0.$

Proof. On setting $\beta_n = 0$ in Theorem 3.1, we get the desired result. □

Theorem 4.2. *Assume that H is a Hilbert space. Also assume that $T : H \to H$ is a nonexpansive mapping with $F = Fix(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by (4.1) weakly converges to a fixed point of T.*

Proof. The proof follows from Theorem 3.2. □

5. Numerical result

In this section, we present a numerical example to find the fixed point of nonexpansive mapping via inertial Picard normal S-iteration process (3.1).

Example 5.1. Let us consider the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(u,v) = \left(6 + \frac{u}{7}, 4 + \frac{v}{5}\right)$$

and euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^2$. The mapping T is a nonexpansive mapping. Indeed,

$$\begin{aligned} \|T(u_1,v_1) - T(u_2,v_2)\|_2 &= \left\|\left(6 + \frac{u_1}{7}, 4 + \frac{v_1}{5}\right) - \left(6 + \frac{u_2}{7}, 4 + \frac{v_2}{5}\right)\right\|_2 \\ &= \sqrt{\left(\frac{u_1 - u_2}{7}\right)^2 + \left(\frac{v_1 - v_2}{5}\right)^2} \\ &\leq \sqrt{(u_1 - u_2)^2 + (u_1 - u_2)^2} \\ &= \|(u_1,v_1) - (u_2,v_2)\|_2. \end{aligned}$$

The point $p = (7,5)$ is fixed point of mapping T. Assume that $\alpha_n = \beta_n = \frac{2n}{3n+21}$, $\gamma_n = \begin{cases} \frac{1}{(n+1)^2} & n < 10^9 \\ 0.15 & n \geq 10^9 \end{cases}$ and initial guesses are $x_0 = (600, -5)$ and $x_1 = (900, 10)$.

Table 1. Comparison of convergence of inertial Picard normal S-iteration process, inertial normal S-iteration process, modified S-iteration process and inertial S-iteration process

n	**InerPNSP**	**InerNSP**	**MSP**	**InerSP**
1	(900,10)	(900,10)	(900,10)	(900,10)
2	(177.4285714, 9)	(177.4285714, 9)	(134.5714286, 6)	(177.4285714,9)
3	(5.741826322, 5.653333333)	(5.645043732, 5.7)	(23.00066812, 5.183333333)	(5.549501944, 5.745833333)
4	(4.785991504, 5.043742763)	(4.450542359, 5.058765432)	(9.01375912, 5.033652898)	(4.119769899, 5.075502464)
5	(6.777000325, 5.00079639)	(6.689390591, 5.00313963)	(7.257121648, 5.006200872)	(6.590318015, 5.006506274)
6	(6.987146952, 4.999880259)	(6.974982589, 5.000147444)	(7.03295356, 5.001138952)	(6.957828452, 5.000714057)
7	(6.999418204, 4.999982434)	(6.998140484, 5.000010007)	(7.004208033, 5.000207654)	(6.995735861, 5.000103803)
8	(6.999974339, 4.99999824)	(6.999854125, 5.000001086)	(7.000533379, 5.000037503)	(6.999541819, 5.000016886)
9	(6.999998763, 4.999999837)	(6.999987847, 5.000000139)	(7.000066987, 5.000006704)	(6.999948466, 5.00000283)
10	(6.999999935, 4.999999985)	(6.999998957, 5.000000018)	(7.000008329, 5.000001186)	(6.999994075, 5.000000477)
11	(6.999999997, 4.999999999)	(6.99999991, 5.000000002)	(7.000001025, 5.000000208)	(6.999999313, 5.000000081)
12	(7,5)	(6.999999992, 5)	(7.000000125, 5.000000036)	(6.99999992, 5.000000014)
13		(6.999999999, 5)	(7.000000015, 5.000000006)	(6.999999991, 5.000000002)
14		(7,5)	(7.000000002, 5.000000001)	(6.999999999, 5)
15			(7,5)	(7,5)

TABLE 2. Error estimates of InerPNSP, InerNSP, MSP and InerSP

n	**InerPNSP**	**InerNSP**	**MSP**	**InerSP**
1	893.0139977	893.0139977	893.0139977	893.0139977
2	170.4755055	170.4755055	127.5753479	170.4755055
3	1.417690180	1.525092289	16.00171839	1.631015626
4	2.214440572	2.550134827	2.014040295	2.881219543
5	0.223001097	0.310625277	0.257196409	0.409733646
6	0.012853605	0.025017845	0.032973237	0.042177593
7	0.000582061	0.001859543	0.004213153	0.004265403
8	0.000001248	0.000145880	0.000534696	0.000458492
9	0.000000066	0.000012154	0.000067322	0.000051612
10	0.000000004	0.000001043	0.000008413	0.000005944
11	0	0.000000090	0.000001046	0.000000692
12		0.000000008	0.000000130	0.000000081
13		0	0.000000016	0.000000010
14			0.000000002	0.000000001
15			0	0

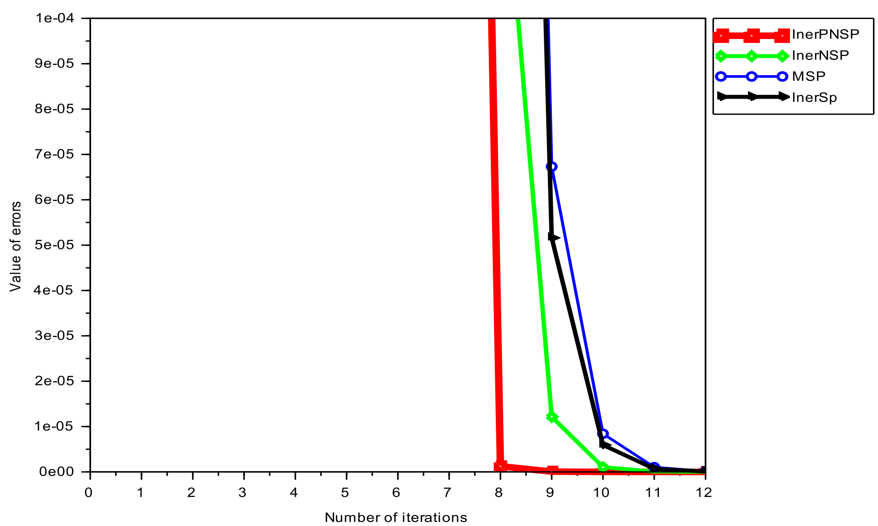

FIGURE 1. Comparison among errors

From Table 1 and Figure 1, it is clear that InerPNSP takes less iterations to approximate fixed point $p = (7, 5)$ of nonexpansive mapping T defined in Example 1 than InerNSP, MSP and InerSP. Assume that $\{x_n\}, \{s_n\}, \{t_n\}$ and $\{u_n\}$ are the sequences generated by InerPNSP, InerNSP, MSP and InerSP, respectively. From Table 2,

(1) $||x_n - p||_2 \leq ||s_n - p||_2$ for all $n \geq 2$ and $\lim_{n\to\infty} \frac{||x_n-p||_2}{||s_n-p||_2} = 0$, therefore $\{x_n\}$ converges faster than $\{s_n\}$,

(2) $||x_n - p||_2 \leq ||t_n - p||_2$ for all $n \geq 2$ and $\lim_{n\to\infty} \frac{||x_n-p||_2}{||t_n-p||_2} = 0$, therefore $\{x_n\}$ converges faster than $\{t_n\}$,

(3) $||x_n - p||_2 \leq ||u_n - p||_2$ for all $n \geq 2$ and $\lim_{n\to\infty} \frac{||x_n-p||_2}{||u_n-p||_2} = 0$, therefore $\{x_n\}$ converges faster than $\{u_n\}$,

(4) $||s_n - p||_2 \leq ||t_n - p||_2$ for all $n \geq 4$ and $\lim_{n\to\infty} \frac{||s_n-p||_2}{||t_n-p||_2} = 0$, therefore $\{s_n\}$ converges faster than $\{t_n\}$,

(5) $||s_n - p||_2 \leq ||u_n - p||_2$ for all $n \geq 2$ and $\lim_{n\to\infty} \frac{||s_n-p||_2}{||u_n-p||_2} = 0$, therefore $\{s_n\}$ converges faster than $\{u_n\}$.

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