Nonlinear Functional Analysis and Applications
Vol. 16, No. 1 (2011), pp. 9-16

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON APPROXIMATE N-RING HOMOMORPHISMS AND N-RING DERIVATIONS

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Abstract. In this paper, we investigate the Hyers-Ulam-Rassias stability of n-ring homomorphisms and n-ring derivations on Banach algebras.

1. Introduction and statement of results

Let A, B be two rings (algebras). An additive (linear) map $h : A \to B$ is called a n-ring homomorphism (n-homomorphism) if $h(\Pi_{i=1}^n a_i) = \Pi_{i=1}^n h(a_i)$, for all $a_1, a_2, \cdots, a_n \in A$. The concept of n-homomorphisms was studied for complex algebras by Hejazian, Mirzavaziri, and Moslehian [12] (see also [7], [9], [10], [22]).

Let A be a ring and let X be an $A-$module. An additive map $D : A \to X$ is called an n-ring derivation if

$$D(\Pi_{i=1}^n a_i) = D(a_1)a_2 \cdots a_n + a_1 D(a_2) a_3 \cdots a_n + \cdots + a_1 a_2 \cdots a_{n-1} D(a_n),$$

for all $a_1, a_2, \cdots, a_n \in A$. A 2-ring derivation is then a ring derivation, in the usual sense, from an algebra into its module. Furthermore, every ring derivation is clearly also an n-ring derivation for all $n \geq 2$, but the converse is not true, in general. For instance, let

$$\mathcal{A} := \begin{bmatrix} 0 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[0]Received March September 3, 2009. Revised October 11, 2009.
[0]2000 Mathematics Subject Classification: Primary 39B52, Secondary 39B82, 46H25.
[0]Keywords: Hyers-Ulam-Rassias stability, n-ring derivation, n-ring homomorphism, Banach algebra.

then $\mathcal{A}$ is an algebra equipped with the usual matrix-like operations. It is easy to see that

$$\mathcal{A}^3 \neq 0 = \mathcal{A}^4.$$

Then every additive map $f : \mathcal{A} \to \mathcal{A}$ is a 4-ring derivation.

We say that a functional equation (*) is stable if any function f approximately satisfying the equation (*) is near to an exact solution of (*). Such a problem was formulated by S. M. Ulam [26] in 1940 and solved in the next year for the Cauchy functional equation by D. H. Hyers [13] in the framework of Banach spaces. Later, T. Aoki [2] and Th. M. Rassias [25] considered mappings f from a normed space into a Banach space such that the norm of the Cauchy difference $f(x + y) - f(x) - f(y)$ is bounded by the expression

$$\epsilon(\|x\|^p + \|y\|^p)$$

for all x, y and some $\epsilon \geq 0$ and $p \in [0, 1)$. The terminology "Hyers-Ulam-Rassias stability" was indeed originated from Th. M. Rassias's paper [25] (see also [8], [23], [15], [18]).

D. G. Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [3, 5, 14, 6, 16, 20, 23, 24] and references therein.

It seems that approximate derivations was first investigated by K.-W. Jun and D.-W. Park [17]. Recently, the stability of derivations have been investigated by some authors; see [1, 4, 11, 17, 19, 21] and references therein. In this paper we investigate the Hyers-Ulam-Rassias stability of n-ring homomorphisms and n-ring derivations.

2. Main results

We start our work with a result concerning approximate n-ring homomorphisms, which can be regarded as an extension of Theorem 1 of [3].

Theorem 2.1. *Let A be a ring, B be a Banach algebra and let δ and ε be nonnegative real numbers. Suppose f is a mapping from A to B such that*

$$\|f(a + b) - f(a) - f(b)\| \leq \varepsilon \tag{2.1}$$

and that

$$\|f(\Pi_{i=1}^n a_i) - \Pi_{i=1}^n f(a_i)\| \leq \delta \tag{2.2}$$

for all $a, b, a_1, a_2, ..., a_n \in A$. Then there exists a unique n-ring homomorphism $h : A \to B$ such that

$$\|f(a) - h(a)\| \leq \varepsilon \tag{2.3}$$

for all $a \in A$. Furthermore,

$$\begin{aligned}(\Pi_{i=1}^{k}h(a_i))(\Pi_{i=k+1}^{n}f(a_i) - \Pi_{i=k+1}^{n}h(a_i))&\\ &= (\Pi_{i=1}^{k}f(a_i) - \Pi_{i=1}^{k}h(a_i))(\Pi_{i=k+1}^{n}h(a_i))\\ &= 0 \qquad (2.4)\end{aligned}$$

for all $a_1, a_2, ..., a_n \in A$ *and all* $k \in \{1, 2, ..., n-1\}$.

Proof. Put $h(a) = \lim_m \frac{1}{2^m} f(2^m a)$ for all $a \in A$. Then by Hyers' Theorem, h is additive. We will show that h is an n-ring homomorphism. For every $a_1, a_2, ..., a_n \in A$ we have

$$\begin{aligned}&\|h(a_1a_2...a_n) - h(a_1)(\Pi_{i=2}^{n}f(a_i))\|\\ &= \lim_m \|\frac{1}{2^m} f(2^m(a_1a_2...a_n)) - h(a_1)(\Pi_{i=2}^{n}f(a_i))\|\\ &= \lim_m \|\frac{1}{2^m} f((2^m a_1)a_2...a_n) - h(a_1)(\Pi_{i=2}^{n}f(a_i))\|\\ &= \lim_m \|\frac{1}{2^m}\{f((2^m a_1)a_2...a_n) - f(2^m a_1)(\Pi_{i=2}^{n}f(a_i))\\ &\quad + f(2^m a_1)(\Pi_{i=2}^{n}f(a_i))\} - h(a_1)(\Pi_{i=2}^{n}f(a_i))\|\\ &\le \lim_m \frac{1}{2^m}\delta = 0.\end{aligned}$$

Hence,

$$h(a_1a_2...a_n) = h(a_1)(\Pi_{i=2}^{n}f(a_i)). \qquad (2.5)$$

By (2.5) it follows that

$$h(a_1)f(2^m a_2)(\Pi_{i=3}^{n}f(a_i)) = h(2^m a_1a_2...a_n) = 2^m h(a_1a_2...a_n)$$

for all $a_1, a_2, ..., a_n \in A, m \in \mathbb{N}$. Dividing both sides of above equality by 2^m and taking the limit $m \to \infty$. Then we have

$$h(a_1)h(a_2)(\Pi_{i=3}^{n}f(a_i)) = \lim_m h(a_1)\frac{1}{2^m} f(2^m a_2)(\Pi_{i=3}^{n}f(a_i)) = h(a_1a_2...a_n).$$

Hence by (2.5) we have

$$h(a_1)h(a_2)(\Pi_{i=3}^{n}f(a_i)) = h(a_1a_2...a_n) = h(a_1)(\Pi_{i=2}^{n}f(a_i)).$$

Now, proceed in this way to prove that

$$(\Pi_{i=1}^{k}h(a_i))(\Pi_{i=k+1}^{n}f(a_i)) = h(a_1a_2...a_n) \qquad (2.6)$$

for all $a_1, a_2, ..., a_n \in A$ and all $k \in \{1, 2, ..., n-1\}$. Put $k = n-1$ in (2.6), we obtain

$$(\Pi_{i=1}^{n-1}h(a_i))f(2^m a_n) = h(2^m(a_1a_2...a_n)) = 2^m h(a_1a_2...a_n) \qquad (2.7)$$

for all $a_1, a_2, ..., a_n \in A, m \in \mathbb{N}$. Dividing both sides of (2.7) by 2^m and taking the limit $m \to \infty$, it follows that h is an n-homomorphism. On the other hand h is additive and $h(a) = \lim_m \frac{1}{2^m} f(2^m a)$ for all $a \in A$. Then we have

$$(\Pi_{i=1}^k f(a_i))(\Pi_{i=k+1}^n h(a_i)) = h(a_1 a_2 ... a_n) = (\Pi_{i=1}^n h(a_i)) \tag{2.8}$$

for all $a_1, a_2, ..., a_n \in A$ and all $k \in \{1, 2, ..., n-1\}$, and (2.4) follows (2.6) and (2.8). Obviously the uniqueness property of h follows from additivity. □

Similarly to the proof of Theorem 2 of [3], we can prove the Hyers-Ulam-Rassias type stability of n-ring homomorphisms as follows.

Theorem 2.2. *Let A be a normed algebra, B be a Banach algebra, δ and ε be nonnegative real numbers and let p, q be two real numbers such that $p, q < 1$ or $p, q > 1$. Assume that $f : A \to B$ satisfies the system of functional inequalities*

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon(\|a\|^p + \|b\|^p)$$

and

$$\|f(\Pi_{i=1}^n a_i) - \Pi_{i=1}^n f(a_i)\| \leq \delta(\Pi_{i=1}^n \|a_i\|^q)$$

for all $a, b, a_1, a_2, ..., a_n \in A$. Then there exists a unique n-ring homomorphism $h : A \to B$ and a constant k such that

$$\|f(a) - h(a)\| \leq k\varepsilon\|a\|^p$$

for all $a \in A$.

Now we will prove the stability of n-ring derivations from a normed algebra into a Banach module.

Theorem 2.3. *Let $\mathcal{A}$ be a normed algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}-$ module. Suppose the map $f : \mathcal{A} \longrightarrow \mathcal{X}$ satisfying the system of inequalities:*

$$\|f(a+b) - f(a) - f(b)\| \leq \epsilon(\|a\|^p + \|b\|^p) \quad (a, b \in \mathcal{A}), \tag{2.9}$$

$$\|f(\Pi_{i=1}^n a_i) - f(a_1)\Pi_{i=2}^n a_i - a_1 f(a_2)\Pi_{i=3}^k a_i - \cdots - \Pi_{i=1}^{n-1} a_i f(a_n)\| \leq \epsilon(\sum_{i=1}^n \|a_i\|^p) \tag{2.10}$$

for all $a_1, a_2, ..., a_n \in \mathcal{A}$, where ϵ and p are constants in $\mathbb{R}^+ \cup \{0\}$. If $p < 1$, then there is a unique n-ring derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that

$$\|f(a) - D(a)\| \leq \frac{2\epsilon}{2 - 2^p}\|a\|^p \tag{2.11}$$

for all $a \in \mathcal{A}$. Moreover if for every $c \in \mathbb{C}$ and $a \in \mathcal{A}$, $f(ca) = cf(a)$, then $f = D$.

Proof. By Rassias's Theorem and (2.9), it follows that there exists a unique additive mapping $D : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies (2.11). We have to show that D is an n-derivation. Let $s = \frac{1-p}{|1-p|}$, and let $a, a_1, a_2, ..., a_n \in \mathcal{A}$. For each $m \in \mathbb{N}$, we have $D(a) = m^{-s}D(m^s a)$, therefore

$$\begin{aligned}\|m^{-s}f(m^s a) - D(a)\| &= m^{-s}\|f(m^s a) - D(m^s a)\| \\ &\leq m^{-s}\frac{2\epsilon}{2-2^p}\|a\|^p\|m^s a\|^p \\ &= m^{s(p-1)}\frac{2\epsilon}{2-2^p}\|a\|^p.\end{aligned}$$

Since $s(p-1) \leq 0$, we have

$$\lim_m \|m^{-s}f(m^s a) - D(a)\| = 0. \tag{2.12}$$

Similarly we can show that

$$\|m^{-ns}f(m^{ns}\Pi_{i=1}^n a_i) - D(\Pi_{i=1}^n a_i)\| \leq m^{ns(p-1)}\frac{2\epsilon}{2-2^p}\|\Pi_{i=1}^n a_i\|^p.$$

Therefore we have

$$\lim_m \|m^{-ns}f(m^{ns}\Pi_{i=1}^n a_i) - D(\Pi_{i=1}^n a_i)\|^p = 0. \tag{2.13}$$

By (2.10), for each $m \in \mathbb{N}$ we have

$$\begin{aligned}\|m^{-ns}f(m^{ns}\Pi_{i=1}^n a_i) - m^{-s}f(m^s a_1)\Pi_{i=2}^n(a_i) - m^{-s}a_1 f(m^s a_2)\Pi_{i=3}^n(a_i) \\ - ... - m^{-s}\Pi_{i=1}^{n-1}(a_i)f(m^s a_n)\| \\ =& m^{-ns}\|f(\Pi_{i=1}^n(m^s a_i)) - f(m^s a_1)(\Pi_{i=2}^n(m^s a_i)) \\ & - \sum_{j=2}^{n-1}\Pi_{l=1}^{j-1}(m^s a_l)f(m^s a_j))\Pi_{l=j+1}^n(m^s a_l) \\ & - \Pi_{l=1}^{n-1}(m^s a_l)f(m^s a_n)\| \\ \leq& m^{-ns}\epsilon\Pi_{i=1}^n\|m^s a_i\|^p \\ =& m^{ns(p-1)}\epsilon\Pi_{i=1}^n\|a_i\|^p.\end{aligned}$$

Thus we have

$$\begin{aligned}\lim_m \|m^{-ns}f(m^{ns}\Pi_{i=1}^n a_i) - m^{-s}f(m^s a_1)\Pi_{i=2}^n(a_i) - m^{-s}a_1 f(m^s a_2)\Pi_{i=3}^n(a_i) \\ - ... - m^{-s}\Pi_{i=1}^{n-1}(a_i)f(m^s a_n)\| = 0\end{aligned} \tag{2.14}$$

for all $a_1, a_2, ..., a_n \in \mathcal{A}$. On the other hand, we have

$$\begin{aligned}
&\|D(\Pi_{i=1}^n a_i) - D(a_1)\Pi_{i=2}^n a_i - a_1 D(a_2)\Pi_{i=3}^n a_i - \cdots - \Pi_{i=1}^{n-1} a_i D(a_n)\| \\
&\quad \le \|D(\Pi_{i=1}^n a_i) - m^{-ns} f(m^{ns}\Pi_{i=1}^n a_i)\| + \|m^{-ns} f(m^{ns}\Pi_{i=1}^n a_i) \\
&\qquad - m^{-s} f(m^s a_1)\Pi_{i=2}^n (a_i) - m^{-s} a_1 f(m^s a_2)\Pi_{i=3}^n (a_i) \\
&\qquad - ... - m^{-s}\Pi_{i=1}^{n-1}(a_i) f(m^s a_n)\| \\
&\qquad + \|m^{-s} f(m^s a_1)\Pi_{i=2}^n a_i - D(a_1)\Pi_{i=2}^n a_i\| \\
&\qquad + \|m^{-s} a_1 f(m^s a_2)\Pi_{i=3}^n a_i - a_1 D(a_2)\Pi_{i=3}^n a_i\| \\
&\qquad + ... \\
&\qquad + \|m^{-s}\Pi_{i=1}^{n-1} a_i f(m^s a_n) - \Pi_{i=1}^{n-1} a_i D(a_n)\|
\end{aligned}$$

for all $a_1, a_2, ..., a_n \in \mathcal{A}$. According to (2.10), (2.13) and (2.14), if $m \to \infty$, then the right hand side of above inequality tends to 0, so we have

$$D(\Pi_{i=1}^n a_i) = D(a_1)\Pi_{i=2}^n a_i + a_1 D(a_2)\Pi_{i=3}^n a_i + \cdots + \Pi_{i=1}^{n-1} a_i D(a_n),$$

for all $a_1, a_2, ..., a_n \in \mathcal{A}$. Hence D is an n-ring derivation. The uniqueness property of D follows from additivity. Let now for every $c \in \mathbb{C}$ and $a \in \mathcal{A}$, $f(ca) = cf(a)$, then by (2.11), we have

$$\begin{aligned}
\|f(a) - D(a)\| &= \|m^{-s} f(m^s a) - m^{-s} D(m^s a)\| \\
&\le m^{-s}\frac{2\epsilon}{2-2^p}\|m^s a\|^p \\
&= m^{s(p-1)}\frac{2\epsilon}{2-2^p}\|a\|^p
\end{aligned}$$

for all $a \in \mathcal{A}$. Hence by letting $m \to \infty$ in above inequality, we conclude that $f(a) = D(a)$ for all $a \in \mathcal{A}$. □

Similarly we can prove the following Theorem which can be regarded as an extension of Theorem 2.6 of [11].

Theorem 2.4. *Let p, q be real numbers such that $p, q < 1$, or $p, q > 1$. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{A}-$ module. Suppose the map $f : \mathcal{A} \longrightarrow \mathcal{X}$ satisfying the system of inequalities:*

$$\|f(a+b) - f(a) - f(b)\| \le \epsilon(\|a\|^p + \|b\|^p) \quad (a, b \in \mathcal{A}),$$

$$\|f(\Pi_{i=1}^n a_i) - f(a_1)\Pi_{i=2}^n a_i - a_1 f(a_2)\Pi_{i=3}^k a_i - \cdots - \Pi_{i=1}^{n-1} a_i f(a_n)\| \le \epsilon(\Pi_{i=1}^n \|a_i\|^q)$$

for all $a_1, a_2, ..., a_n \in \mathcal{A}$, where ϵ and p are constants in $\mathbb{R}^+ \cup \{0\}$. Then there exists a unique n-ring derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ such that

$$\|f(a) - D(a)\| \le \frac{2\epsilon}{2-2^p}\|a\|^p$$

for all $a \in \mathcal{A}$.

The following counterexample, which is a modification of Luminet's example (see [16]), shows that Theorem 2.2 is failed for $p = 1$ (see [3]).

Example 2.5. *Define a function* $\varphi : \mathbb{R} \to \mathbb{R}$ *by*

$$\varphi(x) = \begin{cases} 0 & |x| \leq 1, \\ \\ x\ln(|x|) & |x| > 1. \end{cases}$$

Let $f : \mathbb{R} \to \mathcal{M}_3(\mathbb{R})$ *be defined by*

$$f(x) = \begin{bmatrix} 0 & 0 & 0 \\ \varphi(x) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for all $x \in \mathbb{R}$. *Then*

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon(|a| + |b|)$$

and

$$\|f(\Pi_{i=1}^{n} a_i) - (\Pi_{i=1}^{n} f(a_i))\| \leq \delta(\Pi_{i=1}^{n} |a_i|^2)$$

for some $\delta > 0, \varepsilon > 0$ *and all* $a_1, a_2, ..a_n \in \mathbb{R}$*; see [3]. Therefore* f *satisfies the conditions of Theorem 2.2 with* $p = 1$, $q = 2$. *There is however no n-ring homomorphism* $h : \mathbb{R} \to \mathcal{M}_3(\mathbb{R})$ *and no constant* $k > 0$ *such that*

$$\|f(a) - h(a)\| \leq k\varepsilon|a| \qquad (a \in \mathbb{R}).$$

Also example 2.7 of [11] shows that Theorem 2.4 above is failed for $p = 1$.

Acknowledgement. The author like to express his sincere thanks to professor Th. M. Rassias for valuable comments. The authors would like to thank the Semnan University for its financial support.

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