Nonlinear Functional Analysis and Applications
Vol. 14, No. 1 (2009), pp. 81–97

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON CONSTRAINT QUALIFICATIONS AND OPTIMALITY CONDITIONS IN LOCALLY LIPSCHITZ MULTIOBJECTIVE PROGRAMMING PROBLEMS

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Abstract. In this paper we consider a multiobjective optimization problem with locally Lipschitz functions defined on a Banach space involving inequality, equality and set constraints. Some constraint qualifications in terms of Clarke's generalized gradients and directional derivatives are studied, and necessary and sufficient conditions for efficiency with positive Lagrange multipliers associated with all components of the objective are established.

1. Introduction

Constraint qualifications (called also regularity conditions) play an important role in the theory of extremum problems. They allow us to get Kuhn-Tucker necessary conditions for efficiency from Fritz John conditions. A lot of studies dealt with constraint qualifications under which we can obtain positive Lagrange multipliers associated with all components of objective functions (see, e.g., [4], [6], [9]-[13], and references therein), which leads to that none of components of the objective vanishes in necessary conditions for efficiency.

Maeda [10] studies multiobjective optimization problems involving Fréchet differentiable inequality constraints and introduces a constraint qualification under which he derives Kuhn-Tucker necessary conditions for efficiency with Lagrange multipliers corresponding to all components of objective functions to be positive. Preda-Chitescu [13] give a constraint qualification of Maeda type in the semidifferentiable case for a multiobjective optimization problem

[0]Received: September 17, 2007. Revised May 6, 2008.

[0]2000 Mathematics Subject Classification: 90C46, 90C29.

[0]Keywords: Constraint qualification, Kuhn-Tucker conditions for efficiency, Clarke's generalized directional derivative, Clarke's generalized gradient.

[0]This research was partially supported by the Natural Science Council of Vietnam.

with inequality constraints and develop results similar to those obtained by Maeda [10].

Giorgi et al. [4] consider multiobjective minimization problems involving inequality and equality constraints in finite dimensions where all functions are Dini or Hadamard differentiable. They introduce several constraint qualifications which generalize the constraint qualification introduced by Maeda [10] and the classical ones, and under which they establish Kuhn-Tucker necessary conditions for efficiency with positive Lagrange multipliers associated with all components of objective functions. Recently, Luu-Hung [9] develop Kuhn-Tucker necessary conditions for efficiency to mathematical programs in normed spaces involving inequality, equality and set constraints with positive Lagrange multipliers corresponding to all components of objective functions, while Nguyen-Luu [12] establish a theorem of the alternative of Tucker type to a system comprising inequalities described by sup-functions and an inclusion together with Kuhn-Tucker necessary conditions of aforementioned type.

The purpose of this paper is to study some constraint qualifications in terms of Clarke's generalized gradients and directional derivatives, and establish necessary and sufficient conditions for efficiency in locally Lipschitz mathematical programming problems involving inequality, equality and set constraints in Banach spaces with positive Lagrange multipliers associated with all components of objective functions along with some properties of the set of Lagrange multipliers.

The remainder of the paper is organized as follows. After some preliminaries, Section 3 presents a Kuhn-Tucker necessary condition for efficiency in terms of Clarke's generalized gradients and normal cones together with necessary and sufficient conditions ensuring the regulaity condition (CQI) holds. In Section 4 we introduce a constraint qualification of Maeda type in terms of Clarke's generalized directional derivatives, and show that (CQI) implies (CQII). Section 5 gives a necessary condition for efficiency under the regularity condition (CQII). When imposing some generalized convexity hypotheses on the data of the problem, the necessary condition mentioned above is also a sufficient one.

2. Preliminaries

Let X be a Banach space, and let f, g, h be mappings from X into $\mathbb{R}^r$, $\mathbb{R}^m$, $\mathbb{R}^\ell$, respectively. Then f, g, h can be expressed as follows: $f = (f_1, \ldots, f_r)$, $g = (g_1, \ldots, g_m)$, $h = (h_1, \ldots, h_\ell)$, where $f_1, \ldots, f_r$, $g_1, \ldots, g_m$, $h_1, \ldots, h_\ell$ are locally Lipschitz functions defined on X. Let C be a nonempty closed subset of X. For the sake of simplicity, we define the sets: $I = \{1, \ldots, m\}$ $J = \{1, \ldots, \ell\}$ and $L = \{1, \ldots, r\}$.

Consider the following multiobjective programming problem:

$$\text{(MP)} \qquad \begin{aligned} &\min f(x),\\ &\text{subject to}\\ &\quad g_i(x) \leqslant 0, \quad i = 1, \ldots, m,\\ &\quad h_j(x) = 0, \quad j = 1, \ldots, \ell,\\ &\quad x \in C. \end{aligned}$$

Denote by $I(\overline{x})$ the set of active indexes at a point $\overline{x}$

$$I(\overline{x}) = \Big\{ i \in I \ : \ g_i(\overline{x}) = 0 \Big\},$$

and M the feasible set of Problem (MP)

$$M = \Big\{ x \in C \ : \ g_i(x) \leqslant 0 \ (\forall i \in I), h_j(x) = 0 \ (\forall j \in J) \Big\}.$$

Recall that a point $\overline{x} \in M$ is said to be a local weak efficient minimizer of (MP) if there exists a number $\delta > 0$ such that there is no $x \in M \cap B(\overline{x}; \delta)$ satisfying

$$f_k(x) < f_k(\overline{x}) \quad (\forall\, k \in L), \tag{2.1}$$

where $B(\overline{x}; \delta)$ stands for the open ball of radius δ around $\overline{x}$. The point $\overline{x}$ is called a local Pareto minimizer of (MP) if (2.1) is replaced by the following

$$f_k(x) \leqslant f_k(\overline{x}) \quad (\forall\, k \in L), \tag{2.2}$$

$$f_j(x) < f_j(\overline{x}) \quad \text{at least one } j \in L. \tag{2.3}$$

Following [2], for a locally Lipschitz real-valued function f defined on X, the Clarke generalized directional derivative of f at $\overline{x}$, with respect to a direction v, is defined as

$$f^0(\overline{x}; v) = \limsup_{x \to \overline{x},\ t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

where $t \downarrow 0$ means $t \to 0_+$. The function $f^0(\overline{x}; .)$ is finite, positively homogeneous and subadditive. The following set of the topological dual X^* of X is called the Clarke generalized gradient of f at $\overline{x}$

$$\partial f(\overline{x}) = \Big\{ x^* \in X^* \ : \ \langle x^*, v \rangle \leqslant f^0(\overline{x}; v), \forall\, v \in X \Big\}.$$

Now let f be a real-valued function defined on X. The upper Dini derivative of f at $\overline{x}$ in a direction v is

$$\overline{D} f(\overline{x}; v) = \limsup_{t \downarrow 0} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}; \tag{2.4}$$

The upper Hadamard derivative of f at $\overline{x}$ in the direction v is

$$\overline{d}f(\overline{x};v) = \limsup_{u\to v,\ t\downarrow 0} \frac{f(\overline{x}+tu)-f(\overline{x})}{t} \cdot \tag{2.5}$$

Replacing "lim sup" by "lim inf" in (2.4) and (2.5), we obtain the lower Dini derivative $\underline{D}f(\overline{x};v)$ and the lower Hadamard derivative $\underline{d}f(\overline{x};v)$, respectively, of f at $\overline{x}$ in the direction v. In case $\overline{D}f(\overline{x};v) = \underline{D}f(\overline{x};v)$ (resp. $\overline{d}f(\overline{x};v) = \underline{d}f(\overline{x};v)$), we denote their common value by $Df(\overline{x};v)$ (resp. $df(\overline{x};v)$), which is called the Dini derivative (resp. Hadamard derivative) of f at $\overline{x}$ in the direction v. The function f is Dini differentiable (resp. Hadamard differentiable) at $\overline{x}$ if its Dini derivative (resp. Hadamard derivative) at $\overline{x}$ exists in all directions. Note that if $df(\overline{x};v)$ exists, then also $Df(\overline{x};v)$ exists and they are equal. In case f is Fréchet differentiable at $\overline{x}$ with Fréchet derivative $\nabla f(\overline{x})$, then

$$Df(\overline{x};v) = df(\overline{x};v) = \langle \nabla f(\overline{x}), v\rangle.$$

Following [2], the Clarke tangent cone and the contingent cone to a set $C \subset X$ at a point $\overline{x} \in C$ are respectively defined as

$$T(C;\overline{x}) = \Big\{ v \in C : \forall\, x_n \in C, x_n \to \overline{x}, \forall\, t_n \downarrow 0, \exists\, v_n \to v \\ \text{such that } x_n + t_n v_n \in C, \forall\, n \Big\},$$

$$K(C;\overline{x}) = \Big\{ v \in X \ : \ \exists\, v_n \to v, \exists\, t_n \downarrow 0 \text{ such that } \overline{x} + t_n v_n \in C, \forall\, n \Big\}.$$

Note that $T(C;\overline{x})$ is convex, while $K(C;\overline{x})$ is not necessarily convex. Both $T(C;\overline{x})$ and $K(C;\overline{x})$ are nonempty closed, and $T(C.\overline{x}) \subset K(C;\overline{x})$. In case C is convex, they are equal.

The normal cones associated with $T(C;\overline{x})$ and $K(C;\overline{x})$ are

$$N_T(C;\overline{x}) = \Big\{ x^* \in X^* : \langle x^*, v\rangle \leqslant 0, \forall\, v \in T(C;\overline{x}) \Big\},$$
$$N_K(C;\overline{x}) = \Big\{ x^* \in X^* : \langle x^*, v\rangle \leqslant 0, \forall\, v \in K(C;\overline{x}) \Big\}.$$

Both these cones are weakly* closed convex. The cone $N_T(C;\overline{x})$ is called Clarke's normal cone, and $N_K(C;\overline{x}) \subset N_T(C;\overline{x})$.

For any set $K \subset X$, the polar cone of K is given by

$$K^0 = \Big\{ \xi \in X^* \ : \ \langle \xi, v\rangle \leqslant 0, \forall\, v \in K \Big\}.$$

Thus, $N_T(C;\overline{x}) = T^0(C;\overline{x})$, $N_K(C;\overline{x}) = K^0(C;\overline{x})$.

The Dini subdifferential of a Dini differentiable function defined on X at $\overline{x}$ is

$$\partial_D f(\overline{x}) = \Big\{ \xi \in X^* : \langle \xi, v\rangle \leqslant Df(\overline{x};v), \forall\, v \in X \Big\}.$$

Note that in case f is locally Lipschitz at $\overline{x}$,

$$\partial_D f(\overline{x}) \subset \partial f(\overline{x}),$$

as $Df(\overline{x};v) \leqslant f^0(\overline{x};v)$.

Adapting the definition of Giorgi and Guerraggio [3], a locally Lipschitz function defined on X is called ∂-quasiconvex (resp. ∂-pseudoconvex) at $\overline{x}$ if

$$f(y) \leqslant f(\overline{x}) \text{ implies } \langle \xi, y-\overline{x}\rangle \leqslant 0 \quad (\forall \xi \in \partial f(\overline{x}))$$
$$(\text{resp. } f(y) < f(\overline{x}) \text{ implies } \langle \xi, y-\overline{x}\rangle < 0, \quad \forall \xi \in \partial f(\overline{x})).$$

In case f is Fréchet continuously differentiable, the ∂-quasiconvexity (resp. ∂-pseudoconvexity) at $\overline{x}$ reduced to the quasiconvexity (resp. pseudoconvexity) at $\overline{x}$.

3. The constraint qualification (CQI) and Lagrange multipliers for problem (MP)

Adapting the definition of Chandra-Dutta-Lalitha [1], we shall say that Problem (MP) satisfies the constraint qualification (CQI) at $\overline{x} \in M$ if for each $s \in L$, there is no scalars $\tau_k \geqslant 0$, $k \in L$, $k \neq s$, $\lambda_i \geqslant 0$, $i \in I(\overline{x})$, $\mu_j \in \mathbb{R}$, $j \in J$, not all zero, satisfying

$$0 \in \sum_{k\in L,\ k\neq s} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \partial h_j(\overline{x}) + N_T(C;\overline{x}). \tag{3.1}$$

Note that (CQI) is equivalent to that for every $s \in L$,

$$\left.\begin{array}{r} 0 \in \displaystyle\sum_{k\neq s} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \partial h_j(\overline{x}) + N_T(C;\overline{x}) \\ \tau_k \geqslant 0, \forall\, k \in L, k \neq s, \lambda_i \geqslant 0, \forall\, i \in I(\overline{x}), \mu_j \in \mathbb{R}, \forall\, j \in J \end{array}\right\} \Rightarrow$$
$$\Rightarrow \tau_k = \lambda_i = \mu_j = 0$$
$$(\forall\, k \in L, k \neq s, \forall\, i \in I(\overline{x}), \forall\, j \in J).$$

A Kuhn-Tucker necessary condition for efficiency with positive Lagrange multipliers corresponding to all components of the objective can be stated as follows.

Theorem 3.1. *Let $\overline{x}$ be a local Pareto minimizer of (MP). Assume that the constraint qualification (CQI) holds at $\overline{x}$. Then there exist scalars $\overline{\tau}_k > 0$, $k \in L$, $\overline{\lambda}_i \geqslant 0$, $i \in I$ and $\overline{\mu}_j \in \mathbb{R}$, $j \in J$ such that*

$$0 \in \sum_{k\in L} \overline{\tau}_k \partial f_k(\overline{x}) + \sum_{i\in I} \overline{\lambda}_i \partial g_i(\overline{x}) + \sum_{j\in J} \overline{\mu}_j \partial h_j(\overline{x}) + N_T(C;\overline{x}), \tag{3.2}$$

$$\overline{\lambda}_i g_i(\overline{x}) = 0, \quad i \in I. \tag{3.3}$$

Proof. It is easy to see that for each $s \in L$, $\overline{x}$ is a local minimizer of the following scalar optimization problem:

$$(P_s) \qquad \begin{aligned} &\min f_s(x),\\ &\text{s.t. } f_k(x) \leqslant f_k(\overline{x}), \quad k \in L, k \neq s,\\ &\quad g_i(x) \leqslant 0, \quad i \in I,\\ &\quad h_j(x) = 0, \quad j \in J,\\ &\quad x \in C. \end{aligned}$$

Applying Theorem 6.1.1 [2] to each Problem (P_s) yields the existence of numbers $\tau_k^{(s)} \geqslant 0$, $k \in L$, $k \neq s$, $\lambda_i^{(s)} \geqslant 0$, $i \in I$, $\mu_j^{(s)} \in \mathbb{R}$, $j \in J$, not all zero, such that

$$0 \in \sum_{k\in L} \tau_k^{(s)} \partial f_k(\overline{x}) + \sum_{i\in I} \lambda_i^{(s)} \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j^{(s)} \partial h_j(\overline{x}) + N_T(C.\overline{x}), \tag{3.4.s}$$

$$\lambda_i^{(s)} g_i(\overline{x}) = 0, \quad i \in I. \tag{3.5.s}$$

In view of the constraint qualification (CQI), one get $\tau_s^{(s)} > 0$.

Summing up the inclusion (3.4.1),...,(3.4.r), we obtain

$$0 \in \sum_{k\in L} \overline{\tau}_k \partial f_k(\overline{x}) + \sum_{i\in I} \overline{\lambda}_i \partial g_i(\overline{x}) + \sum_{j\in J} \overline{\mu}_j \partial h_j(\overline{x}) + N_T(C;\overline{x}),$$

where $\overline{\tau}_k = \sum_{s\in L} \tau_k^{(s)} > 0 \ \ (k \in L)$, $\overline{\lambda}_i = \sum_{s\in L} \lambda_i^{(s)} \geqslant 0 \ \ (i \in I)$, and $\overline{\mu}_j = \sum_{s\in L} \mu_j^{(s)} \in \mathbb{R} \ \ (j \in J)$. Moreover, it follows readily from (3.5.s) that

$$\overline{\lambda}_i g_i(\overline{x}) = 0, \quad i \in I,$$

as was to be shown. □

Without loss of generality we can suppose that $I(\overline{x}) = \{1, \ldots, p\}$ $(p \leqslant m)$. For $s \in L$ we denote by $\Lambda_s(\overline{x})$ the set of vectors $(\tau_1, \ldots, \tau_{s-1}, \tau_{s+1}, \ldots, \tau_r, \lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_\ell)$ with $\tau_k \geqslant 0$ $(\forall\, k \in L, k \neq s)$, $\lambda_i \geqslant 0$ $(\forall\, i \in I(\overline{x}))$, $\mu_j \in \mathbb{R}$ $(\forall\, j \in J)$ such that

$$0 \in \partial f_s(\overline{x}) + \sum_{k\in L,\ k\neq s} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \partial h_j(\overline{x}) + N_T(C;\overline{x}),$$

and $\Lambda(\overline{x}) = \bigcup_{s\in L} \Lambda_s(\overline{x})$.

The following result shows that the boundedness of $\Lambda(\overline{x})$ is a sufficient condition ensuring the regularity condition (CQI) holds.

Theorem 3.2. *Let $\overline{x} \in M$. Assume that $\Lambda_s(\overline{x})$ is a nonempty and bounded for all $s \in L$. Then the constraint qualification (CQI) holds at $\overline{x}$.*

Proof. Assume the contrary, that the regularity condition (CQI) does not holds at $\overline{x}$. Then there exist $s_0 \in L$, $\tau'_k \geqslant 0$ $(k \in L, k \neq s_0)$, $\lambda'_i \geqslant 0$ $(i \in I(\overline{x}))$ and $\mu'_j \in \mathbb{R}$ $(j \in J)$, not all zero, such that

$$0 \in \sum_{k\in L,\ k\neq s_0} \tau'_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda'_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu'_j \partial h_j(\overline{x}) + N_T(C;\overline{x}). \tag{3.6}$$

Since $\Lambda_{s_0}(\overline{x}) \neq \emptyset$, there exist $\tau_k \geqslant 0$ $(k \in L, k \neq s_0)$, $\lambda_i \geqslant 0$ $(i \in I(\overline{x}))$, and $\mu_j \in \mathbb{R}$ $(j \in J)$ such that

$$0 \in \partial f_{s_0}(\overline{x}) + \sum_{k\in L,\ k\neq s_0} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \partial h_j(\overline{x}) + N_T(C;\overline{x}). \tag{3.7}$$

Combining (3.6) and (3.7) yields that for any $\gamma > 0$,

$$0 \in \partial f_{s_0}(\overline{x}) + \sum_{k\in L,\ k\neq s_0} (\tau_k + \gamma\tau'_k) \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} (\lambda_i + \gamma\lambda'_i) \partial g_i(\overline{x}) + \\ + \sum_{j\in J} (\mu_j + \gamma\mu'_j) \partial h_j(\overline{x}) + N_T(C;\overline{x}),$$

which implies that

$$(\tau_1+\gamma\tau'_1, \ldots, \tau_{s_0-1} + \gamma\tau'_{s_0-1}, \tau_{s_0+1} + \gamma\tau'_{s_0+1}, \ldots, \tau_r + \gamma\tau'_r, \\ \lambda_1 + \gamma\lambda'_1, \ldots, \lambda_p + \gamma\lambda'_p, \mu_1 + \gamma\mu'_1, \ldots, \mu_\ell + \gamma\mu'_\ell) \in \Lambda_{s_0}(\overline{x}),$$

and hence, $\Lambda_{s_0}(\overline{x})$ is unbounded, which contradicts the hypothesis. □

In what follows we can see that whenever the regularity condition (CQI) holds at a local Pareto minimum $\overline{x}$ the set $\Lambda(\overline{x})$ will be nonempty and bounded.

Theorem 3.3. *Let $\overline{x}$ be a local Pareto minimizer of (MP). Assume that the regularity condition (CQI) holds at $\overline{x}$. Then $\Lambda(\overline{x})$ is nonempty and bounded.*

Proof. Since $\overline{x}$ is a local Pareto minimizer for (MP), it also is a local minimizer of the scalar optimization problem (P_s) for all $s \in L$. Applying Theorem 6.1.1 [2] to each Problem (P_s) $(s \in L)$ yields the existence of scalas $\tau_k \geqslant 0$ $(k \in L)$, $\lambda_i \geqslant 0$ $(i \in I(\overline{x}))$ and $\mu_j \in \mathbb{R}$ $(j \in J)$, not all zero, such that

$$0 \in \sum_{k\in L} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \partial h_j(\overline{x}) + N_T(C;\overline{x}).$$

By virtue of the regularity condition (CQI), each the following constraint qualification holds to (P_s) at $\overline{x}$: for all $(\tau_1, \ldots, \tau_{s-1}, \tau_{s+1}, \ldots, \tau_r, \lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_\ell) \in \mathbb{R}^{r-1}_+ \times \mathbb{R}^p_+ \times \mathbb{R}^\ell \setminus \{0\}$,

$$0 \notin \sum_{k\in L,\ k\neq s} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \partial h_j(\overline{x}) + N_T(C;\overline{x}). \tag{3.8}$$

for all $s \in L$. Hence $\tau_s > 0$, and we can take $\tau_s = 1$ ($\forall\, s \in L$). Consequently, $\Lambda_s(\overline{x}) \neq \emptyset$, and so is $\Lambda(\overline{x})$.

It is obvious that the regularity condition (3.8) implies that for all $(\tau_1, \ldots, \tau_{s-1}, \tau_{s+1}, \ldots, \tau_r, \lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_\ell) \in \mathbb{R}_+^{r-1} \times \mathbb{R}_+^p \times \mathbb{R}^\ell \setminus \{0\}$,

$$0 \notin \partial\Big(\sum_{k\in L,\ k\neq s} \tau_k f_k\Big)(\overline{x}) + \partial\Big(\sum_{i\in I(\overline{x})} \lambda_i g_i\Big)(\overline{x}) + \partial\Big(\sum_{j\in J} \mu_j h_j\Big)(\overline{x}) + N_T(C;\overline{x}). \tag{3.9}$$

This is the constraint qualification (CQII) for Problem (P_s) in the Jourani sense in [8]. Under the gerularity condition (3.9) Theorem 3.2 in [8] shows that $\Lambda_s(\overline{x})$ is bounded. Hence, $\Lambda(\overline{x})$ is bounded as well. □

4. The constraint qualification (CQII)

In this section we shall be concerned with a constraint qualification of Maeda [10] type in terms of the Clarke generalized directional derivatives and the relationship between this constraint qualification and (CQI)

For $\overline{x} \in X$, we set

$$Q_s = \Big\{x \in C : f_k(x) \leqslant f_k(\overline{x})(\forall k \in L, k \neq s), g_i(x) \leqslant 0(\forall i \in I), \\ h_j(x) = 0(\forall j \in J)\Big\},$$

$$P_s = \Big\{x \in C : f_k(x) \leqslant f_k(\overline{x})(\forall k \in L, k \neq s), g_i(x) \leqslant 0(\forall i \in I), \\ h_j(x) = 0(\forall j \in J)\Big\},$$

$$Q = \Big\{x \in C : f_k(x) \leqslant f_k(\overline{x})(\forall k \in L), g_i(x) \leqslant 0(\forall i \in I), \\ h_j(x) = 0(\forall j \in J)\Big\},$$

For a nonempty closed convex subcone T of $K(C;\overline{x})$, we set

$$C_T(Q_s;\overline{x}) = \Big\{v \in T : f_k^0(\overline{x};v) \leqslant 0\ (\forall k \in L, k \neq s), \\ g_i^0(\overline{x},v) \leqslant 0\ (\forall i \in I(\overline{x})), h_j^0(\overline{x};v) = 0\ (\forall j \in J)\Big\},$$

$$C_T(Q;\overline{x}) = \Big\{v \in T : f_k^0(\overline{x};v) \leqslant 0\ (\forall k \in L), \\ g_i^0(\overline{x};v) \leqslant 0\ (\forall i \in I(\overline{x})), h_j^0(\overline{x};v) = 0\ (\forall j \in J)\Big\}. \tag{4.1}$$

Thus $Q_s = P_s \cap C$ and $C_T(Q;\overline{x}) = \bigcap_{s\in L} C_T(Q_s;\overline{x})$.

Maeda [10] considered Problem (MP) consists only of Fréchet continuously differentiable constraints of inequality type. To derive necessary conditions with positive Lagrange multipliers associated with all components of the objective, Maeda [10] introduced the following constraint qualification:

$$C(Q;\overline{x}) \subset \bigcap_{s\in L} \overline{\text{co}}\, K(Q_s;\overline{x}), \tag{4.2}$$

where $\overline{\text{co}}$ indicates the closed convex hull, $X=\mathbb{R}^n$, and

$$C(Q;\overline{x}) = \Big\{ v\in X : \langle \nabla f_k(\overline{x}), v\rangle \leqslant 0 \ (\forall k\in L), \\ \langle \nabla g_i(\overline{x}), v\rangle \leqslant 0 \ (\forall i\in I(\overline{x})), \langle \nabla h_j(\overline{x}), v\rangle = 0 \ (\forall j\in J)\Big\}.$$

Chandra et al. [1] studied Problem (MP) without set constraints, and introduced the constraint qualification of the form

$$C(Q_s;\overline{x}) \subset \overline{\text{co}}\, K(Q_s;\overline{x}) \quad \text{for some } s\in L. \tag{4.3}$$

Note here that $T=X$ and $C(Q_s;\overline{x}) = C_X(Q_s;\overline{x})$.

Under condition (4.3) Chandra et al. only obtain necessary conditions for efficiency with Lagrange multipliers corresponding to the objective to be nonzero. Motivated by the results due to Maeda [10] and Chandra et a. [1], we introduce the following constraint qualification: there exist nonempty closed convex subcones T of $K(C;\overline{x})$ and T_s of $K(Q_s;\overline{x})$ $(\forall s\in L)$ such that

$$C_T(Q;\overline{x}) \subset \bigcap_{s\in L} T_s, \tag{4.4}$$

which is called the constraint qualification (CQII). For example, it can be taken T and T_s as the Clarke tangent cones $T(C;\overline{x})$ and $T(Q_s;\overline{x})$ $(s\in L)$, respectively.

Theorem 4.1 below will show that the constraint qualification (CQI) implies the constraint qualification (CQII) under suitable hypothesis.

Theorem 4.1. *Let $\overline{x}$ be a feasible point of Problem (MP) and C a convex set. Assume that h_j is Fréchet continuously differentiable at $\overline{x}$ with Fréchet derivatives ∇h_j $(\forall j\in J)$; f_k and g_i are Dini differentiable at $\overline{x}$ with convex derivatives $(\forall k\in L, \forall i\in I(\overline{x}))$. Suppose, in addition, that the constraint qualification (CQI) holds at $\overline{x}$. Then the constraint qualification (CQII) holds at $\overline{x}$.*

Proof. Since f_k and g_i are Dini differentiable and locally Lipschitz at $\overline{x}$, they are Hadamard differentiable at $\overline{x}$ for all $k\in L$, $i\in I(\overline{x})$. By virtue of the regularity condition (CQI), it follows that for all $s\in L$,

$$\left.\begin{array}{l} 0\in \displaystyle\sum_{k\in L,\ k\neq s} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \nabla h_j(\overline{x}) + N_T(C;\overline{x}) \\ \qquad \tau_k \geqslant 0 \ (\forall k\in L, k\neq s), \lambda_i\geqslant 0 \ (\forall i\in I(\overline{x})), \mu_j\in\mathbb{R} \ (\forall j\in J) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \tau_k = \lambda_i = \mu_j = 0 \quad (\forall k \in L, k \neq s, \forall i \in I(\overline{x}), \forall j \in J). \tag{4.5}$$

Since $\partial_D f_k(\overline{x}) \subset \partial f_k(\overline{x})$, $\partial_D g_i(\overline{x}) \subset \partial g_i(\overline{x})$, it follows from (4.5) that for all $s \in L$,

$$\left.\begin{array}{l} 0 \in \displaystyle\sum_{k\in L,\ k\neq s} \tau_k \partial_D f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial_D g_i(\overline{x}) + \sum_{j\in J} \mu_j \nabla h_j(\overline{x}) + N_T(C;\overline{x}) \\ \qquad \tau_k \geqslant 0\ (\forall k \in L, k \neq s), \lambda_i \geqslant 0\ (\forall i \in I(\overline{x})), \mu_j \in \mathbb{R}\ (\forall j \in J) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \tau_k = \lambda_i = \mu_j = 0 \quad (\forall k \in L, k \neq s, \forall i \in I(\overline{x}), \forall j \in J),$$

which implies that for all $s \in L$,

$$\left.\begin{array}{l} 0 \in \displaystyle\sum_{k\in L,\ k\neq s} \tau_k \partial_D f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial_D g_i(\overline{x}) + \sum_{j\in J} \mu_j \nabla h_j(\overline{x}) \\ \qquad \tau_k \geqslant 0\ (\forall k \in L, k \neq s), \lambda_i \geqslant 0\ (\forall i \in I(\overline{x})), \mu_j \in \mathbb{R}\ (\forall j \in J) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \tau_k = \lambda_i = \mu_j = 0 \quad (\forall k \in L, k \neq s, \forall i \in I(\overline{x}), \forall j \in J), \tag{4.6}$$

as $0 \in N_T(C;\overline{x})$.

Moreover, one has, according to Theorem 3.1 [7], that

$$K(Q_s;\overline{x}) = K(P_s;\overline{x}) \cap K(C;\overline{x}). \tag{4.7}$$

Note here that $K(C;\overline{x}) = T(C;\overline{x})$, as C is convex. Under the regularity condition (4.6) we can invoke Lemma 2.1 [14] to deduce that for all $s \in L$,

$$\begin{aligned} K(P_s;\overline{x}) = \Big\{ v \in X : & Df_k(\overline{x};v) \leqslant 0\ (\forall k \in L, k \neq s), \\ & Dg_i(\overline{x};v) \leqslant 0\ (\forall i \in I(\overline{x})), \langle \nabla h_j(\overline{x}), v\rangle = 0\ (\forall j \in J) \Big\}. \end{aligned}$$

In view of the convexity of the functions $Df_k(\overline{x};.)$ $(k \in L)$ and $Dg_i(\overline{x};.)$ $(i \in I(\overline{x}))$, we deduce that $K(P_s;\overline{x})$ is convex. This along with (4.7) yields that $K(Q_s;\overline{x})$ is a closed convex cone.

On the other hand, it is clear that for each $s \in L$,

$$\begin{aligned} C_K(Q_s;\overline{x}) \subset \Big\{ v \in K(C;\overline{x}) : & Df_k(\overline{x};v) \leqslant 0\ (\forall k \in L, k \neq s) \\ & Dg_i(\overline{x};v) \leqslant 0\ (\forall i \in I(\overline{x})), \langle \nabla h_j(\overline{x}), v\rangle = 0\ (\forall j \in J) \Big\} \\ = K(P_s;\overline{x}) \cap K(C;\overline{x}) & \\ = K(Q_s;\overline{x}). & \end{aligned}$$

Hence,

$$\bigcap_{s\in L} C_k(Q_s;\overline{x}) = C_K(Q;\overline{x}) \subset \bigcap_{s\in L} K(Q_s;\overline{x}).$$

Thus the constraint qualification (CQII) holds at $\overline{x}$ with $T = K(C;\overline{x})$ and $T_s = K(Q_s;\overline{x})$. □

5. Optimality conditions

In this section, under the constraint qualification (CQII) we shall derive necessary conditions for efficiency of Problem (MP) with positive Lagrange multipliers corresponding to all components of the objective in terms of the Clarke generalized gradients and the contingent cone $K(C;\overline{x})$. At this point we also assume that h_j $(j \in J)$ is Fréchet continuously differentiable at $\overline{x}$ with Fréchet derivative $\nabla h_j(\overline{x})$ for all $j \in J$. For an arbitrary nonempty closed convex subcone T of $K(C;\overline{x})$, we set

$$B_T(\overline{x}) = \bigcup \Big\{ \sum_{k\in L} \tau_k \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j\in J} \mu_j \nabla h_j(\overline{x}) + T^0 : \tau_k \geqslant 0, \\ \lambda_i \geqslant 0, \mu_j \in \mathbb{R} \text{ for all } k \in L, i \in I(\overline{x}), j \in J \Big\}.$$

Theorem 5.1. *Let $\overline{x}$ be a local Pareto minimizer of Problem (MP). Assume that the constraint qualification (CQII) holds at $\overline{x}$ for the nonempty closed convex subcones T and T_s of $K(C;\overline{x})$ and $K(Q_s;\overline{x})$, respectively. Then*

$$\Big(-\sum_{k\in L} \partial f_k(\overline{x}) \Big) \bigcap \overline{B_T(\overline{x})} \neq \emptyset, \tag{5.1}$$

where the bar indicates the weak closure.*

Proof. Since $\overline{x}$ is a local Pareto minimizer of (MP), it is a local minimizer of the scalar optimization problem (P_s) for all $s \in L$. For any $v \in K(Q_s;\overline{x})$, there exist sequences $t_n \downarrow 0$ and $v_n \to v$ such that $\overline{x} + t_n v_n \in Q_s$ $(\forall n)$. Hence,

$$\overline{d} f_s(\overline{x}; v) \geqslant \limsup_{n\to\infty} \frac{f_s(\overline{x} + t_n v_n) - f_s(\overline{x})}{t_n} \geqslant 0.$$

As f_s is locally Lipschitz at $\overline{x}$, it results that

$$\overline{D} f_s(\overline{x}; v) = \overline{d} f_s(\overline{x}; v) \geqslant 0 \quad (\forall v \in K(Q_s;\overline{x})),$$

which leads to the following

$$f_s^0(\overline{x}; v) \geqslant 0 \quad (\forall v \in K(Q_s;\overline{x})),$$

as $\overline{D} f_s(\overline{x}; v) \leqslant f_s^0(\overline{x}; v)$. This yields that

$$f_s^0(\overline{x}; v) \geqslant 0 \quad (\forall v \in T_s). \tag{5.2}$$

Let us show that

$$0 \in \partial f_s(\overline{x}) + T_s^0. \tag{5.3}$$

Assume the contrary, that

$$0 \notin \partial f_s(\overline{x}) + T_s^0.$$

It is obvious that the set $\partial f_s(\overline{x}) + T_s^0$ is convex. In view of the weak* compactness of $\partial f_s(\overline{x})$ and the weak* closedness of T_s^0, we infer that the set $\partial f_s(\overline{x}) + T_s^0$

is weakly* closed. Making use of a separation theorem of disjoint convex sets (see, for example, [5, Theorem 3.6]), we claim that there exists $v_0 \in X$, $v_0 \neq 0$, such that

$$\langle \xi, v_0 \rangle < 0 \quad (\forall \xi \in \partial f_s(\overline{x}) + T_s^0). \tag{5.4}$$

As $0 \in T_s^0$, it results that

$$\langle \xi, v_0 \rangle < 0 \quad (\forall \xi \in \partial f_s(\overline{x})),$$

whence,

$$f_s^0(\overline{x}; v_0) < 0. \tag{5.5}$$

We shall prove that $v_0 \in T_s$. If this were not so, there would exist $\xi_0 \in T_s^0$ such that $\langle \xi_0, v_0 \rangle > 0$. For any $\lambda > 0$, $\lambda \xi_0 \in T_s^0$, and so, for λ sufficiently large and any $\xi \in \partial f_s(\overline{x})$, we have

$$\langle \xi, v_0 \rangle + \langle \lambda \xi_0, v_0 \rangle \geqslant 0.$$

We then arrive at a contradiction with (5.4). Hence, $v_0 \in T_s$, and so, (5.5) conflicts with (5.2). Thus one gets (5.3).

It follows from (5.3) that

$$0 \in \sum_{s \in L} \partial f_s(\overline{x}) + \sum_{s \in L} T_s^0 \subset \sum_{s \in L} \partial f_s(\overline{x}) + \overline{\sum_{s \in L} T_s^0}. \tag{5.6}$$

On the other hand, the constraint qualification (CQII) yields that

$$\Big(\bigcap_{s \in L} T_s \Big)^0 \subset C_T^0(Q; \overline{x}), \tag{5.7}$$

where $C_T^0(Q; \overline{x})$ is the polar cone of $C_T(Q; \overline{x})$. Taking account of Lemma 5.8 in [5], we get

$$\Big(\bigcap_{s \in L} T_s \Big)^0 = \overline{\sum_{s \in L} T_s^0}. \tag{5.8}$$

Combining (5.6)-(5.8) yields that

$$0 \in \sum_{s \in L} \partial f_s(\overline{x}) + C_T^0(Q; \overline{x}). \tag{5.9}$$

We now prove that

$$C_T^0(Q; \overline{x}) = \overline{B_T(\overline{x})}. \tag{5.10}$$

We first show that

$$B_T(\overline{x}) \subset C_T^0(Q; \overline{x}). \tag{5.11}$$

For $\xi \in B_T(\overline{x})$, there exist $\tau_k \geqslant 0$, $\xi_k \in \partial f_k(\overline{x})$ $(k = 1, \dots, r)$, $\lambda_i \geqslant 0$, $\eta_i \in \partial g_i(\overline{x})$ $(i \in I(\overline{x}))$, $\mu_j \in \mathbb{R}$ $(j = 1, \dots, \ell)$ and $\zeta \in T^0$ such that

$$\xi = \sum_{k \in L} \tau_k \xi_k + \sum_{i \in I(\overline{x})} \lambda_i \eta_i + \sum_{j \in J} \mu_j \nabla h_j(\overline{x}) + \zeta.$$

For any $v \in C_T(Q; \overline{x})$, it follows readily that $v \in T$, and

$$\begin{aligned} &\langle \xi_k, v \rangle \leqslant f_k^0(\overline{x}; v) \leqslant 0 \quad (\forall k \in L), \\ &\langle \eta_i, v \rangle \leqslant g_i^0(\overline{x}; v) \leqslant 0 \quad (\forall i \in I(\overline{x})), \\ &\langle \nabla h_j(\overline{x}), v \rangle = 0 \qquad\qquad (\forall j \in J), \\ &\langle \zeta, v \rangle \leqslant 0. \end{aligned}$$

Hence, $\xi \in C_T^0(Q; \overline{x})$, and (5.11) holds. As $C_T^0(Q; \overline{x})$ is weakly* closed, it results that

$$\overline{B_T(\overline{x})} \subset C_T^0(Q; \overline{x}). \tag{5.12}$$

Let us verify the opposite inclusion of (5.12). If it were false, there would exist $w \in C_T^0(Q; \overline{x})$, but $w \notin \overline{B_T(\overline{x})}$. Observing that $\overline{B_T(\overline{x})}$ is weakly* closed and convex, we invoke a separation theorem of disjoint convex sets (see, e.g., [5, Theorem 3.6]) to deduce that there exists $u_0 \in X$, $u_0 \neq 0$, such that

$$\langle \xi, u_0 \rangle \leqslant 0 < \langle w, u_0 \rangle \quad (\forall \xi \in \overline{B_T(\overline{x})}), \tag{5.13}$$

which implies that $u_0 \in B_T^0(\overline{x})$.

On the other hand, for $\xi_k \in \partial f_k(\overline{x})$, it holds that $\xi_k \in B_T(\overline{x})$ $(\forall k \in L)$. Hence,

$$\langle \xi_k, u_0 \rangle \leqslant 0 \quad (\forall \xi_k \in \partial f_k(\overline{x})),$$

which leads to the following

$$f_k^0(\overline{x}; u_0) \leqslant 0 \quad (\forall k \in L). \tag{5.14}$$

Similarly, one gets

$$g_i^0(\overline{x}; u_0) \leqslant 0 \quad (\forall i \in I(\overline{x})). \tag{5.15}$$

Moreover, since $\pm \nabla h_j(\overline{x}) \in B_T(\overline{x})$ $(\forall j \in J)$, it results that

$$\langle \nabla h_j(\overline{x}), u_0 \rangle = 0 \quad (\forall j \in J). \tag{5.16}$$

For every $\zeta \in T^0$, we also have

$$\langle \zeta, u_0 \rangle \leqslant 0, \tag{5.17}$$

whence, $u_0 \in T^{00} = T$.

Combining (5.14)-(5.17) yields that $u_0 \in C_T(Q; \overline{x})$. Observing that $w \in C_T^0(Q; \overline{x})$. we obtain that

$$\langle w, u_0 \rangle \leqslant 0,$$

which conflicts with (5.13). Hence,

$$C_T^0(Q;\overline{x}) \subset \overline{B_T(\overline{x})},$$

which along with (5.12) yields that (5.10) holds.

Substituting (5.10) into (5.9) yields that

$$0 \in \sum_{s\in L} \partial f_s(\overline{x}) + \overline{B_T(\overline{x})},$$

which implies that

$$\Big(-\sum_{s\in L} \partial f_s(\overline{x})\Big) \cap \overline{B_T(\overline{x})} \neq \emptyset.$$

The proof is complete. □

The following corollaries give us standard Kuhn-Tucker necessary conditions for efficiency of (MP) with positive Lagrangian multipliers corresponding to all components of the objective.

Corollary 5.2. *Let $\overline{x}$ be a local Pareto minimizer of (MP). Assume that the set $B_T(\overline{x})$ is weakly* closed, and the constraint qualification (CQII) holds at $\overline{x}$ for some nonempty closed convex subcones T of $K(C;\overline{x})$ and T_s of $K(Q_s;\overline{x})$ $(s=1,\dots,r)$. Then there exist $\overline{\tau}_k > 0$ $(k=1,\dots,r)$, $\overline{\lambda}_i \geqslant 0$ $(i=1,\dots,m)$ and $\overline{\mu}_j \in \mathbb{R}$ $(j=1,\dots,\ell)$ such that*

$$0 \in \sum_{k=1}^{r} \overline{\tau}_k \partial f_k(\overline{x}) + \sum_{i=1}^{m} \overline{\lambda}_i \partial g_i(\overline{x}) + \sum_{j=1}^{\ell} \overline{\mu}_j \nabla h_j(\overline{x}) + T^0, \qquad (5.18)$$

$$\overline{\lambda}_i g_i(\overline{x}) = 0, \quad i=1,\dots,m. \qquad (5.19)$$

Proof. We invoke Theorem 5.1 to deduce that

$$\Big(-\sum_{k\in L} \partial f_k(\overline{x})\Big) \cap B_T(\overline{x}) \neq \emptyset, \qquad (5.20)$$

as $B_T(\overline{x})$ weakly* closed. But (5.20) is equivalent to the following

$$0 \in \sum_{k\in L} \partial f_k(\overline{x}) + B_T(\overline{x}).$$

Hence, there exist $\tau_k \geqslant 0$ $(k=1,\dots,r)$, $\lambda_i \geqslant 0$ $(i \in I(\overline{x}))$ and $\mu_j \in \mathbb{R}$ $(j=1,\dots,\ell)$ such that

$$0 \in \sum_{k=1}^{r} (1+\tau_k) \partial f_k(\overline{x}) + \sum_{i\in I(\overline{x})} \lambda_i \partial g_i(\overline{x}) + \sum_{j=1}^{\ell} \mu_j \nabla h_j(\overline{x}) + T^0.$$

By taking $\overline{\tau}_k = 1+\tau_k$ $(k=1,\dots,r)$, $\overline{\lambda}_i = \lambda_i$ $(i \in I(\overline{x}))$, $\overline{\lambda}_i = 0$ $(i \notin I(\overline{x}))$ and $\overline{\mu}_j = \mu_j$ $(j=1,\dots,\ell)$, we obtain (5.18) and (5.19). □

Corollary 5.3. *Let $\overline{x}$ be a local Pareto minimizer of (MP). Assume that all the hypotheses of Corollary 5.2 are fulfilled and the constraint qualification (CQII) holds with $T = T(C;\overline{x})$ and $T_s = T(Q_s;\overline{x})$. Then there exists $\overline{\tau}_k > 0$ $(k = 1,\dots,r)$, $\overline{\lambda}_i \geqslant 0$ $(i = 1,\dots,m)$ and $\overline{\mu}_j \in \mathbb{R}$ $(j = 1,\dots,\ell)$ such that*

$$0 \in \sum_{k=1}^{r} \overline{\tau}_k \partial f_k(\overline{x}) + \sum_{i=1}^{m} \overline{\lambda}_i \partial g_i(\overline{x}) + \sum_{j=1}^{\ell} \overline{\mu}_j \nabla h_j(\overline{x}) + N_T(C;\overline{x}),$$
$$\overline{\lambda}_i g_i(\overline{x}) = 0, \quad i = 1,\dots,m.$$

Proof. Since $T^0(C;\overline{x}) = N_T(C;\overline{x})$, applying Corollary 5.2 we obtain desired conclusions. □

We close the paper with a sufficient condition for efficiency.

Theorem 5.4. *Let $\overline{x}$ be a feasible point of (MP). Assume that the function f_{k_0} is ∂-pseudoconvex at $\overline{x}$ for some $k_0 \in L$, while the functions f_k $(k = 1,\dots,r, k \neq k_0)$ and g_i $(i \in I(\overline{x}))$ are ∂-quasiconvex at $\overline{x}$. Suppose, furthermore, that $\pm h_1,\dots,\pm h_\ell$ are quasiconvex at $\overline{x}$, C is convex, and*

$$\Big(- \sum_{k \in L} \partial f_k(\overline{x}) \Big) \cap \overline{B_T(\overline{x})} \neq \emptyset, \tag{5.21}$$

with $T = T(C;\overline{x})$. Then $\overline{x}$ is a weak minimum of (MP).

Proof. It can be rewritten (5.21) in the form

$$0 \in \sum_{kinL} \partial f_k(\overline{x}) + \overline{B_T(\overline{x})},$$

which implies that there exist $\xi_k \in \partial f_k(\overline{x})$ $(k = 1,\dots,r)$ and $\eta \in \overline{B_T(\overline{x})}$ such that

$$0 = \sum_{k \in L} \xi_k + \eta. \tag{5.22}$$

It can be expressed η as follows

$$\eta = w^* - \lim_{n\to\infty} \Big(\sum_{k \in L} \tau_k^{(n)} \xi_k^{(n)} + \sum_{i \in I(\overline{x})} \lambda_i^{(n)} \eta_i^{(n)} + \sum_{j \in J} \mu_j^{(n)} \nabla h_j(\overline{x}) + \sigma^{(n)} \Big), \tag{5.23}$$

where $w^* - \lim$ indicates the limit in weak* topology, $\tau_k^{(n)} \geqslant 0$, $\lambda_i^{(n)} \geqslant 0$, $\mu_j^{(n)} \in \mathbb{R}$, $\xi_k^{(n)} \in \partial f_k(\overline{x})$, $\eta_i^{(n)} \in \partial g_i(\overline{x})$ and $\sigma^{(n)} \in N_T(C;\overline{x})$ (for all $k \in L$, $i \in I(\overline{x})$, $j \in J$).

Contrary to the conclusion, suppose that $\overline{x}$ is not a weak minimizer of (MP). Then there exists $x \in X$ which is a feasible point for (MP) such that

$$f_k(x) < f_k(\overline{x}) \quad (\forall k \in L).$$

In view of the ∂-pseudoconvexity of f_{k_0}, the ∂-quasiconvexity of f_k $(k = 1, \ldots, r, k \neq k_0)$, g_i $(i \in I(\overline{x}))$, and the quasiconvexity of $\pm h_j$ $(j = 1, \ldots, \ell)$, it follows that for $n = 1, 2, \ldots,$

$$f_{k_0}(x) < f_{k_0}(\overline{x}) \Rightarrow \begin{cases} \langle \xi_{k_0}, x - \overline{x} \rangle < 0, \\ \langle \xi_{k_0}^{(n)}, x - \overline{x} \rangle < 0, \end{cases} \tag{5.24}$$

$$f_k(x) < f_k(\overline{x}) \Rightarrow \begin{cases} \langle \xi_k, x - \overline{x} \rangle \leqslant 0, \\ \langle \xi_k^{(n)}, x - \overline{x} \rangle \leqslant 0 \ (\forall k \in L, k \neq k_0), \end{cases} \tag{5.25}$$

$$g_i(x) \leqslant 0 = g_i(\overline{x}) \Rightarrow \langle \eta_i^{(n)}, x - \overline{x} \rangle \leqslant 0 \quad (\forall i \in I(\overline{x})), \tag{5.26}$$

$$h_j(x) = 0 = h_j(\overline{x}) \Rightarrow \langle \nabla h_j(\overline{x}), x - \overline{x} \rangle = 0 \quad (\forall j \in J). \tag{5.27}$$

Combining (5.23)-(5.27) yields that

$$\langle \sum_{k \in L} \xi_k + \eta, x - \overline{x} \rangle < 0,$$

which contradicts (5.22). Hence $\overline{x}$ is a weak minimum of (MP). □

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