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FIXED POINT THEOREMS FOR (α, p)-NONEXPANSIVE MAPPINGS IN $CAT(0)$ SPACES

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Abstract. We present some fixed point theorems for $((\alpha_1, \alpha_2, \cdots, \alpha_n), p)$-nonexpansive mappings in $CAT(0)$ spaces. Moreover the properties of the fixed points set are studied. Many of them have been derived from new condition on these mappings, which makes the nonexpansive mapping $T_\alpha := \alpha_1 T \oplus \alpha_2 T^2 \oplus \cdots \oplus \alpha_n T^n$.

1. Introduction

In 2007, Goebel and Pineda introduced the class of (α, p)-nonexpansive mappings in Banach spaces [4] and the fixed point theorems, due to these mappings, developed and many articles were stablished in such direction (see [3, 5] for more details).

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Many of these results are not verifiable in metric spaces. Therefore, we need to find metric spaces that have similar properties to the Banach spaces. In this paper, we study the existence of fixed points for (α, p)-nonexpansive mappings in $CAT(0)$ spaces.

2. Preliminaries

Throughout this paper, (M, d) will stand for a metric space. A metric space M is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in M is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. Other examples include Euclidean buildings, pre-Hilbert spaces, the complex Hilbert ball with a hyperbolic metric, $\mathbb{R}$-trees and many others.

$CAT(0)$ spaces are convex metric spaces, that is, for each $x, y \in M$ and $t \in [0,1]$, there exists $z \in [x, y]$ such that

$$d(x,z) = td(x,y), \quad d(y,z) = (1-t)d(x,y). \tag{2.1}$$

For the sake of readerships, denote the unique element $(1-t)x \oplus ty$ in (2.1) by z. If (M, d) is a $CAT(0)$ space, then

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z),$$

for all $x, y, z \in M$ and $t \in [0,1]$.

Lemma 2.1. ([1, Proposition 2.2]) *Let (M, d) be a $CAT(0)$ space, $p, q, r, s \in M$ and $t \in [0,1]$. Then*

$$d(tp \oplus (1-t)q, tr \oplus (1-t)s) \leq td(p,r) + (1-t)d(q,s).$$

Moreover by [6, Proposition 2] for each convex metric space, when $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$ for all i, we have

$$d(\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \cdots \oplus \alpha_n x_n, z) \leq \sum_{i=1}^{n} \alpha_i d(x_i, z),$$

for all $x_1, \cdots, x_n$ and $z \in X$. Therefore Lemma 2.1, yields the following inequality:

$$d(\alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \cdots \oplus \alpha_n x_n, \alpha_1 y_1 \oplus \alpha_2 y_2 \oplus \cdots \oplus \alpha_n y_n) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j d(x_i, y_j), \tag{2.2}$$

for all $x_1, \cdots, x_n$ and $y_1, \cdots, y_n \in X$.

A subset C of a $CAT(0)$ space (M, d) is called convex if the geodesic segment joining any two points of C, is entirely contained in C. Clearly, a convex subset of a $CAT(0)$ space is itself a $CAT(0)$ space when endowed with the induced metric. We say that C has the fixed point property for nonexpansive mappings if each nonexpansive mapping $T : C \to C$ has at least one fixed point.

Theorem 2.2. ([2]) *Let (M, d) be a complete locally compact $CAT(0)$ space. Then the following statements are equivalent:*

(i) *A nonempty closed and convex subset C of M has the fixed point property for nonexpansive mappings.*
(ii) *C is geodesically bounded.*
(iii) *C is bounded.*

Goebel and Pineda introduced the class of (α, p)-nonexpansive mappings in Banach spaces [4]. Similarly in $CAT(0)$ spaces we define:

Definition 2.3. A function $T : C \to C$ is called (α, p)-nonexpansive if, for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$ for all $1 \leq i \leq n$, $\alpha_1, \alpha_n > 0$, and for some $p \in [1, \infty)$,

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^i y)^p \leq d(x, y)^p,$$

for all $x, y \in C$.

Note that, their proofs rely on the nonexpansive property of the following mapping

$$T_\alpha := \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_n T^n.$$

As an anology, in $CAT(0)$ space we can write

$$T_\alpha := \alpha_1 T \oplus \alpha_2 T^2 \oplus \cdots \oplus \alpha_n T^n.$$

Equivalently, $T : C \to C$ is called (α, p)-nonexpansive, if it is nonexpansive with respect to the metric ρ defined, for all $x, y \in C$ as

$$\rho(x, y) = \Big[\sum_{j=1}^{n} \Big(\sum_{i=j}^{n} \alpha_i \Big) d(T^{j-1} x, T^{j-1} y)^p \Big]^{\frac{1}{p}}. \tag{2.3}$$

Moreover, proving a fixed point theorem for nonexpansive mappings, we get a fixed point theorem for some the class of (α, p)-nonexpansive mappings [5].

It is easy to see that ρ is equivalent to the metric d on C. When $p = 1$, T is called α-nonexpansive. Clearly $d(Tx, Ty)^p \leq \frac{1}{\alpha_1} d(x, y)^p$, that is $d(Tx, Ty) \leq \Big(\frac{1}{\alpha_1^{1/p}}\Big) d(x, y)$, and so $k(T) \leq \frac{1}{\alpha_1^{1/p}}$, hence

$$d(T^j x, T^j y) \leq \Big(\frac{1}{\alpha_1^{1/p}} \Big)^j d(x, y). \tag{2.4}$$

For simplicity, we will generally discuss the case $n = 2$. That is, $T : C \to C$ is $((\alpha_1, \alpha_2), p)$-nonexpansive if for some $p \in [1, \infty)$, we have

$$\alpha_1 d(Tx, Ty)^p + \alpha_2 d(T^2x, T^2y)^p \leq d(x, y)^p,$$

for all $x, y \in C$. When the multi-index α isn't specified, we say T is mean nonexpansive. As one can imagine, (α, p)-nonexpansive mappings are natural to study in L^p spaces. It is very well known that (real) Hilbert spaces are the only Banach spaces which are $CAT(0)$. The following is an example of a $\left(\frac{1}{2}, \frac{1}{2}\right)$-2-nonexpansive map which is defined on $(l^2, \|.\|_2)$ in which none of its iterates are nonexpansive [3].

Example 2.4. ([3]) Let $(l^2, \|.\|_2)$ be a $CAT(0)$ space and let $\tau : [-1, 1] \to [-1, 1]$ be given by

$$\tau(t) := \begin{cases} \sqrt{2}t + (\sqrt{2} - 1), & -1 \leq t \leq -t_0; \\ 0, & -t_0 \leq t \leq t_0; \\ \sqrt{2}t - (\sqrt{2} - 1), & t_0 \leq t \leq 1, \end{cases}$$

where $t_0 := \dfrac{(\sqrt{2} - 1)}{\sqrt{2}}$. Then the following facts that τ are considerable:

(1) τ is Lipschitz with $k(\tau) = \sqrt{2}$, and
(2) $|\tau(t)| \leq |t|$ for all $t \in [-1, 1]$.

Let B_{l^2} denotes the closed unit ball of $(l^2, \|.\|_2)$ and for any $x \in l^2$, define T by

$$T(x_1, x_2, \cdots) := \left(\tau(x_2), \sqrt{\frac{2}{3}}x_3, x_4, x_5, \cdots\right),$$

is a $\left(\frac{1}{2}, \frac{1}{2}\right)$-2-nonexpansive map in which each iterate T^j is not nonexpansive.

Example 2.5. The following are examples of (α, p)-nonexpansive mappings.

(1) If T is identity or constant mapping, then T is (α, p)-nonexpansive mapping.
(2) Let $C = [0, 1] \times [0, 1]$ and $T(x, y) = (x, 0)$ for all $(x, y) \in C$ be the projection. Therefore, T is an (α, p)-nonexpansive mapping.
(3) Let $C = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0,\ x^2 + y^2 \leq 1\}$ and $T(x, y) = (y, x)$ for all $(x, y) \in C$ with Euclidian metric. Then T is an $((\alpha_1, \alpha_2), p)$-nonexpansive mapping.

3. Methods

The properties of the fixed points set are studied for nonexpansive mappings in $CAT(0)$ spaces. Many of them have been derived from new condition on these mappings, which makes the nonexpansive mapping $T_\alpha := \alpha_1 T \oplus \alpha_2 T^2 \oplus \cdots \oplus \alpha_n T^n$.

The goal of this article is to introduce some fixed point theorems for (α, p)-nonexpansive mappings for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in $CAT(0)$ spaces.

4. Discussion and main results

At first we start with $(\alpha, 1)$-nonexpansive mappings and next, we get the new results for (α, p)-nonexpansive mappings.

4.1. $(\alpha, 1)$**-nonexpansive mappings.** Let (M, d) be a complete $CAT(0)$ space and C be a nonempty convex subset of M.

Definition 4.1. We say that the (α, p)-nonexpansive mapping $T : C \to C$ is (α, p)-metrically invariant, if for any two points x and y in C, we have

$$d(T^i x, T^i y) - d(T^i x, T^j y) - d(T^j x, T^i y) + d(T^j x, T^j y) \geq 0,$$

for all $i, j \in \{1, 2, \cdots, n\}$ and $i \neq j$.

Specially, when T is (α_1, α_2)-metrically invariant, then

$$d(Tx, Ty) - d(Tx, T^2 y) - d(T^2 x, Ty) + d(T^2 x, T^2 y) \geq 0. \tag{4.1}$$

Example 4.2. In the following, we examine the correctness of (4.1) for the following mappings.

(1) If T is an identity or constant mapping, then T is an (α, p)-metrically invariant mapping.

(2) Let $C = [0, 1]$ and $Tx = \frac{i}{n}$ for all $x \in [\frac{i}{n}, \frac{i+1}{n})$, $i = 0, 1, \cdots, n-1$ and for some $n \in \mathbb{N}$. If $x, y \in [\frac{i}{n}, \frac{i+1}{n})$, then $Tx = Ty = \frac{i}{n}$ and $T^2 = T$. If $x \in [\frac{i}{n}, \frac{i+1}{n})$ and $y \in [\frac{j}{n}, \frac{j+1}{n})$ such that $i \neq j$, then $Tx = \frac{i}{n}$, $Ty = \frac{j}{n}$ and $T^2 = T$. Therefore, the relation (4.1) is established.

(3) Let $C = [0, 1] \times [0, 1]$ and $T(x, y) = (x, 0)$ for all $(x, y) \in C$ be a projective operator. Then all distances in (4.1) are equal. Therefore T is an (α, p)-metrically invariant mapping.

(4) Let $C = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0,\ x^2 + y^2 \leq 2^2\}$ and $T(x, y) = (y, x)$ for all $(x, y) \in C$. If for all $(x_1, y_1), (x_2, y_2) \in C$, we have $(x_2 - y_2)(y_1 - x_1) \geq 0$, then the relation (4.1) is established and then T is an (α, p)-metrically invariant mapping.

The following lemma plays an important role in this article.

Lemma 4.3. *If T is an (α, p)-metrically invariant mapping, then the mapping T_α is a nonexpansive mapping.*

Proof. Suppose that $\alpha = (\alpha_1, \alpha_2)$. So, by Lemma 2.1, we have

$$\begin{aligned} d(T_\alpha x, T_\alpha y) &= d(\alpha_1 Tx \oplus \alpha_2 T^2 x, \alpha_1 Ty \oplus \alpha_2 T^2 y) \\ &\leq \alpha_1 d(Tx, Ty) + \alpha_2 d(T^2 x, T^2 y) \\ &\leq d(x, y) \end{aligned}$$

for all $x, y \in C$.

Similarly by Definition 4.1 and (2.2) for any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, we have

$$\begin{aligned} d(T_\alpha x, T_\alpha y) &= d(\alpha_1 Tx \oplus \alpha_2 T^2 x \oplus \cdots \oplus \alpha_n T^n x, \alpha_1 Ty \oplus \alpha_2 T^2 y \oplus \cdots \oplus \alpha_n T^n y) \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j d(T^i x, T^j y) \\ &= \sum_{i=1}^{n} \alpha_i^2 d(T^i x, T^i y) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \alpha_i \alpha_j d(T^i x, T^j y), \\ &= \sum_{i=1}^{n} \alpha_i d(T^i x, T^i y) - \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \alpha_j d(T^i x, T^i y) \\ &\quad + \sum_{i=1}^{n} \sum_{j=1j, j\neq i}^{n} \alpha_i \alpha_j d(T^i x, T^j y) \\ &\leq d(x, y) - \sum_{i=1}^{n} \sum_{j=1j, j\neq i}^{n} \alpha_i \alpha_j \\ &\quad \times \Big(d(T^i x, T^i y) - d(T^i x, T^j y) - d(T^j x, T^i y) + d(T^j x, T^j y) \Big) \\ &\leq d(x, y) \end{aligned}$$

for all $x, y \in C$, where $\alpha_i^2 = \alpha_i (1 - \sum_{j=1j, j\neq i}^{n} \alpha_j)$. □

In all the next results, we assume that C is a nonempty, bounded, closed and convex subset of complete $CAT(0)$ space (M, d).

Lemma 4.4. *If $S : C \to C$ is nonexpansive, then*

$$d(S) := \inf\{d(x, Sx) : x \in C\} = 0.$$

Proof. Define $S_\epsilon : C \to C$ by

$$S_\epsilon x = \epsilon z \oplus (1 - \epsilon) Sx,$$

where, for each $Sx, Sy \in C$ there exists $z \in [Sx, Sy]$ such that $d(Sx, z) + d(z, Sy) = d(Sx, Sy)$ and $\epsilon \in (0, 1)$. From Lemma 2.1, S_ϵ is a contraction. Indeed

$$\begin{aligned} d(S_\epsilon x, S_\epsilon y) &= d(\epsilon z \oplus (1-\epsilon)Sx, \epsilon z \oplus (1-\epsilon)Sy) \\ &\leq \epsilon d(z, z) + (1-\epsilon) d(Sx, Sy) \\ &\leq (1-\epsilon) d(x, y), \end{aligned}$$

for all $x, y \in C$. By the Banach's contraction principle, there exists a unique point $x_\epsilon \in C$ such that $S_\epsilon x_\epsilon = x_\epsilon$. Thus

$$\begin{aligned} d(x_\epsilon, Sx_\epsilon) &= d(S_\epsilon x_\epsilon, Sx_\epsilon) \\ &= d(\epsilon z \oplus (1-\epsilon)Sx_\epsilon, Sx_\epsilon) \\ &\leq \epsilon d(z, Sx_\epsilon) + (1-\epsilon) d(Sx_\epsilon, Sx_\epsilon) \\ &\leq \epsilon \, \mathrm{diam} C. \end{aligned}$$

When $\epsilon \to 0$, desired result is obtained. □

Theorem 4.5. *Any $((\alpha_1, \alpha_2), 1)$-metrically invariant mapping $T : C \to C$ with $\alpha_1 \geq \frac{1}{2}$, has an approximate fixed point sequence.*

Proof. Fix $\epsilon > 0$. Since $T_\alpha : C \to C$ is nonexpansive, by the Lemma 4.4, we have $\inf d(T_\alpha x, x) = 0$ and thus there exists $x_\epsilon \in C$ such that

$$d(T_\alpha x_\epsilon, x_\epsilon) \leq \alpha_2 \epsilon.$$

Since T is (α_1, α_2)-nonexpansive, so

$$\begin{aligned} \alpha_1 d(T(Tx_\epsilon), Tx_\epsilon) + \alpha_2 d(T^2(Tx_\epsilon), T^2 x_\epsilon) &\leq d(Tx_\epsilon, x_\epsilon) \\ &\leq d(Tx_\epsilon, T_\alpha x_\epsilon) + d(T_\alpha x_\epsilon, x_\epsilon) \\ &\leq d(Tx_\epsilon, \alpha_1 Tx_\epsilon \oplus \alpha_2 T^2 x_\epsilon) + \alpha_2 \epsilon \\ &\leq \alpha_1 d(Tx_\epsilon, Tx_\epsilon) + \alpha_2 d(Tx_\epsilon, T^2 x_\epsilon) \\ &\quad + \alpha_2 \epsilon, \end{aligned}$$

hence

$$\alpha_1 d(T^2 x_\epsilon, Tx_\epsilon) + \alpha_2 d(T^3 x_\epsilon, T^2 x_\epsilon) \leq \alpha_2 d(Tx_\epsilon, T^2 x_\epsilon) + \alpha_2 \epsilon.$$

Thus

$$(\alpha_1 - \alpha_2) d(T^2 x_\epsilon, Tx_\epsilon) + \alpha_2 d(T^3 x_\epsilon, T^2 x_\epsilon) \leq \alpha_2 \epsilon,$$

if and only if $(2\alpha_1 - 1) d(Tx_\epsilon, T^2 x_\epsilon) + \alpha_2 d(T^3 x_\epsilon, T^2 x_\epsilon) \leq \alpha_2 \epsilon$. Since $\alpha_1 \geq \frac{1}{2}$, we know $2\alpha_1 - 1 \geq 0$, so $d(Tz_\epsilon, z_\epsilon) \leq \epsilon$ where $z_\epsilon = T^2 x_\epsilon \in C$. □

When C be a nonempty bounded subset of a complete locally compact $CAT(0)$ space, then by Theorem 2.2, C has the fixed point property for nonexpansive mappings and we have the following result.

Theorem 4.6. *Let (M, d) be a complete locally compact $CAT(0)$ space. Then any $((\alpha_1, \alpha_2), 1)$-metrically invariant mapping $T : C \to C$ has a fixed point, provided with $\alpha_1 \geq \frac{1}{2}$.*

Theorem 4.7. *Let (M, d) be a complete $CAT(0)$ space. If $T : C \to C$ is an $(\alpha, 1)$-metrically invariant mapping with $n \geq 2$ for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\alpha_1 \geq 2^{\frac{1}{1-n}}$, then*

$$d(T) = \inf\{d(x, Tx) : x \in C\} = 0.$$

Proof. According to the proof of Theorem 4.5, there exists $x \in C$ such that $d(T_\alpha x, x) \leq \epsilon$. Then,

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i d(T^i x, T^{i+1} x) &\leq d(x, Tx) \leq d(x, T_\alpha x) + d(T_\alpha x, Tx) \\
&\leq d(\alpha_1 Tx \oplus \alpha_2 T^2 x \oplus \cdots \oplus \alpha_n T^n x, Tx) + \epsilon \\
&\leq \sum_{i=2}^{n} \alpha_i d(T^i x, Tx) + \epsilon.
\end{aligned}$$

Thus by triangle inequality, we have,

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i d(T^i x, T^{i+1} x) &\leq \sum_{i=2}^{n} \Big[\alpha_i \Big(\sum_{j=2}^{i} d(T^{j-1} x, T^j x)\Big)\Big] + \epsilon \\
&= \sum_{i=2}^{n} \Big(\sum_{j=2}^{i} \alpha_i d(T^{j-1} x, T^j x)\Big) + \epsilon \\
&= \sum_{j=2}^{n} \Big[\Big(\sum_{i=j}^{n} \alpha_i\Big) d(T^{j-1} x, T^j x)\Big] + \epsilon.
\end{aligned}$$

So we have

$$\alpha_n d(T^n x, T^{n+1} x) \leq \sum_{j=2}^{n} \Big[\Big(\sum_{i=j}^{n} \alpha_i\Big) - \alpha_{j-1}\Big] d(T^{j-1} x, T^j x) + \epsilon.$$

The first term on the right-hand side equals to

$$\Big[\Big(\sum_{i=2}^{n} \alpha_i\Big) - \alpha_1\Big] d(Tx, T^2 x) = (1 - 2\alpha_1) d(Tx, T^2 x).$$

Therefore, for any $j = 3, \cdots, n$, we have

$$\Big(\sum_{i=j}^{n} \alpha_i\Big) - \alpha_{j-1} = 1 - 2\alpha_{j-1} - \Big(\sum_{i=1}^{j-2} \alpha_i\Big) \leq 1 - \alpha_1.$$

By the above relation and (2.4), we have the following

$$\begin{aligned}
\alpha_n d(T^n x, T^{n+1}x) &\leq (1-2\alpha_1) d(Tx, T^2x) + \sum_{j=3}^{n} (1-\alpha_1) d(T^{j-1}x, T^j x) + \epsilon \\
&\leq \Big[1 - 2\alpha_1 + (1-\alpha_1)\Big(\sum_{j=3}^{n} \frac{1}{\alpha_1^{j-2}}\Big)\Big] d(Tx, T^2x) + \epsilon \\
&= \Big[1 - 2\alpha_1 + (1-\alpha_1)\Big(\frac{\sum_{j=3}^{n} \alpha_1^{j-3}}{\alpha_1^{n-2}}\Big)\Big] d(Tx, T^2x) + \epsilon \\
&= \Big(1 - 2\alpha_1 + \frac{1-\alpha_1^{n-2}}{\alpha_1^{n-2}}\Big) d(Tx, T^2x) + \epsilon.
\end{aligned}$$

On the other hand

$$1 - 2\alpha_1 + \frac{1-\alpha_1^{n-2}}{\alpha_1^{n-2}} \leq 0$$

holds if

$$\alpha_1 \geq 2^{\frac{1}{1-n}}.$$

□

If $x = T_\alpha x$ is a fixed point of T_α, then with $\epsilon = 0$ we see that $T^n x$ is a fixed point of T. So we have the following theorem by Theorem 2.2.

Theorem 4.8. *Let (M, d) be a complete locally compact $CAT(0)$ space. Then C has the fixed point property for $(\alpha, 1)$-metrically invariant mappings with $n \geq 2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\alpha_1 \geq 2^{\frac{1}{1-n}}$. In other words, if for some α and p satisfying the above condition, a mapping $T : C \to C$ is nonexpansive with respect to the metric ρ given by formula (2.3) with $p = 1$, then T has a fixed point.*

Example 4.9. Let $M = \ell^1$ with a metric inherited from the standard norm, $\|x\| = \|(x_1, x_2, \cdots)\| = \sum_{i=1}^{\infty} |x_i|$, $C := B_{\ell^1}$ the closed unit ball of ℓ^1 and $\alpha = (1-k, k)$ for all $0 < k < 1$. Suppose $T : C \to C$ is a mapping defined for every $(x_1, x_2, \cdots) \in C$ by

$$T(x) = (s_2 x_2, s_3 x_3, \cdots, s_j x_j, \cdots),$$

where $s_j = \dfrac{1-(-k)^{j-1}}{1-(-k)^j}$ for $j \in \{2, 3, \cdots\}$. Then

$$T^2(x) = (r_3x_3, r_4x_4, \cdots, r_jx_j, \cdots),$$

where $r_j = \dfrac{1-(-k)^{j-2}}{1-(-k)^j}$ for $j \in \{3, 4, \cdots\}$. Therefore, for any $(x_1, x_2, \cdots) \in C$, we get

$$\begin{aligned}(1-k)\|Tx\| + k\|T^2x\| &= (1-k)\sum_{i=2}^{\infty} s_i|x_i| + k\sum_{i=3}^{\infty} r_i|x_i| \\ &= (1-k)\frac{(1+k)}{(1-k^2)}|x_2| + \sum_{i=3}^{\infty}((1-k)s_i + kr_i)|x_i| \\ &= |x_2| + \sum_{i=3}^{\infty}|x_i| \\ &\leq \|x\|.\end{aligned}$$

Since T is linear, so T^2 is also linear and therefore T is $(1-k, k)$-nonexpansive for $0 < k < 1$. As the proof of Theorem 4.5, when $k \leq \frac{1}{2}$ we have $d(T) = 0$. Moreover, T is nonexpansive with respect to the metric ρ given by formula (2.3) with $p = 1$. That is,

$$\begin{aligned}\rho(Tx, Ty) &= \|Tx - Ty\| + k\|T^2x - T^2y\| \\ &= ((1-k) + k)\|Tx - Ty\| + k\|T^2x - T^2y\| \\ &= k\|Tx - Ty\| + (1-k)\|Tx - Ty\| + k\|T^2x - T^2y\| \\ &\leq k\|Tx - Ty\| + \|x - y\| \\ &= \rho(x, y).\end{aligned}$$

Clearly, $(a, \dfrac{a}{s_2}, \dfrac{a}{s_2s_3}, \dfrac{a}{s_2s_3s_4}, \cdots, \dfrac{a}{s_2s_3\cdots s_j}, \cdots)$ is a fixed point of T for all $a \in (0, 1)$.

5. (α, p)-nonexpansive mappings

Here, we generalize the results of Section 3 for (α, p)-nonexpansive mappings, when $p \geq 1$.

Theorem 5.1. *Let (M, d) be a complete $CAT(0)$ space. If $diam(C) > 0$ and $T : C \to C$ is (α, p)-metrically invariant with $n \geq 2$ for some $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $p \geq 1$ such that*

$$(1-\alpha_1)\Big(1-\alpha_1^{\frac{n-1}{p}}\Big) \leq \alpha_1^{\frac{n-1}{p}}\Big(1-\alpha_1^{\frac{1}{p}}\Big), \tag{5.1}$$

then $d(T) = \inf\{d(x, Tx) : x \in C\} = 0$.

Proof. If $p = 1$, then the conclusion follows from Theorem 4.7. Suppose $p > 1$. As the proof of Theorem 4.5, there exists $x \in C$ such that $d(T_\alpha x, x) \le \epsilon$. Then, from (2.4) and triangle inequality we have

$$\begin{aligned}
\Big(\sum_{i=1}^{n} \alpha_i d(T^i x, T^{i+1}x)^p\Big)^{\frac{1}{p}} &\le d(x, Tx) \le d(x, T_\alpha x) + d(T_\alpha x, Tx) \\
&\le d(\alpha_1 Tx \oplus \alpha_2 T^2 x \oplus \cdots \oplus \alpha_n T^n x, Tx) + \epsilon \\
&\le \sum_{i=2}^{n} \alpha_i d(T^i x, Tx) + \epsilon \\
&\le \sum_{i=2}^{n} \alpha_i \sum_{j=2}^{i} d(T^{j-1}x, T^j x) + \epsilon \\
&\le \Big[\sum_{i=2}^{n} \sum_{j=2}^{i} \alpha_i \frac{1}{\alpha_1^{(j-2)/p}}\Big] d(Tx, T^2 x) + \epsilon \\
&\le \Big[\sum_{j=2}^{n} \Big(\sum_{i=j}^{n} \alpha_i\Big) \frac{1}{\alpha_1^{(j-2)/p}}\Big] d(Tx, T^2 x) + \epsilon \\
&\le (1 - \alpha_1)\Big(\sum_{j=2}^{n} \frac{1}{\alpha_1^{(j-2)/p}}\Big) d(Tx, T^2 x) + \epsilon \\
&= (1 - \alpha_1). \frac{1}{\alpha_1^{(n-2)/p}} \Big(\sum_{j=2}^{n} \alpha_1^{(j-2)/p}\Big) d(Tx, T^2 x) + \epsilon \\
&= (1 - \alpha_1). \frac{1}{\alpha_1^{(n-2)/p}} . \frac{1 - \alpha_1^{(n-1)/p}}{1 - \alpha_1^{1/p}} d(Tx, T^2 x) + \epsilon.
\end{aligned}$$

Put

$$K = (1 - \alpha_1). \frac{1}{\alpha_1^{(n-2)/p}} . \frac{1 - \alpha_1^{(n-1)/p}}{1 - \alpha_1^{1/p}},$$

then by (5.1) we have

$$\Big(\sum_{i=1}^{n} \alpha_i d(T^i x, T^{i+1}x)^p\Big)^{\frac{1}{p}} \le K d(Tx, T^2 x) + \epsilon.$$

Thus

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^{i+1}x)^p \le (K d(Tx, T^2 x) + \epsilon)^p. \tag{5.2}$$

We can select $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Using Hölder inequality, we obtain

$$\begin{aligned}(Kd(Tx,T^2x)+\epsilon)^p &= (Kd(Tx,T^2x)+\epsilon^{\frac{1}{p}}.\epsilon^{\frac{1}{q}})^p \\ &\le \Big[\big(K^p d(Tx,T^2x)^p+\epsilon\big)^{\frac{1}{p}}.(1+\epsilon)^{\frac{1}{q}}\Big]^p \\ &= \big(K^p d(Tx,T^2x)^p+\epsilon\big)(1+\epsilon)^{p-1} \\ &= K^p d(Tx,T^2x)^p + \Big[(1+\epsilon)^{p-1}-1\Big]K^p d(Tx,T^2x)^p \\ &\quad + \epsilon(1+\epsilon)^{p-1} \\ &\le K^p d(Tx,T^2x)^p + \Big[(1+\epsilon)^{p-1}-1\Big]K^p(\mathrm{diam}C)^p \\ &\quad + \epsilon(1+\epsilon)^{p-1}.\end{aligned}$$

Thus

$$\begin{aligned}&(\alpha_1 - K^p)d(Tx,T^2x)^p + \sum_{i=2}^{n}\alpha_i d(T^ix,T^{i+1}x)^p \\ &\le \Big[(1+\epsilon)^{p-1}-1\Big]K^p(\mathrm{diam}C)^p + \epsilon(1+\epsilon)^{p-1}.\end{aligned}$$

By assumption $\alpha_1 - K^p \ge 0$, this implies that

$$\alpha_n d(T^nx,T^{n+1}x)^p \le \Big[(1+\epsilon)^{p-1}-1\Big]K^p(\mathrm{diam}C)^p + \epsilon(1+\epsilon)^{p-1}.$$

Since $\alpha_n > 0$ and the right-hand side of the above inequality converges to 0 if $\epsilon \to 0$, we finally obtain $d(T) = 0$. □

For example, if $n = 2$, the condition (5.1) in Theorem 5.1 reduced to $\alpha_2^p + \alpha_2 \le 1$. If C has the fixed point property for nonexpansive mappings, then by Theorem 2.2 we obtain:

Theorem 5.2. *Let (M, d) be a complete locally compact $CAT(0)$ space. Then C has the fixed point property for (α, p)-metrically invariant mappings with $n \ge 2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $p \ge 1$ such that*

$$(1-\alpha_1)\Big(1-\alpha_1^{\frac{n-1}{p}}\Big) \le \alpha_1^{\frac{n-1}{p}}\Big(1-\alpha_1^{\frac{1}{p}}\Big).$$

Equivalently, if for some α and p satisfying the above condition, a mapping $T : C \to C$ is nonexpansive with respect to the metric ρ given by formula (2.3), then T has a fixed point. Moreover, if

$$(1-\alpha_1)\Big(1-\alpha_1^{\frac{n-1}{p}}\Big) < \alpha_1^{\frac{n-1}{p}}\Big(1-\alpha_1^{\frac{1}{p}}\Big),$$

then $Fix(T) = Fix(T_\alpha)$.

Proof. It is clear that $Fix(T) \subseteq Fix(T_\alpha)$. We claim that $Fix(T_\alpha) \subseteq Fix(T)$. Since $T_\alpha : C \to C$ is nonexpansive, there exists $x \in C$ such that

$$T_\alpha x = \alpha_1 Tx \oplus \alpha_2 T^2 x \oplus \cdots \oplus \alpha_n T^n x = x. \tag{5.3}$$

Now inequality (5.2) takes the form

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^{i+1} x)^p \leq (Kd(Tx, T^2 x))^p.$$

Note that the above inequality also holds for $p = 1$. Therefore,

$$d(T^i x, T^{i+1} x) \leq (\alpha_1 - K^p) d(Tx, T^2 x)^p + \sum_{i=2}^{n} \alpha_i d(T^i x, T^{i-1} x)^p \leq 0.$$

Since $(\alpha_1 - K^p) \geq 0$ and $\alpha_n > 0$, this implies that $T^i x$ is fixed under T for all $1 \leq i \leq n$. If $(\alpha_1 - K^p) > 0$, then

$$Tx = T^2 x = \cdots = T^n x.$$

From (5.3), we get that $Tx = x$. Thus, $Fix(T_\alpha) \subseteq Fix(T)$. □

Conclusion: After introducing the class of (α, p)-nonexpansive mappings in Banach spaces [4] and the fixed point theorems, developed and many articles were stablished in such way. Many of these results are not verifiable in metric spaces. Therefore, we need to find metric spaces that have similar properties to the Banach spaces.

In this paper, we studied the existence of fixed points for (α, p)-nonexpansive mappings in $CAT(0)$ spaces.

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