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EXISTENCE OF SUBHARMONIC SOLUTIONS FOR NON-AUTONOMOUS SECOND ORDER HAMILTONIAN SYSTEMS UNDER SOME WEAK CONDITIONS

Zhao-Hong Sun[1], Yong-Qin Ni[2] and Jong Kyu Kim[3]

[1]Department of Computer Science, Guangzhou
Zhong kai University Of Agriculture And Engineering
Guangzhou, guangdong 510225, P. R. China
e-mail: sunzh60@163.com

[2]Department of Physics, Yuxi Normal College,
Yuxi, Yunnan 653100, P. R. China
e-mail: wdjy@yxtc.net

[3]Department of Mathematics Education, Kyungnam University,
Masan, Kyungnam, 631-701, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In the present paper, we study that under some weak conditions, the following non-autonomous second order Hamiltonian systems

$$\ddot{u}(t) + \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R$$

have infinitely distinct subharmonic solutions. The results in this paper develop and generalize some recent results.

1. Introduction and Preliminaries

Consider the second order Hamiltonian systems

$$\ddot{u}(t) + \bigtriangledown F(t, u(t)) = 0 \qquad a.\ e.\ t \in R \tag{1}$$

where $F : R \times R^N \to R$ is $T-$periodic $(T > 0)$ in t for all $x \in R^N$, that is

$$F(t + T, x) = F(t, x) \tag{2}$$

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for all $x \in R^N$ and a. e. $t \in R$, and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in R^N$ and continuously differentiable in x for a. e. $t \in [0,T]$, and there exist $a \in L^1(R^+;R^+), b \in L^1(0,T;R^+)$, such that $|F(t,x)| \le a(|x|)b(t)$, $|\bigtriangledown F(t,x)| \le a(|x|)b(t)$ for all $x \in R^N$ and a. e. $t \in R$.

A solution of problem (1) is called to be subharmonic if it is $kT-$periodic solution for some positive integer k.

A function $G : R^N \to R$ is called to be $(\lambda,\mu)-$subconvex if

$$G(\lambda(x+y)) \le \mu(G(x)+G(y))$$

for some $\lambda, \mu > 0$ and all $x, y \in R^N$.

Let $H^1_{kT} = \{u : [0,kT] \to R^N | u$ is absolutely continuous, $u(0) = u(kT)$ and $\dot{u} \in L^2(0,kT;R^N)\}$ is a Hilbert space with the norm defined by

$$\|u\| = [\int_0^{kT} |u(t)|^2 dt + \int_0^{kT} |\dot{u}(t)|^2 dt]^{\frac{1}{2}}$$

and $\|u\|_\infty = \max_{0\le t\le kT} |u(t)|$ for $u \in H^1_{kT}$.

The corresponding functional φ_k on H^1_{kT} given by

$$\varphi_k(u) = \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} F(t,u(t))dt$$

is continuously differentiable and weakly lower semi-continuous on H^1_{kT} (see [1]). Moreover one has

$$< \varphi'_k(u), v >= \int_0^{kT} [(\dot{u}(t),\dot{v}(t)) - (\bigtriangledown F(t,u(t)), v(t))]dt$$

for all $u, v \in H^1_{kT}$, where $(\cdot,\cdot)$ denotes the inner product in R^N. It is well known that the $kT-$periodic solutions of problem (1) correspond to the critical points of functional φ_k.

For $u \in H^1_{kT}$, let $\overline{u} = (kT)^{-1}\int_0^{kT} u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}$. Then one has Sobolev's inequality

$$\|\widetilde{u}\|^2_\infty \le \frac{kT}{12}\int_0^{kT} |\dot{u}(t)|^2 dt \tag{3}$$

and Wertinger's inequality

$$\int_0^{kT} |\widetilde{u}(t)|^2 dt \le \frac{k^2T^2}{4\pi^2}\int_0^{kT} |\dot{u}(t)|^2 dt. \tag{4}$$

Under the conditions that there exists $h \in L^1(0,T;R^+)$ such that

$$|\bigtriangledown F(t,x)| \le h(t) \tag{5}$$

for all $x \in R^N$ and a. e. $t \in [0,T]$, and that

$$\int_0^T F(t,x)dt \to +\infty \tag{6}$$

as $|x| \to +\infty$, the existence of $T-$periodic solutions is proved in [1]. Meanwhile, [2] proves that problem has infinitely distinct subharmonic solutions under (5) and the condition that

$$F(t,x) \to +\infty \tag{7}$$

as $|x| \to +\infty$ uniformly for $t \in [0,T]$. Motivated by the results of [1, 2], a natural question is whether problem (1) has infinitely distinct subharmonic solutions under (5) and (6). In [3] a positive answer was given if in addition $F(t,x)$ is convex in x for every $t \in [0,T]$. Tang in [4] generalizes the existence result of $T-$periodic solutions in [1] to the sublinear case. The existence of $T-$periodic solutions is proved in [4] under the conditions that there exist $g, h \in L^1(0,T;R^+)$ and $\alpha \in [0,1)$ such that

$$| \bigtriangledown (F(t,x)| \leq g(t)|x|^\alpha + h(t) \tag{8}$$

for all $x \in R^N$ and a. e. $t \in [0,T]$, and that

$$|x|^{-2\alpha} \int_0^{kT} F(t,x)dt \to +\infty \quad as \quad |x| \to +\infty. \tag{9}$$

It has been proved that problem (1) has infinitely distinct subharmonic solutions under suitable conditions (see [1-4]). Recently, Tang-Wu [5] considered the nonconvex case and generalized the existence result of subharmonic solutions to the sublinear case under a condition weaker than (6) but stronger than (7) and Zhao-Wu [6] consider the existence of $T-$periodic solutions with saddle point character. Inspired and motivated by the results due to Mawhin-Willem [1], F. Giannoni [2], Fonda-Ramos [3],Tang[4], Tang-Wu[5] and Zhao-Wu [6, 7, 8], we have studied in [9] the existence of subharmonic solutions with saddle point character under condition (7) and in [10] the existence of subharmonic solutions under some else other conditions. In this paper, we shall continue to consider the existence of subharmonic solutions under some weak conditions by using the critical point reduction method and using the minimax methods. Therefore the results in this paper develop and generalize the corresponding results.

In the sequel, we set

$$e_k(t) = k(\cos k^{-1}\omega t)x_0$$

for all $t \in R$ and some $x_0 \in R^N$ with $|x_0| = 1$, where $\omega = 2\pi/T$.

2. MAIN RESULTS AND PROOF

Now we state and prove our main results.

Theorem 2.1. *Suppose that F satisfies assumption (A), (2) and the following conditions:*

(i) there exists a function $\lambda \in L^1(0,T;R)$ with $\int_0^T \lambda(t)dt > 0$ such that $\nabla F(t,\cdot)$ is $\lambda(t)-$monotone, that is

$$(\nabla F(t,x) - \nabla F(t,y), x-y) \geq \lambda(t)|x-y|^2; \qquad (10)$$

for all $x, y \in R^N$ and a. e. $t \in [0,T]$;
(ii) there exist $g, h \in L^1(0,T;R), M > 1$ and $\alpha \in [1,2)$ such that

$$F(t,x) \leq g(t)|x|^\alpha + h(t)$$

for all $x \in R^N$ and $|x| \geq M$ and a. e. $t \in [0,T]$;
(iii) there exists some $e_k(t) = k(cosk^{-1}\omega t)x_0$ such that

$$(\bigtriangledown F(t, x+se_k), e_k) \geq k^{-1}(e_k, e_k)$$

for all $x \in R^N$ and $s \in [0,1]$;
(iv) there exists some $x \in R^N$ such that

$$\int_0^T F(t,x)dt \geq 0.$$

Then problem (1) has $kT-$periodic solutions u_k with saddle point character in H^1_{kT} for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Proof. Without loss of generality, we may assume that functions b in assumption(A), λ in (10) and g, h in (8) are $T-$ periodic and assumptions (A), (10), (8) and (9) hold for all $t \in R$ by the $T-$ periodicity of $F(t,x)$ in the first variable.

Set $\widetilde{H}^1_{kT} = \{u \in H^1_{kT} | \overline{u} = 0\}$, then $H^1_{kT} = R^N \bigoplus \widetilde{H}^1_{kT}$, obviously. Define the function Ψ as follows:

$$\Psi(u) = \sup_{x \in R^N} \varphi_k(u+x) \qquad \forall u \in \widetilde{H}^1_{kT}.$$

For each fixed $u \in \widetilde{H}^1_{kT}$ and any $x_1, x_2 \in R^N$, one has

$$\int_0^{kT} (\nabla F(t, u(t)+x_1) - \nabla F(t, u(t)+x_2), x_1 - x_2)dt \geq |x_1 - x_2|^2 \int_0^{kT} \lambda(t)dt$$

Consequently,

$$\langle -\varphi'_k(u(t)+x_1) - (-\varphi'_k(u(t)+x_2)), x_1 - x_2 \rangle \geq |x_1 - x_2|^2 \int_0^{kT} \lambda(t)dt.$$

By virtue of Theorem 2.3 in [11] there exists a continuous mapping $\theta : \widetilde{H}^1_{kT} \to R^N$ such that $\varphi_k(u+\theta(u)) = \Psi(u)$ for all $u \in \widetilde{H}^1_{kT}, \Psi : \widetilde{H}^1_{kT} \to R$ is continuously differentiable, and $\Psi'(u) = \varphi'_k(u+\theta(u))|_{\widetilde{H}^1_{kT}}$ for all $u \in \widetilde{H}^1_{kT}$. Hence, $u \in \widetilde{H}^1_{kT}$ is a critical point of Ψ implies $u+\theta(u)$ is a critical point of φ_k.

Moreover, for each $u \in \widetilde{H}^1_{kT}$, by condition (ii) and Sobolev's inequality one has

$$\begin{aligned}\Psi(u) \geq & \varphi_k(u) = \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} F(t,u(t))dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} g(t)|u(t)|^\alpha dt - \int_0^{kT} h(t)dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - \|u\|_\infty^\alpha \int_0^{kT} |g(t)|dt - \int_0^{kT} h(t)dt \\ & \geq \frac{1}{2}\int_0^{kT} |\dot{u}(t)|^2 dt - C_1(\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha}{2}} - C_2\end{aligned} \tag{11}$$

for all $u \in \widetilde{H}^1_{kT}$ and some positive constants C_1 and C_2. By Wertinger's inequality, one has

$$\|u\| \to +\infty \Leftrightarrow \|\dot{u}\|_2 \to +\infty$$

on $\widetilde{H}^1_{kT}$, then (11) implies that $\Psi(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, there exists a point $u_0 \in \widetilde{H}^1_{kT}$ such that $\Psi(u_0) = \min_{\widetilde{H}^1_{kT}} \Psi(u)$, and hence $u_k = u_0 + \theta(u_0)$ is a solution with saddle point character of problem (1) in H^1_{kT}.

By the definition of u_k, we have

$$\varphi_k(u_k) = \min_{u \in \widetilde{H}^1_{kT}} \sup_{x \in R^N} \varphi_k(x+u) \leq \sup_{x \in R^N} \varphi_k(x+e_k) = \sup_{R^N + e_k} \varphi_k. \tag{12}$$

Now we prove that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

For $e_k(t) = k(\cos k^{-1}\omega t)x_0$ we have

$$\dot{e}_k(t) = -\omega(\sin k^{-1}\omega t)x_0$$

for all $t \in R$ which implies that

$$\int_0^{kT} |\dot{e}_k(t)|^2 dt = \frac{1}{2}kT\omega^2.$$

Hence one has

$$\varphi_k(x+e_k) = \frac{1}{4}kT\omega^2 - \int_0^{kT} F(t, x+e_k)dt$$

for all $x \in R^N$. By condition (iii) we have

$$\begin{aligned}
&k^{-1}\varphi_k(x+e_k)\\
&\quad= \frac{1}{4}T\omega^2 - k^{-1}\int_0^{kT}[F(t,x+e_k)-F(t,x)]dt - k^{-1}\int_0^{kT}F(t,x)dt\\
&\quad= \frac{1}{4}T\omega^2 - k^{-1}\int_0^{kT}\int_0^1(\bigtriangledown F(t,x+se_k),e_k)dsdt - \int_0^T F(t,x)dt\\
&\quad\le \frac{1}{4}T\omega^2 - k^{-2}\int_0^{kT}(e_k,e_k)dt - \int_0^T F(t,x)dt\\
&\quad\le \frac{1}{4}T\omega^2 - \int_0^{kT}\cos^2(k^{-1}\omega t)dt - \int_0^T F(t,x)dt\\
&\quad= \frac{1}{4}T\omega^2 - \frac{Tk}{2} - \int_0^T F(t,x)dt
\end{aligned} \tag{13}$$

Hence by assumption (A) and condition (iv) there exists some constant C such that

$$\sup_{x\in R^N} k^{-1}\varphi_k(x+e_k) \le C - \frac{Tk}{2}$$

for all k, so we obtain

$$\limsup_{k\to+\infty}\sup_{x\in R^N} k^{-1}\varphi_k(x+e_k) = -\infty. \tag{14}$$

Then following the same way in [5] we complete our proof. □

Remark 2.2. *There indeed exist functions $F(t,u)$ satisfy the condition (iii), for example,*

$$F(t,u) = (e_k,u).$$

Remark 2.3. *Theorem 2.1 is not required any coercive condition on the function $F(t,x)$, so our result is a real improvement to some extent.*

Theorem 2.4. *Suppose that F satisfies assumption (A), (2) and the following conditions:*
(i) there exist $g,h \in L^1(0,T;R^+)$ and $\alpha \in [0,1)$ such that

$$|\bigtriangledown F(t,x)| \le g(t)|x|^\alpha + h(t)$$

for all $x \in R^N$ and a. e. $t \in [0,T]$;
(ii) there exists some $e_k(t) = k(\cos k^{-1}\omega t)x_0$ such that

$$(\bigtriangledown F(t,x+se_k),e_k) \ge k^{-1}(e_k,e_k)$$

for all $x \in R^N$ and $s \in [0,1]$;

(iii)

$$|x|^{-2\alpha}\int_0^T F(t,x) \to +\infty \quad as \quad |x| \to +\infty.$$

Then problem (1) has kT−periodic solutions $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$

Proof. It is well known that φ_k satisfies the (PS) condition under conditions (i), (iii)(see [5]). To complete our theorem, we now prove that φ_k satisfies the other conditions of the saddle point theorem. Since

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \to +\infty$$

as $|x| \to +\infty$, so for every $\beta > 0$ there exists $M \geq 1$ such that

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \geq \beta \tag{15}$$

which implies that

$$\int_0^T F(t,x)dt \geq \beta M^{2\alpha} \tag{16}$$

for all $|x| \geq M$.

For $e_k(t) = k(\cos k^{-1}\omega t)x_0$ we have $\dot{e}_k(t) = -\omega(\sin k^{-1}\omega t)x_0$ for all $t \in R$ which implies that

$$\int_0^{kT} |\dot{e}_k(t)|^2 dt = \frac{1}{2}kT\omega^2.$$

Hence one has

$$\varphi_k(x+e_k) = \frac{1}{4}kT\omega^2 - \int_0^{kT} F(t, x + k(\cos k^{-1}\omega t)x_0)dt$$

for all $x \in R^N$. So by (16) one has

$$\begin{aligned}\varphi_k(x+e_k) &= \frac{1}{4}kT\omega^2 - \sum_{i=0}^{k-1}\int_0^T F(t, x + k(\cos k^{-1}\omega(t+iT))x_0)dt \\ &\leq \frac{1}{4}kT\omega^2 - k\beta M^{2\alpha}\end{aligned}$$

for all $|x| \geq M + k$, which implies that

$$\varphi_k(x+e_k) \to -\infty \tag{17}$$

as $|x| \to +\infty$ by the arbitrariness of β.

On the other hand, we have

$$\varphi_k(u) \to +\infty \tag{18}$$

as $\|u\| \to \infty$ in $\widetilde{H}^1_{kT} = \{u \in H^1_{kT} | \overline{u} = 0\}$. In fact, we have

$$\begin{aligned} &| \int_0^{kT} [F(t, u(t)) - F(t, 0)]dt| \\ &\le C_3 (\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} + C_4 (\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{1}{2}} \end{aligned}$$

for all $u \in \widetilde{H}^1_{kT}$ and some positive constants C_3 and C_4. Hence we have

$$\begin{aligned} \varphi_k(u) &= \frac{1}{2} \int_0^{kT} |\dot{u}(t)|^2 dt - \int_0^{kT} [F(t, u(t)) - F(t, 0)]dt - \int_0^{kT} F(t, 0)dt \\ &\ge \frac{1}{2} \int_0^{kT} |\dot{u}(t)|^2 dt - C_3 (\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} \\ &\quad - C_4 (\int_0^{kT} |\dot{u}(t)|^2 dt)^{\frac{1}{2}} - \int_0^{kT} F(t, 0)dt \end{aligned}$$

for all $u \in \widetilde{H}^1_{kT}$. By Wertinger's inequality, one has

$$\|u\| \to \infty \Leftrightarrow \|\dot{u}\|_2 \to \infty$$

on $\widetilde{H}^1_{kT}$. Hence (18) follows from the above inequality.

So by (17), (18) and the saddle point Theorem (see Theorem 4.6 in [1]), there exists a critical point $u_k \in \widetilde{H}^1_{kT}$ for φ_k such that

$$-\infty < \inf_{\widetilde{H}^1_{kT}} \varphi_k \le \varphi_k(u_k) \le \sup_{R^N + e^k} \varphi_k.$$

By the condition (ii) we can prove Theorem 2.4 in the same way as in Theorem 2.1. □

Theorem 2.5. *Suppose that F satisfies assumption (A), (2) and the following conditions:*
(i) there exists a function $\gamma \in L^1(0, T; R)$ with $\int_0^T \gamma(t)dt > 0$ and $\alpha \in [1, 2)$ such that

$$(\nabla F(t, x) - \nabla F(t, y), x - y) \le \gamma(t)|x - y|^\alpha \tag{19}$$

for all $x, y \in R^N$ and a. e. $t \in [0, T]$;
(ii) $F(t, \cdot)$ is $(\lambda, \mu)-$subconvex, and $\nabla F(t, 0) = 0$, and there exist $g, h \in L^1(0, T; R^+)$ and $\delta \in [1, 2)$ such that

$$F(t, x) \le g(t)|x|^\delta + h(t) \tag{20}$$

for all $x \in R^N$ and a. e. $t \in [0, T]$;
(iii) there exists some $e_k(t) = k(\cos k^{-1}\omega t)x_0$ such that

$$(\nabla F(t, x + se_k), e_k) \ge k^{-1}(e_k, e_k)$$

for all $x \in R^N$ and $s \in [0, 1]$;

(iv) assume that $a(t)$ is bounded and that

$$\int_0^T F(t,x)dt \to +\infty \qquad as \qquad |x| \to +\infty. \tag{21}$$

Then problem (1) has $kT-$periodic solutions $u_k \in H^1_{kT}$ for every positive integer k such that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$.

Proof. Without loss of generality, we may assume that γ in (19) and g, h in (20) are $T-$ periodic and assumption (A) , (19) and (20) hold for all $t \in R$ by the $T-$ periodicity of $F(t,x)$ in the first variable.

Let us prove that φ_k satisfies the (PS) condition. Suppose that $\{u_n\}$ is a (PS) sequence for φ_k. As $a(t)$ is bounded function, we can assume that $a_0 = \max_{t\in R^+} |a(t)| < +\infty$. By condition (i), (ii) and Sobolev's inequality, it follows that

$$\begin{aligned}
\|\widetilde{u}_n(t)\| \geq &< \varphi_k'(u_n), \widetilde{u}_n >= \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} (\bigtriangledown F(t, u_n(t)), \widetilde{u}_n(t))dt \\
&= \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} (\bigtriangledown F(t, u_n(t)) - \bigtriangledown F(t, \overline{u}_n), \widetilde{u}_n(t))dt \\
&\quad - \int_0^{kT} (\bigtriangledown F(t, \overline{u}_n), \widetilde{u}_n(t))dt \\
&\geq \int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} \gamma(t)|\widetilde{u}_n(t)|^\alpha dt - a_0\|\widetilde{u}_n\|_\infty \int_0^{kT} b(t)dt \\
&\geq \int_0^{kT} |\dot{u}_n(t)|^2 dt - C_1'\|\widetilde{u}_n\|_\infty^\alpha - C_2'\|\widetilde{u}_n\|_\infty
\end{aligned} \tag{22}$$

for large n. By Wertinger's inequality, we have

$$\int_0^{kT} |\dot{u}(t)|^2 dt \leq \|\widetilde{u}\|^2 \leq (\frac{k^2T^2}{4\pi^2} + 1) \int_0^{kT} |\dot{u}(t)|^2 dt. \tag{23}$$

By (22) and (23) we have

$$C(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\alpha/2} \geq \int_0^{kT} |\dot{u}_n(t)|^2 dt - C_1(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{1/2},$$

that is

$$(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{1/2} - C(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\alpha/4} \leq C_2$$

which implies

$$\int_0^{kT} |\dot{u}_n(t)|^2 dt \leq C_3 \tag{24}$$

for large n and some constant C_3 as $\alpha \in [1,2)$. Then by the boundedness of $\{\varphi_k(u_n)\}$, condition (ii) and Sobolev's inequality one has

$$\begin{aligned} C_4 \le \varphi_k(u_n) &= \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \int_0^{kT} F(t,u_n) \\ &\le \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \frac{1}{\mu}\int_0^{kT} F(t,\lambda\overline{u}_n)dt + \int_0^{kT} F(t,-\widetilde{u}_n(t))dt \\ &\le \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \frac{1}{\mu}\int_0^{kT} F(t,\lambda\overline{u}_n)dt + \int_0^{kT} [g(t)|\widetilde{u}_n(t)|^\delta + h(t)]dt \\ &\le \frac{1}{2}\int_0^{kT} |\dot{u}_n(t)|^2 dt - \frac{1}{\mu}\int_0^{kT} F(t,\lambda\overline{u}_n)dt + C_5(\int_0^{kT} |\dot{u}_n(t)|^2 dt)^{\delta/2} + C_6 \end{aligned} \tag{25}$$

for all large n and some constants C_4, C_5 and C_6. Hence by (21), (24) and (25) we obtain $|\overline{u}_n| \le C_7$ for all large n and some constant C_7. Hence $\{u_n\}$ is a bounded sequence, and (PS) condition is satisfied.

Then the rest of proof continue as similar as in Theorem 2.4. We omit the details. So we complete our proof. □

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