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GENERAL BICONVEX FUNCTIONS AND BIVARIATIONAL-LIKE INEQUALITIES

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Abstract. In this paper, we consider and introduce some new concepts of the biconvex functions involving an arbitrary bifunction and function. Some new relationships among various concepts of biconvex functions have been established. We have shown that the optimality conditions for the general biconvex functions can be characterized by a class of bivariational-like inequalities. Auxiliary principle technique is used to propose proximal point methods for solving general bivariational-like inequalities. We also discussed the conversance criteria for the suggested methods under pseudo-monotonicity. Our method of proof is very simple compared with methods. Several special cases are discussed as applications of our main concepts and results. It is a challenging problem to explore the applications of the general bivariational-like inequalities in pure and applied sciences.

1. Introduction

Convexity theory is a branch of mathematical sciences, which have important and novel applications in industry, physical, social, regional, financial and engineering sciences. For more details, see [1, 3, 8, 11, 12, 16, 20, 21, 22, 26] and the references therein. It is worth mentioning that variational inequalities represent the optimality conditions for the differentiable convex functions

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on the convex sets in normed spaces, which were introduced and considered by Stampacchia [23]. Variational inequalities combine both theoretical and algorithmic advances with new and novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. In recent years, considerable interest has been shown in developing various generalizations of variational inequalities and generalized convexity, both for their own sake and their applications.

Inspired by the research work going in this field, we introduce and consider another class of nonconvex functions with respect to an arbitrary bifunction and function. This class of nonconvex functions is called the general biconvex functions. Relationship with other classes of convexity is discussed. Several new concepts of monotonicity are introduced and are discussed. We derive some new results under some mild conditions. It is shown that the optimality conditions of the differentiable general biconvex functions can be characterized by a class of variational-like inequalities, which is called general bivariational-like inequality. Some iterative methods are suggested for solving general bivariational-like inequalities using the auxiliary principle technique [4, 6, 12, 16, 17, 18, 19, 21, 25, 26] involving Bregman distance functions. Convergence criteria is also discussed using the pseudo monotonicity which is a weaker condition than monotonicity. We have discussed only the theoretical aspects of these new classes of bivariational-like inequalities. Implementation of the these iterative methods and comparison with other techniques is an open problem. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminaries

Let K be a nonempty closed set in a real Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ is the inner product and norm, respectively. Let $F : K_\beta \to R$ be a continuous function and let $\beta(. - .) : K_\beta \times K_\beta \to R$ be an arbitrary continuous bifunction.

Definition 2.1. A set $K_{g\beta}$ in H is said to be a general biconvex set with respect to an arbitrary function g and bifunction $\beta(\cdot - \cdot)$, if

$$g(u) + \lambda\beta(g(v) - g(u)) \in K_{g\beta}, \quad \forall u, v \in K_{g\beta},\ \lambda \in [0, 1].$$

The general biconvex set $K_{b\beta}$ is also called a $g\beta$-connected set. Note that the general biconvex set with $\beta(v, u) = g(v) - g(u)$ is a convex set K_g, but the converse is not true. For example, the set $K_\beta = R - (-\frac{1}{2}, \frac{1}{2})$ is an general biconvex set with respect to η, where

$$\beta(g(v)-g(u)) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0, \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases}$$

It is clear that $K_{g\beta}$ is not a convex set. From now onward $K_{g\beta}$ is a nonempty closed general biconvex set in H with respect to the function g and bifunction $\beta(\cdot - \cdot)$, unless otherwise specified.

If $g = I$, the identity operator, then Definition 2.1 reduces to:

Definition 2.2. A set K_β in H is said to be a biconvex set with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if

$$u + \lambda\beta(v-u) \in K_\beta, \quad \forall u, v \in K_\beta,\ \lambda \in [0,1].$$

The biconvex set K_β is also called β-connected set, which was introduced and studied by Noor et al [20]. We would like to point the β-biconvex set is quite different from the invex set considered in [1].

We now introduce some new concepts of general biconvex functions and their variants forms, which is the main motivation of this paper.

Definition 2.3. A function F on the general biconvex set $K_{g\beta}$ is said to be general biconvex with respect to a function g and the bifunction $\beta(\cdot - \cdot)$, if

$$F(g(u) + \lambda\beta(g(v)-g(u))) \quad \leq \quad (1-\lambda)F(g(u)) + \lambda F(g(v)), \qquad (2.1)$$

for all $u, v \in K_{g\beta}, \lambda \in [0,1]$.

The function F is said to be general biconcave if and only if $-F$ is a general biconvex function. Consequently, we have a new concept.

Definition 2.4. A function F is said to be general affine biconvex involving an arbitrary function g and a bifunction $\beta(\cdot - \cdot)$, if

$$F(g(u) + \lambda\beta(g(v)-g(u))) = (1-\lambda)F(g(u)) + \lambda F(g(v)),$$

for all $u, v \in K_{g\beta}, \lambda \in [0,1]$.

Note that every convex function is a general biconvex, but the converse is not true. If $\beta(g(v)-g(u)) = g(v)-g(u)$, then the general biconvex function becomes general convex functions, that is,

Definition 2.5. A function F on the biconvex set $K_{g\beta}$ is said to be a general biconvex with respect to a function g and the bifunction $\beta(\cdot - \cdot)$, if

$$F(g(u) + \lambda(g(v)-g(u))) \leq (1-\lambda)F(g(u)) + \lambda F(g(v)),$$

for all $u, v \in K_g, \lambda \in [0,1]$.

For the properties of the general convex functions in variational inequalities and equilibrium problems, see Noor [15, 16, 17, 18, 19].

Definition 2.6. A function F on the biconvex set K_β is said to be general quasi biconvex with respect to the function g and the bifunction $\beta(\cdot - \cdot)$, if

$$F(g(u) + \lambda\beta(g(v) - gu))) \leq \max\{F(g(u)), F(g(v))\},$$

for all $u, v \in K_{g\beta}, \lambda \in [0, 1]$.

Definition 2.7. A function F on the biconvex set K_β is said to be general log-biconvex with respect to the function g and the bifunction $\beta(\cdot - \cdot)$, if

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq (F(g(u)))^{1-\lambda}(F(g(v)))^{\lambda},$$

for all $u, v \in K_{g\beta}, \lambda \in [0, 1]$, where $F(\cdot) > 0$.

We can rewrite the Definition 2.7 in the following equivalent form:

Definition 2.8. A function F on the biconvex set K_β is said to be general log-biconvex with respect to the function g and the bifunction $\beta(\cdot - \cdot)$, if

$$\log F(g(u) + \lambda\beta(g(v(-g(u))) \leq (1 - \lambda)\log F(g(u)) + \lambda \log F(g(v)),$$

for all $u, v \in K_{g\beta}, \lambda \in [0, 1]$, where $F(\cdot) > 0$.

This equivalent definition can be used to discus the properties of the differentiable log-biconvex functions.

From the above definitions, we have

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq (F(g(u)))^{1-\lambda}(F(g(v)))^{\lambda} \\ &\leq (1 - \lambda)F(g(u)) + \lambda F(g(v)) \\ &\leq \max\{F(g(u)), F(g(v))\}. \end{aligned}$$

This shows that every log-biconvex function is a general biconvex function and every general biconvex function is a general quasi-biconvex function. However, the converse is not true.

For $\lambda = 1$, Definition 2.3 and 2.7 reduce to the following condition.

Condition A.

$$F(g(u) + \beta(g(v) - g(u))) \leq F(g(v)), \quad \forall\, v \in K_{g\beta}.$$

We now define the biconvex functions on the interval:

$$K_\beta = I_\beta = [g(a), g(a) + \beta(g(b) - g(a))].$$

Definition 2.9. Let $I_g = [g(a), g(a) + \beta(g(b) - g(a))]$. Then F is a general biconvex function if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ g(a) & g(x) & g(a) + \beta(g(b) - g(a)) \\ F(g(a)) & F(g(x)) & F(g(b)) \end{vmatrix} \geq 0,$$

where $g(a) \leq g(x) \leq g(a) + \beta(g(b) - g(a))$.

One can easily show that the following are equivalent:

(1) F is a general biconvex function.

(2) $F(g(x)) \leq F(g(a)) + \frac{F(g(b))-F(g(a))}{\beta(g(b)-g(a))}(g(x) - g(a))$.

(3) $\frac{F(g(x))-F(g(a))}{g(x)-g(a)} \leq \frac{F(g(b))-F(g(a))}{\beta(g(b)-g(a))}$.

(4) $\frac{F(g(a))}{(\beta(g(b)-g(a)))(g(a)-g(x))} + \frac{F(g(x))}{(g(x)-g(a)-\beta(g(b)-g(a)))(g(a)-g(x))}$
$+ \frac{F(g(b))}{\beta(g(b)-g(a))(g(x)-g(b))} \leq 0$,

where $g(x) = g(a) + \lambda\beta(g(b) - g(a)) \in [g(a), g(a) + \beta(g(b) - g(a)]$.

3. Properties of biconvex functions

In this section, we consider some basic properties of general biconvex functions and their variant forms.

Theorem 3.1. *Let F be a strictly general biconvex function. Then any local minimum of F is a global minimum.*

Proof. Let the biconvex function F have a local minimum at $u \in K_{g\beta}$. Assume the contrary, that is, $F(g(v)) < F(g(u))$ for some $v \in K_{g\beta}$. Since F is a strictly general biconvex function, so we have

$$F(g(u) + \lambda\beta(g(v) - g(u))) < \lambda F(g(v)) + (1 - \lambda)F(g(u)), \quad 0 < \lambda < 1.$$

Thus

$$F(g(u) + \lambda\beta(g(v) - g(u))) - F(g(u)) < \lambda[F(g(v)) - F(g(u))] < 0,$$

from which it follows that

$$F(g(u) + \lambda\beta(g(v) - g(u))) < F(g(u)),$$

for arbitrary small $\lambda > 0$, contradicting the local minimum. □

Theorem 3.2. *If the function F on the convex set K_β is general biconvex, then the level set*

$$L_\alpha = \{g(u) \in K_{g\beta} : F(g(u)) \leq \alpha, \quad \alpha \in R\}$$

is a general biconvex set.

Proof. Let $u, v \in L_\alpha$. Then $F(u) \leq \alpha$ and $F(g(v)) \leq \alpha$.

Now, for $\lambda \in (0,1)$, $g(w) = g(u) + \lambda\beta(g(v) - g(u)) \in K_{g\beta}$, since $K_{g\beta}$ is a general biconvex set. Thus, by the general biconvexity of F, we have

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq (1-\lambda)F(g(u)) + \lambda F(g(v)) \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which it follows that $g(u) + t\beta(g(v) - g(u)) \in L_\alpha$ Hence L_α is a general biconvex set. □

Theorem 3.3. *A positive function F is a general biconvex if and only if*

$$epi(F) = \{(g(u), \alpha) : g(u) \in K_{b\beta} : F(g(u)) \leq \alpha, \alpha \in R\}$$

is a general biconvex set.

Proof. Assume that$F$ is a general biconvex function. Let

$$(g(u), \alpha),\ (g(v), \beta_1) \in epi(F).$$

Then it follows that $F(g(u)) \leq \alpha$ and $F(g(v)) \leq \beta_1$. Thus, for $\lambda \in [0,1]$, $u, v \in K_{g\beta}$, we have

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &\leq (1-\lambda)F(g(u)) + \lambda F(g(v)) \\ &\leq (1-t)\alpha + t\beta_1, \end{aligned}$$

which implies that

$$(g(u) + \lambda\beta(g(v) - g(u)), (1-\lambda)\alpha + \lambda\beta_1) \in epi(F).$$

Thus $epi(F)$ is a general biconvex set.

Conversely, let $epi(F)$ be a general biconvex set. Let $u, v \in K_{g\beta}$. Then $(g(u), F(g(u))) \in epi(F)$ and $(g(v), F(g(v))) \in epi(F)$. Since $epi(F)$ is a general biconvex set, we must have

$$(g(u) + \lambda\beta(g(v) - g(u)), (1-\lambda)F(g(u)) + \lambda F(g(v))) \in epi(F),$$

which implies that

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq (1-\lambda)F(g(u)) + \lambda F(g(v)).$$

This shows that F is a general biconvex function. □

Theorem 3.4. *A positive function F is general quasi-biconvex if and only if the level set*

$$L_\alpha = \{g(u) \in K_\beta, \alpha \in R : F(g(u)) \leq \alpha\}$$

is a general biconvex set.

Proof. Let $u, v \in L_\alpha$. Then $g(u), g(v) \in K_{g\beta}$ and $\max(F(g(u)), F(g(v))) \leq \alpha$. Now for $\lambda \in (0,1), g(w) = g(u) + \lambda\beta(g(v) - g((u)) \in K_{g\beta}$, we have to prove that $g(u) + \lambda\beta(g(v) - g(u)) \in L_\alpha$. By the quasi-biconvexity of F, we have

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq \max\left(F(g(u)), F(g(v))\right) \leq \alpha,$$

which implies that $g(u) + \lambda\beta(g(v) - g(u)) \in L_\alpha$, showing that the level set L_α is indeed a general biconvex set.

Conversely, assume that L_α is a general biconvex set. Then for all $u, v \in L_\alpha, \lambda \in [0,1]$,

$$g(u) + \lambda\beta(g(v) - g(u)) \in L_\alpha.$$

Let $u, v \in L_\alpha$ for

$$\alpha = \max\left(F(g(u)), F(g(v))\right. \quad \text{and} \quad F(g(v)) \leq F(g(u)).$$

From the definition of the level set L_α, it follows that

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq \max\left(F(g(u)), F(g(v))\right) \leq \alpha.$$

Thus F is a general quasi-biconvex function. This completes the proof. □

Theorem 3.5. *Let F be a general biconvex function. Let $\mu = \inf_{g(u)\in K_{g\beta}} F(u)$. Then the set*

$$E = \{g(u) \in K_{g\beta} : F(g(u)) = \mu\}$$

is a general biconvex set of $K_{g\beta}$. If F is strictly general biconvex, then E is a singleton.

Proof. Let $u, v \in E$. For $0 < \lambda < 1$, let $g(w) = g(u) + \lambda\beta(g(v) - g(u))$. Since F is a general biconvex function, we have

$$\begin{aligned} F(w) &= F(g(u) + \lambda\beta(g(v) - g(u))) \\ &\leq (1-\lambda)F(g(u)) + \lambda F(g(v)) \\ &= \lambda\mu + (1-\lambda)\mu \\ &= \mu, \end{aligned}$$

which implies that $w \in E$ and hence E is a general biconvex set. For the second part, assume to the contrary that $F(g(u)) = F(g(v)) = \mu$. Since $K_{g\beta}$ is a general biconvex set, for $0 < \lambda < 1, g(u) + \lambda\beta(g(v) - g(u)) \in K_{g\beta}$. Further, since F is strictly general biconvex function,

$$\begin{aligned} F(g(u) + \lambda\beta(g(v) - g(u))) &< (1-\lambda)F(g(u)) + \lambda F(g(v)) \\ &= (1-t)\mu + t\mu \\ &= \mu. \end{aligned}$$

This contradicts the fact that $\mu = \inf_{g(u)\in K_{g\beta}} F(u)$ and hence the result follows. □

Theorem 3.6. *If F is a general biconvex function such that*

$$F(g(v)) < F(g(u)), \ \forall\, u, v \in K_{g\beta},$$

then F is a strictly general quasi-biconvex function.

Proof. By the general biconvexity of the function F, $\forall u, v \in K_{g\beta}$, $\lambda \in [0,1]$, we have

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq (1-\lambda)F(g(u)) + \lambda F(g(v)) < F(g(u)),$$

since $F(g(v)) < F(g(u))$, which shows that the function F is strictly general quasi-biconvex. □

4. Properties of log-biconvex functions

We now discuss some properties of the differentiable log-biconvex functions. To obtain the main results, we need the following assumption regarding the bifunction $\beta(\cdot - \cdot)$.

Condition M. We assume that the bifunction $\beta(, -,)$ is homogeneous, that is,

$$\beta(\gamma(v-u)) = \gamma\beta(v-u), \ \forall\, u, v \in K_\beta, \quad \gamma \in R^n.$$

Remark 4.1. Let $\beta(\cdot - \cdot) : K_\beta \times K_\beta \to H$ satisfy the assumption:

$$\beta(g(v) - g(u)) = \beta(g(v) - g(z)) + \beta(g(z) - g(u)), \ \forall\, u, v, z \in K_{g\beta}.$$

Then we can easily show that $\beta(g(v) - g(u)) = 0$ for all $u, v \in K_{g\beta}$. Consequently $\beta(0) = 0$, for $v = u \in K_{g\beta}$. Also $\beta(g(v) - g(u)) + \beta(g(u) - g(v)) = 0$. This implies that the bifunction $\beta(. - .)$ is skew symmetric.

Theorem 4.2. *Let F be a differentiable function on the biconvex set K_β and let the condition M hold. Then the function F is a general* log*-biconvex function if and only if*

$$\log F(g(v)) - \log F(g(u)) \geq \Big\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u)) \Big\rangle, \ \forall\, v, u \in K_{g\beta}. \tag{4.1}$$

Proof. Let F be a general log-biconvex function. Then, for all $u, v \in K_{g\beta}$,

$$\log F(g(u) + \lambda\beta(g(v) - g(u))) \leq (1-\lambda)\log F(g(u)) + \lambda \log F(g(v)),$$

which can be written as

$$\log F(g(v)) - \log F(g(u)) \geq \Big\{ \frac{\log F(g(u) + \lambda\beta(g(v) - g(u))) - \log F(g(u))}{\lambda} \Big\}.$$

Taking the limit in the above inequality as $\lambda \to 0$, we have

$$\log F(g(v)) - \log F(g(u)) \geq \Big\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u)) \Big\rangle,$$

which is (4.1), the required result.

Conversely, let (4.1) hold. Then for all $u, v \in K_{g\beta}, \lambda \in [0, 1]$, $g(v_\lambda) = g(u) + \lambda\beta(g(v) - g(u)) \in K_{g\beta}$ and using the condition M, we have

$$\begin{aligned} \log F(g(v)) - \log F(g(v_\lambda)) &\geq \Big\langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(v) - g(v_\lambda)))\Big\rangle \\ &= (1-\lambda)\Big\langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(v) - g(u))\Big\rangle. \quad (4.2) \end{aligned}$$

In a similar way, we have

$$\begin{aligned} \log F(g(u)) - \log F(g(v_\lambda)) &\geq \Big\langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(u) - g(v_\lambda))\Big\rangle \\ &= -\lambda\Big\langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(v) - g(u))\Big\rangle. \quad (4.3) \end{aligned}$$

Multiplying (4.2) by λ and (4.3) by $(1-\lambda)$ and adding the resultant, we have

$$\log F(g(u) + \lambda\beta(g(v) - g(u))) \leq (1-\lambda)\log F(g(u)) + \lambda \log F(g(v)),$$

showing that F is a general log-biconvex function. □

Remark 4.3. From (4.1), we have

$$F(g(v)) \geq F(g(u))exp\{\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u))\rangle\}, \quad u, v \in K_{g\beta}.$$

Changing the role of u and v in the above inequality, we also have

$$F(g(u)) \geq F(g(v))exp\{\langle \frac{F'(g(v))}{F(g(v))}, \beta(g(u) - g(v))\rangle\}, \quad u, v \in K_{g\beta}.$$

Thus, we can obtain the following inequality:

$$\begin{aligned} F(g(u)) + F(g(v)) &\geq F(g(v))exp\{\langle \frac{F'(g(v))}{F(g(v))}, \beta(g(u) - g(v))\rangle\}, \\ &+F(g(u))exp\{\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u))\rangle\}. \end{aligned}$$

Definition 4.4. The differentiable function F on the general biconvex set $K_{g\beta}$ is said to be a general biconvex function with respect to the bifunction $\beta(\cdot - \cdot)$, if

$$F(g(v)) - F(g(u)) \geq \Big\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u))\Big\rangle, \quad \forall\, u, v \in K_{g\beta},$$

where $F'(g(u)))$ is the differential of F at $g(u)$.

Theorem 4.5. *Let F be a differentiable function on the general biconvex set K_β and Condition M hold. Then the function F is general* log-*biconvex function if and only if*

$$\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle + \langle \frac{F'(g(v))}{F(g(v))}, \beta(g(u)-g(v))\rangle \leq 0, \forall v,u \in K_{g\beta}. \quad (4.4)$$

Proof. Let F be a differentiable function on the general biconvex set K_β. Then from Theorem 4.2, it follows that

$$\log F(g(v)) - \log F(g(u)) \geq \langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle, \quad \forall\, v,u \in K_{g\beta}. \quad (4.5)$$

Changing the role of u and v in (4.5), we have

$$\log F(g(u)) - \log F(g(v)) \geq \langle \frac{F'(g(v))}{F(g(v))}, \beta(g(v)-g(u))\rangle, \quad \forall\, v,u \in K_{g\beta}. \quad (4.6)$$

Adding (4.5) and (4.6), we have

$$\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle + \langle \frac{F'(g(v))}{F(g(v))}, \beta(u-v)\rangle \leq 0, \quad \forall\, v,u \in K_{g\beta},$$

which is the required (4.4).

Since K_β is a general biconvex set, so, for all $u,v \in K_\beta$, $\lambda \in [0,1]$,

$$g(v_\lambda) = g(u) + \lambda\beta(g(v)-g(u)) \in K_{g\beta}.$$

Taking $g(v) = g(v_\lambda)$ in (4.4), we have

$$\begin{aligned} \langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(u)-g(v_\lambda))\rangle &\leq \langle \frac{F'(u)}{F(u)}, \beta(u-g(v_\lambda))\rangle \\ &= -\lambda\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle, \quad (4.7) \end{aligned}$$

which implies that

$$\langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(v)-g(u))\rangle \geq \langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle. \quad (4.8)$$

Consider the auxiliary function

$$\xi(\lambda) = \log F(g(u) + \lambda(g(v)-g(u))) = F(g(v_\lambda)),$$

from which, we have

$$\xi(1) = \log F(g(v)), \quad \xi(0) = \log F(g(u)).$$

Then, from (4.8), we have

$$\begin{aligned} \xi'(\lambda) &= \langle \frac{F'(g(v_\lambda))}{F(g(v_\lambda))}, \beta(g(v) - g(u)) \rangle \\ &\geq \langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u)) \rangle. \end{aligned} \tag{4.9}$$

Integrating (4.9) between 0 and 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g((u)) \rangle.$$

Thus it follows that

$$\log F(g(v)) - \log F(g(u)) \geq \langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v) - g(u)) \rangle,$$

which is the required (4.1). □

Definition 4.6. An operator $T : K_\beta \to H$ with respect to the operator g is said to be:

(1) $g\beta$-monotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle + \langle Tv, \beta(g(u) - g(v)) \rangle \leq 0, \quad \forall\, u, v \in K_{g\beta}.$$

(2) $g\beta$-pseudomonotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle \geq 0 \Rightarrow -\langle Tv, \beta(g(u) - g(v)) \rangle \geq 0, \quad \forall\, u, v \in K_{g\beta}.$$

(3) relaxed $g\beta$-pseudomonotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle \geq 0 \Rightarrow -\langle Tv, \beta(g(u) - g(v)) \rangle \geq 0, \quad \forall\, u, v \in K_{g\beta}.$$

(4) strictly $g\beta$-monotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle + \langle Tv, \beta(g(u) - g(v)) \rangle < 0, \quad \forall\, u, v \in K_{g\beta}.$$

(5) $g\beta$-pseudomonotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle \geq 0 \Rightarrow \langle Tv, \eta(g(u) - g(v)) \rangle \leq 0, \quad \forall\, u, v \in K_{g\beta}.$$

(6) quasi $g\beta$-monotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle > 0 \Rightarrow \langle Tv, \beta(g(u) - g(v)) \rangle \leq 0, \quad \forall\, u, v \in K_{g\beta}.$$

(7) strictly $g\beta$-pseudomonotone, if

$$\langle Tu, \beta(g(v) - g(u)) \rangle \geq 0 \Rightarrow \langle Tv, \beta(g(u) - g(v)) \rangle < 0, \quad \forall\, u, v \in K_{g\beta}.$$

Definition 4.7. A differentiable function F on the general biconvex set K_η is said to be a general pseudo β-biconvex function, if

$$\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle \geq 0$$

then

$$F(g(v)) - F(g(u)) \geq 0, \ \ \forall\, u, v \in K_{g\beta}.$$

Definition 4.8. A differentiable function F on $K_{g\beta}$ is said to be a general quasi-biconvex function, if $F(g(v)) \leq F(g(u))$ then

$$\langle \frac{F'(g(u))}{F(g(u))}, \beta(v-u)\rangle \leq 0, \ \ \forall\, u, v \in K_{g\beta}.$$

Definition 4.9. The function F on the set $K_{g\beta}$ is said to be general pseudo-biconvex, if

$$\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle \geq 0$$

then

$$F(g(v)) \geq F(g(u)), \ \ \forall\, u, v \in K_{g\beta}.$$

Definition 4.10. The differentiable function F on the K_β is said to be general quasi-biconvex function, if $F(g(v)) \leq F(g(u))$ then

$$\langle \frac{F'(g(u))}{F(g(u))}, \beta(g(v)-g(u))\rangle \leq 0, \ \ \forall\, u, v \in K_{g\beta}.$$

We remark that the concepts introduced in this paper represent significant improvement of the previously known ones. All these new concepts may play important and fundamental part in the development of mathematical programming and optimization theory.

Theorem 4.11. *Let F be a differentiable function on the general biconvex set $K_{g\beta}$ in H and let the condition M hold. Then the function F is a general biconvex function if and only if F is a general biconvex function.*

Proof. Let F be a general biconvex function on the general biconvex set $K_{g\beta}$. Then, for all $u, v \in K_{g\beta}, \lambda \in [0,1]$,

$$F(g(u) + \lambda\beta(g(v)-g(u))) \leq (1-\lambda)F(g(u)) + \lambda F(g(v)),$$

which can be written as

$$F(g(v)) - F(g(u)) \geq \left\{\frac{F(g(u) + \lambda\beta(g(v)-g(u))) - F(g(u))}{\lambda}\right\}.$$

Taking the limit in the above inequality as $\lambda \to 0$, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle.$$

This shows that F is a general biconvex function.

Conversely, let F be a biconvex function on the biconvex set K_β. Then, for all $u, v \in K_{g\beta}$, $\lambda \in [0,1]$, $v_t = u + \lambda\beta(v-u) \in K_{g\beta}$ and using the condition M, we have

$$\begin{aligned} & F(g(v)) - F(g(u) + \lambda\beta(g(v)-g(u))) \\ & \geq \langle F'(g(u) + \lambda\beta(g(v)-g(u))), \beta(g(v)-g(u)+\lambda\beta(g(v)-g(u)))\rangle \\ & = (1-\lambda)F'(g(u)+\lambda\beta(g(v)-g(u))), \beta(g(v)-g(u))\rangle. \end{aligned} \tag{4.10}$$

In a similar way, we have

$$\begin{aligned} & F(g(u)) - F(g(u) + \lambda\beta(g(v)-g(u))) \\ & \geq \langle F'(g(u) + \lambda\beta(g(v)-g(u))), \beta(g(u)-g(u)+\lambda\beta(g(v)-g(u)))\rangle \\ & = -\lambda F'(g(u)+\lambda\beta(g(v)-gu))), \beta(g(v)-g(u))\rangle. \end{aligned} \tag{4.11}$$

Multiplying (4.10) by λ and (4.11) by $(1-\lambda)$ and adding the resultant, we have

$$F(g(u)+\lambda\beta(g(v)-g(u))) \leq (1-\lambda)F(g(u)) + \lambda F(g(v)),$$

showing that F is a general biconvex function. □

Theorem 4.12. *Let F be a differentiable general biconvex function on the general biconvex set $K_{g\beta}$. If F is a general biconvex function, then*

$$\begin{aligned} \langle F'(g(u)), \beta(g(v)-g(u)))\rangle \quad + \quad & \langle F'(g(v)), \beta(g(u)-g(v))\rangle \\ & \leq 0, \quad \forall\, u, v \in K_{g\beta}. \end{aligned} \tag{4.12}$$

Proof. Let F be a general biconvex function on the general biconvex set $K_{g\beta}$. Then

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v)-g(u)))\rangle, \quad \forall\, u, v \in K_{g\beta}. \tag{4.13}$$

Changing the role of u and v in (4.13), we have

$$F(g(u)) - F(g(v)) \geq \langle F'(g(v)), \beta(g(u)-g(v))\rangle, \quad \forall\, u, v \in K_{g\beta}. \tag{4.14}$$

Adding (4.13) and (4.14), we have

$$\begin{aligned} \langle F'(g(u)), \beta(g(v)-g(u)))\rangle \quad + \quad & \langle F'(g(v)), \beta(g(u)-g(v))\rangle \\ & \leq 0, \quad \forall\, u, v \in K_{g\beta}, \end{aligned}$$

which shows that $F'(.)$ is a $g\beta$-monotone operator. □

Theorem 4.13. *If the differential $F'(.)$ is a $g\beta$-monotone, then*

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle.$$

Proof. Let $F'(.)$ be a $g\beta$-monotone. From (4.15), we have

$$\langle F'(g(v)), \beta(g(u)-g(v))\rangle \geq \langle F'(g(u)), \beta(g(v)-g(u)))\rangle. \tag{4.15}$$

Since $K_{g\beta}$ is a general biconvex set, for all $u, v \in K_{g\beta}$, $\lambda \in [0,1]$,

$$g(v_\lambda) = g(u) + \lambda\beta(g(v)-g(u)) \in K_{g\beta}.$$

Taking $g(v) = g(v_\lambda)$ in (4.15) and using Condition M, we have

$$\begin{aligned} \langle F'(g(v_\lambda)), \beta(-\lambda\beta(g(v)-g(u)))\rangle &\leq \langle F'(g(u)), \eta(\lambda\beta(g(v)-g(u))))\rangle \\ &\quad + \|\beta(-\lambda\beta(g(v)-g(u))\|^2\} \\ &= -\lambda\langle F'(g(u)), \beta(g(v)-g(u))\rangle, \end{aligned}$$

which implies that

$$\langle F'(g(v_\lambda)), \beta(g(v)-g(u))\rangle \geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle. \tag{4.16}$$

Let $\xi(\lambda) = F(g(u) + \lambda\beta(g(v)-g(u)))$. Then, from (4.16), we have

$$\begin{aligned} \xi'(\lambda) &= \langle F'(g(u) + \lambda\beta(g(v)-g(u))), \beta(g(v)-g(u))\rangle \\ &\geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle. \end{aligned} \tag{4.17}$$

Integrating (4.17) between 0 and 1, we have

$$\xi(1) - \xi(0) \geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle,$$

that is,

$$F(g(u) + \beta(g(v)-g(u))) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle.$$

By using Condition A, we have

$$F(g(v)) - F(g(u)) \geq \langle F'(g(u)), \beta(g(v)-g(u))\rangle.$$

This completes the proof. □

We now give a necessary condition for general $g\beta$-pseudo-biconvex function.

Theorem 4.14. *Let $F'(.)$ be a relaxed general $g\beta$-pseudomonotone operator and Conditions A and M hold. Then F is a general $g\beta$-pseudo-biconvex function.*

Proof. Let F' be a relaxed general $g\beta$-pseudomonotone. Then, for all $u, v \in K_{g\beta}$,

$$\langle F'(g(u)), \beta(g(v)-g(u))\rangle \geq 0,$$

implies that

$$-\langle F'(g(v)), \beta(g(u)-g(v))\rangle \geq 0. \tag{4.18}$$

Since $K_{g\beta}$ is a general biconvex set, for all $u, v \in K_{g\eta}$, $\lambda \in [0,1]$,

$$g(v_\lambda) = g(u) + \lambda\beta(g(v)-g(u)) \in K_{g\beta}.$$

Taking $g(v) = g(v_\lambda)$ in (4.18) and using condition Condition M, we have

$$-\langle F'(g(u) + \lambda\beta(g(v) - g(u))), \beta(g(u) - g(v))\rangle \geq 0. \tag{4.19}$$

Let

$$\xi(\lambda) = F(g(u) + \lambda\beta(g(v) - g(u))), \ \ \forall\, u, v \in K_{g\beta},\ \lambda \in [0, 1].$$

Then, using (4.19), we have

$$\xi'(\lambda) = \langle F'(g(u) + \lambda\beta(g(v) - g(u))), \beta(g(u) - g(v))\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) \geq 0,$$

that is,

$$F(g(u) + \lambda\beta(g(v) - g(u))) - F(g(u)) \geq 0,$$

which implies, using Condition A,

$$F(v) - F(u) \geq 0,$$

showing that F is a general $g\beta$-pseudo-biconvex function. □

Definition 4.15. The function F is said to be sharply general pseudo biconvex, if $\langle F'(g(u)), \beta(g(v) - g(u))\rangle \geq 0$, then

$$F(g(v)) \geq F(g(v) + \lambda\beta(g(v) - g(u))), \ \ \forall\, u, v \in K_{g\beta},\ \lambda \in [0, 1].$$

Theorem 4.16. *Let F be a sharply general pseudo biconvex function on $K_{g\beta}$. Then*

$$-\langle F'(g(v)), \beta(g(v) - g(u))\rangle \geq 0, \ \ \forall\, u, v \in K_{g\beta}.$$

Proof. Let F be a sharply general pseudo biconvex function on $K_{g\beta}$. Then

$$F(g(v)) \geq F(g(v) + \lambda\beta(g(v) - g(u))), \ \ \forall\, u, v \in K_{g\beta},\ \lambda \in [0, 1],$$

from which we have

$$\frac{F(g(v) + \lambda\beta(g(v) - g(u))) - F(g(v))}{\lambda} \leq 0.$$

Taking limit in the above mentioned inequality, as $\lambda \to 0$, we have

$$-\langle F'(g(v)), \beta(g(v) - g(u))\rangle \geq 0,$$

the required result. □

Definition 4.17. A function F is said to be a pseudo general biconvex function with respect to strictly positive bifunction $W(.,.)$, if $F(g(v)) < F(g(u))$ then

$$F(g(u)+\lambda\beta(g(v)-g(u))) < F(g(u))+\lambda(\lambda-1)W(g(v),g(u)),$$

for all $u, v \in K_{g\beta}, \lambda \in [0,1]$.

Theorem 4.18. *If the function F is a general biconvex function such that $F(g(v)) < F(g(u))$, then the function F is pseudo general biconvex.*

Proof. Since $F(g(v)) < F(g(u))$ and F is biconvex function, then for all, $u, v \in K_{g\eta}, \lambda \in [0,1]$, we have

$$\begin{aligned} F(g(u)+\lambda\beta(g(v)-g(u))) &\leq F(g(u))+\lambda(F(g(v))-F(g(u))) \\ &< F(g(u))+\lambda(1-\lambda)(F(g(v))-F(g(u))) \\ &= F(g(u))+\lambda(\lambda-1)(F(g(u))-F(g(v))) \\ &< F(g(u))+\lambda(\lambda-1)W(g(u),g(v)), \end{aligned}$$

where $W(g(u),g(v)) = F(g(u)) - F(g(v)) > 0$. This shows that the function F is a pseudo general biconvex. □

5. Bivariational-like inequalities

In this section, we consider the bivariational-like inequalities and suggest some iterative methods by using the auxiliary principle techniques involving the Bregman distance functions.

For the readers, we recall some basic properties of the Bregman [2] convex functions. For strongly convex function F, we define the Bregman distance function as

$$B(v,u) = F(v) - F(u) - \langle F'(u), v-u\rangle \geq \alpha\|v-u\|^2, \ \ \forall\, u, v \in K. \quad (5.1)$$

It is important to emphasize that various types of function F gives different Bregman distance. For some practical important types of functions F and their corresponding Bregman distance, see [5, 24].

We now discuss the optimality conditions for the differentiable general biconvex functions.

Theorem 5.1. *Let F be a differentiable general biconvex function with modulus $\mu > 0$. If $u \in K_{g\beta}$ is the minimum of the function F if and only if $u \in K_{g\beta}$ satisfies the*

$$\langle F'(g(u)), \beta(g(v)-g(u))\rangle \geq 0, \ \ \forall\, u, v \in K_{g\beta}. \quad (5.2)$$

Proof. Let $u \in K_{g\beta}$ be a minimum of the general biconvex function F. Then

$$F(g(u)) \leq F(g(v)), \quad \forall\, v \in K_{g\beta}. \tag{5.3}$$

Since $K_{g\beta}$ is a general biconvex set, for all $u, v \in K_{g\beta}, \lambda \in [0,1]$,

$$g(v_\lambda) = g(u) + \lambda\beta(g(v) - g(u)) \in K_{g\beta}.$$

Taking $g(v) = g(v_\lambda)$ in (5.3), we have

$$\begin{aligned} 0 &\leq \lim_{\lambda \to 0}\{\frac{F(g(u) + \lambda\beta((g(v) - g(u))) - F(g(u))}{\lambda}\} \\ &= \langle F'(g(u)), \beta(g(v) - g(u))\rangle, \end{aligned} \tag{5.4}$$

which is the inequality (5.2).

Next, since F is differentiable general biconvex function, we have

$$F(g(u) + \lambda\beta(g(v) - g(u))) \leq F(g(u)) + \lambda(F(g(v)) - F(g(u))), \quad \forall\, u, v \in K_{g\beta},$$

from which, using (5.2), we have

$$\begin{aligned} F(g(v)) - F(g(u)) &\geq \lim_{\lambda \to 0}\{\frac{F(g(u) + \lambda\beta(g(v) - g(u))) - F(g(u))}{\lambda}\} \\ &= \langle F'(g(u)), \beta(g(v), g(u))\rangle \geq 0, \end{aligned}$$

from which, we have

$$F(g(u)) \leq F(g(v)), \quad \forall\, v \in K_{g\beta}. \tag{5.5}$$

This implies that $u \in K_{g\beta}$ is the minimum of the general biconvex functions. □

Remark 5.2. We would like to mention that, if $u \in K_{g\beta}$ satisfies the inequality

$$\langle F'(g(u)), \beta(g(v), g(u))\rangle \geq 0, \quad \forall\, u, v \in K_{g\beta}, \tag{5.6}$$

then $u \in K_{g\beta}$ is the minimum of the differentiable general biconvex function F. The inequality of the type (5.6) is called the bivariational-like inequality and appears to new one. It is worth mentioning that inequalities of the type (5.6) may not arise as the minimization of the biconvex functions. This motivated us to consider a more general bivariational-like inequality of which (5.6) is a special case.

For given operators T, g, and bifunction $\beta(. - .)$, we consider the problem of finding $u \in K_{g\beta}$, such that

$$\langle Tu, \beta(g(v) - g(u))\rangle \geq 0, \quad \forall\, v \in K_{g\beta}, \tag{5.7}$$

which is called the general bivariational-like inequality.

For suitable and appropriate choice of the operators, biconvex sets and spaces, we can obtain a wide class of variational-like inequalities and optimization problems as special cases of the general bivariational-like inequality (5.7). This shows that the general bivariational-like inequalities are quite flexible and unified ones.

We now introduce some iterative methods for solving the problem (5.7). We remark that due to the inherent nonlinearity, the projection method, Wiener-Hopf equations and their variant forms can not be used to consider the iterative methods for solving the general bivariational-like inequalities. To overcome these drawback, one may use the auxiliary principle technique of Glowinski et al. [4] as developed by Noor [9, 10, 12, 13, 14] and Noor et al. [15, 16, 21] to suggest and analyze some iterative methods. This technique does not involve the concept of the projection, which is the main advantage of this technique. We again use the auxiliary principle technique coupled with Bergman functions. These applications are based on the type of convex functions associated with the Bregman distance. We now suggest and analyze some iterative methods for bivariational-like inequalities (5.7) using the auxiliary principle technique coupled with Bregman functions.

For a given $u \in K_{g\beta}$ satisfying the bivariational-like inequality (5.7), we consider the auxiliary problem of finding a $w \in K_{g\beta}$ such that

$$\begin{aligned} \langle \rho Tw, \beta(g(v) - g(w)) \quad &+ \quad \langle E'(g(w)) - E'(g(u)), \beta(v - w) \rangle \\ &\geq 0, \quad \forall v \in K_{g\beta}, \end{aligned} \tag{5.8}$$

where $\rho > 0$ is a constant and $E'(g(u))$ is the differential of a strongly biconvex function $E(g(u))$ at $u \in K_{g\beta.}$

Remark 5.3. The function

$$B(g(w), g(u)) = E(g(w)) - E(g(u)) - \langle E'(g(u)), \beta(g(w) - g(u)) \rangle$$

associated with the general biconvex function $E(g(u))$ is called the generalized Bregman distance function. By the strongly general biconvexity of the function $E(g(u))$, the Bregman function $B(.,.)$ is nonnegative and $B(g(w), g(u)) = 0$, if and only if $g(u) = g(w)$, for all $u, w \in K_{g\beta}$. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [13, 14, 15, 16, 21, 26].

We note that, if $w = u$, then clearly w is solution of the general bivariational-like inequality (5.7). This observation enables us to suggest and analyze the following iterative method for solving (5.7).

Algorithm 5.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\langle \rho T u_{n+1}, \beta(g(v)-g(u_{n+1}))\rangle &+ \langle E'(g(u_{n+1}))-E'(g(u_n)), \beta(g(v)-g(u_{n+1}))\rangle \\ &\geq 0, \ \forall v \in K_{g\beta}, \qquad (5.9)\end{aligned}$$

where $\rho > 0$ is a constant. Algorithm 5.4 is called the proximal method for solving the general bivariational-like inequalities (5.7). In passing we remark that the proximal point method was suggested in the context of convex programming problems as a regularization technique. If $\beta(g(v) - g(u)) = g(v) - g(u)$, then Algorithm 5.4 collapses to:

Algorithm 5.5.. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\langle \rho T(u_{n+1}), g(v) - g(u_{n+1})\rangle &+ \langle E'(g(u_{n+1})) - E'(g(u)), g(v) - g(u_{n+1})\rangle \\ &\geq 0, \ \forall v \in K_g,\end{aligned}$$

for solving the general variational inequality.

For suitable and appropriate choice of the operators and the spaces, we can obtain a number of known and new algorithms for solving variational inequalities and related problems.

Theorem 5.6. *Let the bifunction T be pseudomonotone, If E is a differentiable general biconvex function with module $\beta > 0$, Condition M hold and g^{-1} exists, then the approximate solution u_{n+1} obtained from Algorithm 5.4 converges to a solution $u \in K_{g\beta}$ satisfying the general bivariational-like inequality (5.7).*

Proof. Let $u \in K$ be a solution of the general bivariational-like inequality (5.7). Then

$$\langle Tu, \beta(g(v) - g(u))\rangle \geq 0, \ \forall v \in K_{g\beta},$$

implies that

$$-\langle Tv, \beta(g(u) - g(v)))\rangle \geq 0, \ \forall v \in K_{g\beta}, \qquad (5.10)$$

since T is β-pseudomonotone. Taking $v = u$ in (5.9) and $v = u_{n+1}$ in (5.10), we have

$$\begin{aligned}\langle \rho T(u_{n+1}), \beta(g(u), g(u_{n+1}))\rangle &+ \langle E'(g(u_{n+1})) - E'g(u_n), \beta(g(u) - g(u_{n+1}))\rangle \\ &\geq 0 \qquad (5.11)\end{aligned}$$

and

$$-\langle Tu_{n+1}, \beta(g(u) - g(u)_{n+1}))\rangle \geq 0. \qquad (5.12)$$

We now consider the Bregman function

$$B(g(u),g(w)) = E(g(u)) - E(g(w)) - \langle E'g(w), \beta(u-w)\rangle \geq 0, \qquad (5.13)$$

using higher order strongly biconvexity of E.

Now combining (5.11),(5.12) and (5.13), we have

$$\begin{aligned}
&B(g(u),g(u_n)) - B(g(u),g(u_{n+1})) = E(g(u_{n+1})) - E(g(u_n)) \\
&\quad -\langle E'(g(u_n)), \beta(g(u)-g(u_n))\rangle + \langle E'(g(u_{n+1})), \beta(g(u)-g(u_{n+1}))\rangle \\
&= E(g(u_{n+1})) - E(g(u_n)) - \langle E'(g(u_n)) - E'(g(u_{n+1})), \beta(g(u)-g(u_{n+1}))\rangle \\
&\quad -\langle E'(g(u_n), g(u_{n+1}) - g(u_n)\rangle \\
&\geq \beta\|\beta(g(u_{n+1})-g(u_n))\|^2 + \langle E'(g(u_{n+1})) - E'(g(u_n)), \beta(g(u)-g(u_{n+1}))\rangle \\
&\geq \beta\|\beta(g(u_{n+1})-g(u_n))\|^2 - \rho\langle T(u_{n+1}), \beta(g(u)-g(u_{n+1}))\rangle \\
&\quad -\rho\mu\|\beta(g(u)-g(u_{n+1}))\|^2 \\
&\geq \beta\|\beta(g(u_{n+1})-g(u_n))\|^2.
\end{aligned}$$

If $g(u_{n+1}) = g(u_n)$, then clearly $g(u_n)$ is a solution of the problem (5.7). Otherwise, it follows that $B(g(u),g(u_n)) - B(g(u),g(u_{n+1}))$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(g(u_{n+1}) - g(u_n))\| = 0,$$

from which, we have

$$\lim_{n\to\infty} \|g(u_{n+1}) - g(u_n)\| = 0 \quad \longrightarrow \quad u_{n+1} = u_n,$$

since g^{-1} exists. It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [26], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the bivariational-like inequality(5.7). □

It is well known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the bivariational-like inequality(5.7) using the auxiliary principle technique.

For a given $u \in K_{gb\eta}$, find $w \in K_{g\beta}$ such that

$$\begin{aligned}
\langle \rho T(u, \beta(g(v)-g(w))\rangle \quad + \quad &\langle E'(g(w)) - E'(g(u)), \beta(g(v)-g(w))\rangle \\
&\geq 0, \ \ \forall\, v \in K_{g\beta}, \qquad (5.14)
\end{aligned}$$

where $E'(g(u))$ is the differential of a biconvex function $E(g(u))$ at $u \in K_{g\beta}$. Problem (5.7) has a unique solution, since E is strongly biconvex function.

Note that problems (5.14) and (5.9) are quite different problems. It is clear that for $g(w) = g(u)$, w is a solution of (5.7). This fact allows us to suggest and analyze another iterative method for solving the general bivariational-like inequalities (5.7).

Algorithm 5.7. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_n, \beta(g(v) - g(u_{n+1}))\rangle \quad + \quad \langle E'(g(u_{n+1})) - E'(g(u_n)), \beta(g(v) - g(u_{n+1}))\rangle \geq 0, \quad \forall\, v \in K_{g\beta}, \tag{5.15}$$

for solving the general bivariational-like inequality (5.7).

Remark 5.8. For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving bivariational-like inequality (5.7) and related optimization problems. It is an interesting problem from both analytically and numerically point of views.

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