Nonlinear Functional Analysis and Applications
Vol. 27, No. 1 (2022), pp. 45-58
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.01.03
http://nfaa.kyungnam.ac.kr/journal-nfaa

STRONG AND Δ-CONVERGENCE THEOREMS FOR A COUNTABLE FAMILY OF MULTI-VALUED DEMICONTRACTIVE MAPS IN HADAMARD SPACES

Ma'aruf Shehu Minjibir[1] **and Sani Salisu**[2]

[1]African University of Science and Technology, Abuja, Nigeria
Bayero University, Kano, Nigeria
e-mail: `msminjibir.mth@buk.edu.ng`

[2]African University of Science and Technology, Abuja, Nigeria
e-mail: `ssalisu@aust.edu.ng`

Abstract. In this paper, iterative algorithms for approximating a common fixed point of a countable family of multi-valued demicontractive maps in the setting of Hadamard spaces are presented. Under different mild conditions, the sequences generated are shown to strongly convergent and Δ-convergent to a common fixed point of the considered family, accordingly. Our theorems complement many results in the literature.

1. Introduction

The class of (single-valued) demicontractive maps was introduced by Hicks and Kubicek in [15] as a proper superclass of the class of strictly pseudocontractive maps ([4]) which is itself a superclass of the class of nonexpansive maps. In [7], Chidume et al. introduced a multi-valued analogue of strictly pseudocontractive map. They showed that a Krasnoseslkii-type sequence converges to a fixed point of a strictly pseudocontractive map T in a Hilbert space. Chidume and Ezeora [8] also proved strong convergence theorems for a finite family of multi-valued strictly pseudocontractive maps in the setting of Hilbert spaces.

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[0]Received February 14, 2021. Revised August 27, 2021. Accepted December 9, 2021.

[0]2020 Mathematics Subject Classification: 47H09, 47H10, 47J25.

[0]Keywords: $CAT(0)$ space, multi-valued demicontractive maps, Hausdorff metric, strong convergence, Δ-convergence.

[0]Corresponding author: M. S. Minjibir(`msminjibir.mth@buk.edu.ng`).

Several results concerning finding solutions of equations/inclusions (such as fixed point equations/inclusions, zeros of monotone maps) have been obtained in metric spaces that do not necessarily possess linear structure. Some of these results can be found in, for example, Kirk [19, 20], Reich and Shafrir [25], Kohlenbach and Leustean [21], Chaoha and Pho-on [5], Okeke et al. [24], Dhompongsa and Panyanak [10], Saejung [26], Lerkchaiyaphum and Phuengrattana [22], Khan and Abbas [16], Eskandani et al. [14], Eskandani and Raeisi [13], Kim et al. [17], Tang et al. [27] and Asidi et al. [2]. Dhompongsa et al. [9], proved strong convergence theorems for fixed points of a countable family of multi-valued nonexpansive maps in the setting of $CAT(0)$ spaces. They proved the following theorem, H denotes the Hausdorff metric and $K(C)$ denotes the family of nonempty compact subsets of C.

Theorem 1.1. ([9]) *Let C be a nonempty, closed and convex subset of a complete $CAT(0)$ space X and $U_n, U : C \to K(C)$ be nonexpansive such that $H(U_n, U) \to 0$ uniformly on bounded subsets of C, $Fix(U) = \bigcap_{n=1}^{\infty} Fix(U_n)$ and $U_n(p) = \{p\}$ for all $p \in Fix(U)$. Suppose that $u, z_1 \in C$ are arbitrarily chosen and $\{z_n\}$ is defined by*

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n) u_n, \;\; u_n \in U_n(z_n)$$

such that $d(u_n, u_{n+1}) \leq d(z_n, z_{n+1}) + \varepsilon_n$ for all $n \in \mathbb{N}$, where $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

$$\lim \alpha_n = 0; \sum_n \alpha_n = \infty; \text{ and } \sum_n |\alpha_n - \alpha_{n+1}| < \infty (\text{or } \lim \alpha_n/\alpha_{n+1} = 1).$$

Then $\{z_n\}$ converges strongly to the unique fixed point of U closest to u.

Also in [6], Chidume et al. considered a finite family of demicontractive mappings in a complete CAT(0) space. They developed an iterative algorithm and proved both Δ and strong convergence of the sequence obtained to a common fixed point of the family. They proved the following result.

Theorem 1.2. ([6]) *Let K be a nonempty, closed and convex subset of a complete $CAT(0)$ space. Let $T_i : K \to CB(K)$, $i = 1, 2, \cdots, m$, be a family of demicontractive mappings with constants $k_i \in (0,1)$, $i = 1, \cdots m$ such that $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Suppose for all i, $T_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^{m} F(T_i)$. Let a sequence $\{x_n\}$ be define by*

$$\begin{cases} x_1 \in K; \\ x_{n+1} = \alpha_0 x_n \oplus \alpha_1 y_n^1 \oplus \alpha_2 y_n^2 \oplus \cdots \oplus \alpha_m y_n^m; \quad n \geq 1, \\ y_n^i \in T_i x_n, \; \alpha_0 \in (k, 1), \; \alpha_i \in (0,1), \end{cases} \tag{1.1}$$

where $k = \max\{k_i,\ i = 1, 2, \cdots, m\}$, $\sum_{i=0}^{m} \alpha_i = 1$ *and* $F(T_i)$ *denotes the set of fixed points of* T_i. *Then for every* i, $\lim_{n\to\infty} dist(p, T_i x_n)$ *exists for every* $p \in \bigcap_{i=1}^{m} F(T_i)$. *If in addition* T_i *is* Δ*-demiclosed at* 0 *for* $i = 1, \cdots, m$, *then* $\{x_n\}$ *is* Δ*-convergent to a point* $p \in \bigcap_{i=1}^{m} F(T_i)$. *Furthermore, if at least one of the* T_i*'s is semi-compact, then the convergence is strong.*

Our objective in this paper is two fold: the first is to develop an iterative algorithm and prove Δ and strong convergence of the resulting sequence to a common fixed point of a finite family of multi-valued demicontractive maps in a Hadamard space setting. The second is to develop an iterative algorithm and prove Δ and strong convergence of the resulting sequence to a common fixed point of a countable family of multi-valued demicontractive maps also in Hadamard space setting. The algorithm developed is fashioned after the one of Akbar and Eslamian[1] for a finite family of a subclass of quasi-nonexpansive mappings.

2. PRELIMINARIES

Given a metric space (X, d), a *g*eodesic from x to y is a map $\gamma : [0, l] \subset \mathbb{R} \to X$, for some $l > 0$, such that $\gamma(0) = x$, $\gamma(l) = y$; $d(\gamma(t), \gamma(s)) = |t - s|$, $\forall\ t, s \in [0, l]$. In particular γ is an isometry and $d(x, y) = l$. The image of γ, $\gamma([0, l])$, is called a *geodesic segment* joining x and y. When the geodesic is unique, it is denoted by $[x, y]$. For $x, y \in X$ having unique geodesic and for any $\alpha \in [0, 1]$, we denote by $\alpha x \oplus (1 - \alpha)y$ the unique vector z in $[x, y]$ satisfying $d(x, z) = \alpha d(x, y)$ and $d(z, y) = (1 - \alpha)d(x, y)$. If for every pair of points x, y in the space (X, d) there exists a geodesic joining them, then the space is called a *geodesic space* and if the geodesic is unique for each such pair, it is called a *uniquely geodesic space.* We shall say a subset C of X is *convex* if for every pair of points x, y in C, every segment joining x and y is contained in C.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the *vertices* of $\triangle$) and three geodesic segments-each for a pair of the vertices (these segments are called *edges* of the triangle). A *comparison triangle* for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3)$ which we shall denote by $\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space (X, d) is called a $CAT(0)$ *space* if every geodesic triangle $\triangle$ in (X, d) having comparison triangle $\overline{\triangle}$, the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

holds for all points x, y in $\triangle$ and, respective, comparison points $\bar{x}, \bar{y}$ in $\overline{\triangle}$ (where a point $\bar{z} \in [\bar{x}, \bar{y}]$ is called a *comparison point* of a point $z \in [x, y]$

if $d_{\mathbb{R}^2}(\bar{x}, \bar{z}) = d(x, z)$). A complete $CAT(0)$ space is called Hadamard space. Further details on general $CAT(\kappa)$ spaces can be found in, for example, [3].

For a bounded sequence $\{x_n\}$ in a metric space (X, d), let

$$r(x, \{x_n\}) := \limsup_n d(x, x_n), \quad x \in X.$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is defined as

$$r(\{x_n\}) := \inf \{r(x, \{x_n\}) \ : \ x \in X\}$$

and the *asymptotic centre* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) := \{x \in X \ : \ r(x, \{x_n\}) = r(\{x_n\})\}.$$

Remark 2.1. It is known (see, e.g., [11]) that in a $CAT(0)$ space, $A(\{x_n\})$ is a singleton set.

Let (X, d) be a metric space. A sequence $\{x_n\} \subset X$ is said to be Δ-*convergent* (see [23]) to $x \in X$ if $\limsup_k d(x_{n_k}, x) \leq \limsup_k d(x_{n_k}, y)$, for every $\{x_{n_k}\}$ subsequence of $\{x_n\}$ and for every $y \in X$. In any $CAT(0)$ space, by virtue of Remark 2.1, if the sequence $\{x_n\}$ is bounded, then Δ-convergence of $\{x_n\}$ to x is equivalent to saying that x is the unique asymptotic centre for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. We write $\Delta - \lim_n x_n = x$ or $x_n \xrightarrow{\Delta} x$ to mean $\{x_n\}$ is Δ-convergent to x and we call x the Δ-*limit* of $\{x_n\}$. When a sequence $\{x_n\}$ converges to x in the usual sense, that is when $d(x_n, x) \to 0$, we say it is strongly convergent to x, denoted $x_n \to x$.

Let (X, d) be a metric space. We denote the family of nonempty closed and bounded subsets of X by $\mathcal{CB}(X)$ and define $\operatorname{dist}(b, A) := \inf_{a \in A} d(b, a)$ for any $b \in X$ and for any $A \subseteq X$. Let d_H denote the Hausdorff metric, that is the map $d_H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ defined by

$$d_H(B, D) := \max \left\{ \sup_{b \in B} \operatorname{dist}(b, D), \sup_{d \in D} \operatorname{dist}(d, B) \right\}, \ \forall \ B, D \in \mathcal{CB}(X).$$

Let $T : X \to \mathcal{CB}(X)$ be multi-valued map. We denote by $\mathcal{F}(T)$ the set of all fixed points of T, that is, $\mathcal{F}(T) := \{p \in X : p \in Tp\}$. The map T is called: *nonexpansive* if

$$d_H(Tx, Ty) \leq d(x, y), \ \forall \ x, y \in X;$$

quasinonexpansive if for any $p \in \mathcal{F}(T)$,

$$d_H(Tx, Tp) \leq d(x, p), \ \forall \ x \in X;$$

demicontractive if there exists $k \in [0, 1)$ such that for any $p \in \mathcal{F}(T)$,

$$d_H(Tx, Tp)^2 \leq d(x, p)^2 + k dist(x, Tx)^2, \ \forall \ x \in X.$$

In the sequel, we shall say that the map T has *demiclosedness-type property* if for any sequence $\{x_n\} \subseteq D$ and $x \in D$, $\{x_n\}$ Δ-converges to x and $\operatorname{dist}(x_n, Tx_n) \to 0$, imply $x \in F(T)$.

Lemma 2.2. ([10]) *Let (X, d) be a CAT(0) space. Let $x, y, z \in X$ and $t \in [0,1]$. Then*

(i) $d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z)$,
(ii) $d((1-t)x \oplus ty, z)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$.

Lemma 2.3. ([12]) *Let D be a nonempty, closed and convex subset of a Hadamard space (X, d) and $\{x_n\}$ be a bounded sequence in D. Then the asymptotic centre $A(\{x_n\})$ of $\{x_n\}$ is in D.*

Lemma 2.4. ([10]) *If $\{x_n\}$ is a bounded sequence in a Hadamard space (X, d) with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.5. ([18]) *Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

3. MAIN RESULTS

We first give the algorithm for a finite family of demicontractive maps. Let (X, d) be a Hadamard space and let $D \subseteq X$ be closed, convex and nonempty. Let $T_i : D \to \mathcal{CB}(D)$ be multi-valued demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $m \in \mathbb{N}$, $i = 1, \cdots, m$. Define a sequence $\{x_n\}$ in D by

$$\begin{cases} x_1 \in D; \\ y_n^{(0)} = x_n; \\ y_n^{(i)} = a_{ni} y_n^{(i-1)} \oplus (1-a_{ni}) z_n^{(i-1)}, \qquad i = 1, \cdots, m-1; \\ x_{n+1} = a_{nm} y_n^{(m-1)} \oplus (1-a_{nm}) z_n^{(m-1)}, \quad n = 1, 2, \cdots, \end{cases} \tag{3.1}$$

where $z_n^{(i-1)} \in T_i y_n^{(i-1)}$, $a_{ni} \in [k_i, 1]$, $n \in \mathbb{N}$, $i = 1, \cdots, m$.

Lemma 3.1. *Let (X, d) be a $CAT(0)$ space and let $D \subseteq X$ be nonempty, closed and convex. Let $T_i : D \to \mathcal{CB}(D)$ be multi-valued demicontractive mappings with constants $\{k_i\} \subset (0,1)$, $m \in \mathbb{N}$, $i = 1, \cdots, m$ and $\{x_n\}$ be defined by iterative process (3.1). Suppose $\mathcal{F} := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and for all $i \in \{1, 2, \cdots, m\}$. Then, $\lim_n d(x_n, p)$ exists for all $p \in \mathcal{F}$.*

Proof. Let $p \in \mathcal{F}$ and $i \in \{1, \cdots, m-1\}$. By Lemma 2.2 (ii), the scheme (3.1) and the assumptions on T_i's we have

$$\begin{aligned}
&d(y_n^{(i)}, p)^2 \\
&\le a_{ni} d\left(y_n^{(i-1)}, p\right)^2 + (1-a_{ni}) d\left(z_n^{(i-1)}, p\right)^2 - a_{ni}(1-a_{ni}) d(y_n^{i-1}, z_n^{(i-1)})^2 \\
&\le a_{ni} d(y_n^{(i-1)}, p)^2 + (1-a_{ni}) dist(z_n^{(i-1)}, T_i p)^2 - a_{ni}(1-a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&\le a_{ni} d(y_n^{(i-1)}, p)^2 + (1-a_{ni}) d_H(T_i y_n^{(i-1)}, T_i p)^2 - a_{ni}(1-a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&\le a_{ni} d(y_n^{(i-1)}, p)^2 + (1-a_{ni})[d(y_n^{(i-1)}, p)^2 + k_i d(y_n^{(i-1)}, z_n^{(i-1)})^2] \\
&\quad - a_{ni}(1-a_{ni}) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\
&= d(y_n^{(i-1)}, p)^2 - (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2, \quad i = 1, \cdots, m-1.
\end{aligned}$$

Thus,

$$\begin{aligned}
d(x_{n+1}, p)^2 &\le a_{nm} d(y_n^{(m-1)}, p)^2 + (1-a_{nm}) d(z_n^{(m-1)}, p)^2 \\
&\quad - a_{nm}(1-a_{nm}) d(y_n^{m-1}, z_n^{(m-1)})^2 \\
&\le a_{nm} d(y_n^{(m-1)}, p)^2 + (1-a_{nm}) dist(z_n^{(m-1)}, T_m p)^2 \\
&\quad - a_{nm}(1-a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\le a_{nm} d(y_n^{(m-1)}, p)^2 + (1-a_{nm}) d_H(T_m y_n^{(m-1)}, T_m p)^2 \\
&\quad - a_{nm}(1-a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\le a_{nm} d(y_n^{(m-1)}, p)^2 + (1-a_{nm})[d(y_n^{(m-1)}, p)^2 \\
&\quad + k_m d(y_n^{(m-1)}, z_n^{(m-1)})^2] - a_{nm}(1-a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\le d(y_n^{(m-1)}, p)^2 - (1-a_{nm})(a_{nm}-k_m) d(y_n^{(m-1)}, z_n^{(m-1)})^2.
\end{aligned}$$

So, from the above two inequalities, we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &\le d(y_n^{(m-1)}, p)^2 + (1-a_{nm})(k_m - a_{nm}) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&= d(y_n^{(m-1)}, p)^2 - (1-a_{nm})(a_{nm}-k_m) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\le d(y_n^{(m-2)}, p)^2 - (1-a_{nm})(a_{nm-1}-k_{m-1}) d(y_n^{(m-2)}, z_n^{(m-2)})^2 \\
&\quad - (1-a_{nm})(a_{nm}-k_m) d(y_n^{(m-1)}, z_n^{(m-1)})^2 \\
&\ \vdots \\
&\le d(y_n^{(m-3)}, p)^2 - \sum_{i=m-2}^{m} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2.
\end{aligned}$$

Inductively, we obtain that

$$\begin{aligned} d(x_{n+1}, p)^2 &\leq d(y_n^{(0)}, p)^2 - \sum_{i=1}^{m}(1 - a_{ni})(a_{ni} - k_i)d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\ &= d(x_n, p)^2 - \sum_{i=1}^{m}(1 - a_{ni})(a_{ni} - k_i)d(y_n^{(i-1)}, z_n^{(i-1)})^2 \\ &\leq d(x_n, p)^2. \end{aligned}$$

This implies that $\lim_n d(x_n, p)$ exists (in $\mathbb{R}$). □

Theorem 3.2. *Let X, D, $\{T_i\}$, $\mathcal{F}$, $\{k_i\}$, $\{a_{ni}\}$ and $\{x_n\}$ be as in Lemma 3.1. Let $\liminf_n a_{ni} \in (k_i, 1)$ for each $i \in \{1, \cdots, m\}$ and let $T_1, \cdots, T_m$ be Lipschitzian maps. Then $\lim_n dist(x_n, T_i x_n) = 0$ for all $i = 1, \cdots, m$.*

Proof. As in the proof of Lemma 3.1,

$$\sum_{i=1}^{m}(1 - a_{ni})(a_{ni} - k_i)d(y_n^{(i-1)}, z_n^{(i-1)})^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2$$

and $\lim_n d(x_n, p)$ exists for all $p \in \mathcal{F}$. Thus

$$\lim_n (1 - a_{ni})(a_{ni} - k_i)d(y_n^{(i-1)}, z_n^{(i-1)})^2 = 0$$

for all $i = 1, \cdots, m$.

Since $\liminf_n a_{ni} \in (k_i, 1)$ for each $i \in \{1, \cdots, m\}$, it follows that

$$\lim_n d(y_n^{(i-1)}, z_n^{(i-1)}) = 0 \;\; for\ each \;\; i = 1, \cdots, m. \tag{3.2}$$

Now, let $i \in \{1, \cdots, m\}$. Then,

$$\begin{aligned} & d(x_n, z_n^{(i-1)}) \\ &= d(y_n^{(0)}, z_n^{(i-1)}) \\ &\leq d(y_n^{(0)}, y_n^{(1)}) + d(y_n^{(1)}, y_n^{(2)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\leq d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, y_n^{(2)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\leq d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\;\;\vdots \\ &\leq d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) + \cdots + d(y_n^{(i-2)}, z_n^{(i-2)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\ &\leq \sum_{k=1}^{i} d(y_n^{(k-1)}, z_n^{(k-1)}). \end{aligned}$$

This and (3.2) imply that

$$\lim_n d(x_n, z_n^{(i-1)}) = 0 \ \ for\ each\ \ i = 1, \cdots, m. \tag{3.3}$$

Using $d(x_n, w_n^i) \leq d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, w_n^i)$, we obtain

$$\operatorname{dist}(x_n, T_i x_n) \leq d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, w_n^i), \quad \forall\ w_n^i \in T_i x_n.$$

Thus, using the fact that T_i is L_i-Lipschitzian for each $i \in 1, \cdots, m$, we have the following:

$$\begin{aligned} dist(x_n, T_i x_n) &\leq d(x_n, z_n^{(i-1)}) + dist(z_n^{(i-1)}, T_i x_n) \\ &\leq d(x_n, z_n^{(i-1)}) + d_H(T_i y_n^{(i-1)}, T_i x_n) \\ &\leq d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, x_n) \\ &\leq d(x_n, z_n^{(i-1)}) + L_i[d(y_n^{(i-1)}, z_n^{i-1}) + d(z_n^{i-1}, x_n)]. \end{aligned}$$

Therefore, by (3.2) and (3.3) we have $\lim_n dist(x_n, T_i x_n) = 0$ for all $i = 1, \cdots, m$. □

Corollary 3.3. *Let X, D, $\{T_i\}$ and $\{x_n\}$ be as in Theorem* 3.2*. Suppose T_i is Δ-demiclosed at* 0 *for each $i \in \{1, \cdots, m\}$. Then $\{x_n\}$ is Δ-convergent to a common fixed point.*

Proof. By Lemma 3.1, we have $\lim_n d(x_n, p)$ exists for all $p \in \mathcal{F}$. Hence $\{x_n\}$ is bounded. Now, let $u \in \bigcup A(\{w_n\})$, where the union is taken over subsequences $\{w_n\}$ of $\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.5 there exists $\{v_n\}$, a subsequence of $\{u_n\}$ such that $\Delta - \lim_n v_n = v$ and by Lemma 2.3 we have that $v \in D$.

Using Theorem 3.2 and the fact that T_i is Δ-demiclosed at zero for each i, we have $v \in \mathcal{F}$ and hence $\{d(u_n, v)\}$ converges by Lemma 3.1. Moreover, Lemma 2.4 implies that $u = v \in \mathcal{F}$. Thus

$$\bigcup A(\{w_n\}) \subseteq \mathcal{F}.$$

To conclude, it suffices to show that the set $\bigcup A(\{w_n\})$ is a singleton set. To see this, let $A(\{x_n\}) = \{x\}$ and let $\{u_n\}$ be an arbitrary subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. We have $u \in \mathcal{F}$ and by Lemma 3.1, $\{d(x_n, u)\}$ converges. Lemma 2.4 implies that $u = x$. □

Corollary 3.4. *Let X, D, $\{T_i,\ i = 1, \cdots, m\}$, $\mathcal{F}$ and $\{x_n\}$ be as in Theorem* 3.2*. Suppose D is compact. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i,\ i = 1, \cdots, m\}$.*

Proof. It follows from Theorem 3.2 that $\lim_n \operatorname{dist}(x_n, T_i x_n) = 0$ for all $i = 1, \cdots, m$. Since D is compact, there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $\lim_n d(v_n, w) = 0$ for some $w \in D$. Therefore, for $i \in \{1, \cdots, m\}$,

$$d(w, y_i) \le d(w, v_n) + d(v_n, u_n^i) + d(u_n^i, y_i), \ \ \forall\, u_n^i \in T_i v_n.$$

This implies that

$$\operatorname{dist}(w, T_i w) \le d(w, v_n) + d(v_n, u_n^i) + \operatorname{dist} d(u_n^i, T_i w) \ \ \forall\, y_i \in T_i w, \ \ \forall\, u_n^i \in T_i v_n.$$

Using the fact that T_i is Lipschitzian, we obtain

$$\begin{aligned}\operatorname{dist}(w, T_i w) &\le d(w, v_n) + d(v_n, u_n^i) + \operatorname{dist}(u_n^i, T_i w)\\ &\le d(w, v_n) + d(v_n, u_n^i) + d_H(T_i v_n, T_i w)\\ &\le d(w, v_n) + d(v_n, u_n^i) + L_i d(v_n, w)\\ &\le (1 + L_i) d(w, v_n) + d(v_n, u_n^i),\end{aligned}$$

for all $u_n^i \in T_i v_n$ and i. This implies that

$$\operatorname{dist}(w, T_i w) \le (1 + L_i) d(w, v_n) + \operatorname{dist}(v_n, T_i v_n).$$

Thus, $\operatorname{dist}(w, T_i w) = 0$. Hence, $w \in \mathcal{F}$. By Lemma 3.1 we have that $\lim_n d(x_n, w)$ exists. Thus $\lim_n d(x_n, w) = \lim_n d(v_n, w) = 0$. □

Theorem 3.5. *Let X, D, $\{T_i\}$, $\mathcal{F}$ and $\{x_n\}$ be as in Lemma* 3.1. *Suppose X is complete. Then $\{x_n\}$ converges strongly to a point $p \in \mathcal{F}$ if and only if $\liminf_n \operatorname{dist}(x_n, \mathcal{F}) = 0$.*

Proof. The forward direction is immediate. Suppose that $\liminf_n dist(x_n, \mathcal{F}) = 0$. It is seen in the proof of Lemma 3.1 that $d(x_{n+1}, p) \le d(x_n, p)$ for all $p \in \mathcal{F}$. This implies that $\operatorname{dist}(x_{n+1}, \mathcal{F}) \le \operatorname{dist}(x_n, \mathcal{F})$. So the $\lim_n \operatorname{dist}(x_n, \mathcal{F})$ exists, and sing the hypothesis, $\lim_n \operatorname{dist}(x_{n+1}, \mathcal{F}) = 0$. Therefore we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in $\mathcal{F}$ such that for all $k \in \mathbb{N}$, $d(x_{n_k}, p_k) < \frac{1}{2^k}$. By Lemma 3.1 we have $d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) < \frac{1}{2^k}$. Hence

$$d(p_{k+1}, p_k) \le d(x_{n_{k+1}}, p_{k+1}) + d(x_{n_{k+1}}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Thus $\{p_k\}$ is a Cauchy sequence in D and therefore converges (strongly) to some point $q \in D$. It follows that $\lim_k d(x_{n_k}, q) = 0$. Therefore, for $i \in \{1, \cdots, m\}$,

$$\operatorname{dist}(p_k, T_i q) \le d_H(T_i p_k, T_i q) \le L_i d(p_k, q) \to 0.$$

As $Tq \in \mathcal{CB}(D)$, $q \in \mathcal{F}$. Since $\lim_n d(x_n, q)$ exists, we conclude that

$$\lim_n d(x_n, q) = 0.$$

□

Next we present our convergence theorems for a countable family.

Let (X, d) be a Hadamard space and let D be a nonempty, closed and convex subset of X. Let $T_i : D \to \mathcal{CB}(D)$ be multi-valued demicontractive mappings with constants $\{k_i\} \subset (0, 1)$, $i \in N$. A sequence $\{x_n\}$ is defined iteratively as follows:

$$\begin{cases} x_1 \in D; \\ y_n^{(0)} = x_n; \\ y_n^{(i)} = a_{ni} y_n^{(i-1)} \oplus (1 - a_{ni}) z_n^{(i-1)}, \qquad i = 1, \cdots, n-1; \\ x_{n+1} = a_{nn} y_n^{(n-1)} \oplus (1 - a_{nn}) z_n^{(n-1)}, \quad n = 1, 2, 3, \cdots, \end{cases} \tag{3.4}$$

where $z_n^{(i-1)} \in T_i y_n^{(i-1)}, a_{ni} \in [k_i, 1],\ n \in \mathbb{N},\ i = 1, \cdots, n.$

Lemma 3.6. *Let (X, d) be a $CAT(0)$ space and let D be a nonempty, closed and convex subset of X. Let $T_i : D \to \mathcal{CB}(D)$ be multi-valued demicontractive mappings with constants $\{k_i\} \subset (0, 1)$, $i \in \mathbb{N}$ and let $\{x_n\}$ be defined by the iterative process in (3.4). Suppose $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for all $p \in \mathcal{F}$. Then, $\lim_n d(x_n, p)$ exists for all $p \in \mathcal{F}$.*

Proof. Let $p \in \mathcal{F}$. By lemma 2.2 (ii), the scheme (3.4) and the assumptions on T_i's we have

$$\begin{aligned} d(x_{n+1}, p)^2 &\le a_{nn} d(y_n^{(n-1)}, p)^2 + (1 - a_{nn}) d(z_n^{(n-1)}, p)^2 \\ &\quad - a_{nn}(1 - a_{nn}) d(y_n^{n-1}, z_n^{(n-1)})^2 \\ &\le a_{nn} d(y_n^{(n-1)}, p)^2 + (1 - a_{nn}) dist(z_n^{(n-1)}, T_n p)^2 \\ &\quad - a_{nn}(1 - a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\le a_{nn} d(y_n^{(n-1)}, p)^2 + (1 - a_{nn}) d_H(T_n y_n^{(n-1)}, T_n p)^2 \\ &\quad - a_{nn}(1 - a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\le a_{nn} d(y_n^{(n-1)}, p)^2 + (1 - a_{nn})[d(y_n^{(n-1)}, p)^2 + k_n d(y_n^{(n-1)}, z_n^{(n-1)})^2] \\ &\quad - a_{nn}(1 - a_{nn}) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \\ &\le d(y_n^{(n-1)}, p)^2 - (1 - a_{nn})(a_{nn} - k_n) d(y_n^{(n-1)}, z_n^{(n-1)})^2 \end{aligned}$$

$$
\begin{aligned}
&\le d(y_n^{(n-3)}, p)^2 - \sum_{i=n-2}^{n} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2\\
&\ \vdots\\
&\le d(y_n^{(0)}, p)^2 - \sum_{i=1}^{n} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2\\
&= d(x_n, p)^2 - \sum_{i=1}^{n} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2\\
&\le d(x_n, p)^2.
\end{aligned}
$$

This implies that $\lim_n d(x_n, p)$ exists, as a monotonic nonincreasing sequence of real numbers that is bounded below by 0. □

Theorem 3.7. *Let X, D, $\{T_i\}$, $\mathcal{F}$ and $\{x_n\}$ be as in Lemma* 3.6. *Suppose $\liminf_n a_{ni} > k_i$ for each $i \in \mathbb{N}$ and let T_i be Lipschitzian maps for all $i \in \mathbb{N}$. Then $\lim_n dist(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$.*

Proof. As in the proof of Lemma 3.6,

$$\sum_{i=1}^{n} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \le d(x_n, p)^2 - d(x_{n+1}, p)^2$$

for all $n \in \mathbb{N}$. This implies that

$$\sum_{i=1}^{n} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2 \le d(x_1, p)$$

for all $n \in \mathbb{N}$. And so

$$\lim_n \sum_{i=1}^{n} (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2$$

exists in $\mathbb{R}$. Thus

$$\lim_n (1-a_{ni})(a_{ni}-k_i) d(y_n^{(i-1)}, z_n^{(i-1)})^2 = 0$$

for all $i \in \mathbb{N}$. Since $\liminf_n a_{ni} > k_i$ for each $i \in \mathbb{N}$, it follows that

$$\lim_n d(y_n^{(i-1)}, z_n^{(i-1)}) = 0 \quad for\ each \quad i \in \mathbb{N}. \tag{3.5}$$

Now, let $i \in \mathbb{N}$. Then

$$
\begin{aligned}
& d(x_n, z_n^{(i-1)}) \\
& = d(y_n^{(0)}, z_n^{(i-1)}) \\
& \le d(y_n^{(0)}, y_n^{(1)}) + d(y_n^{(1)}, y_n^{(2)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
& \le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, y_n^{(2)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
& \le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) + \cdots + d(y_n^{(i-2)}, y_n^{(i-1)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
& \quad \vdots \\
& \le d(y_n^{(0)}, z_n^{(0)}) + d(y_n^{(1)}, z_n^{(1)}) + \cdots + d(y_n^{(i-2)}, z_n^{(i-2)}) + d(y_n^{(i-1)}, z_n^{(i-1)}) \\
& \le \sum_{k=1}^{i} d(y_n^{(k-1)}, z_n^{(k-1)}).
\end{aligned}
$$

This and (3.5) imply that

$$\lim_n d(x_n, z_n^{(i-1)}) = 0 \quad for\ each \quad i \in \mathbb{N}. \tag{3.6}$$

Thus, $d(x_n, w_n^i) \le d(x_n, z_n^{(i-1)}) + d(z_n^{(i-1)}, w_n^i)$ for all $w_n^i \in T_i x_n$. Therefore,

$$\operatorname{dist}(x_n, T_i x_n) \le d(x_n, z_n^{(i-1)}) + \operatorname{dist}(z_n^{(i-1)}, T_i x_n).$$

Using the fact that T_i is L_i-Lipschitzian for each $i \in \mathbb{N}$, we have the following

$$
\begin{aligned}
\operatorname{dist}(x_n, T_i x_n) & \le d(x_n, z_n^{(i-1)}) + \operatorname{dist}(z_n^{(i-1)}, T_i x_n) \\
& \le d(x_n, z_n^{(i-1)}) + d_H(T_i y_n^{(i-1)}, T_i x_n) \\
& \le d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, x_n) \\
& \le d(x_n, z_n^{(i-1)}) + L_i[d(y_n^{(i-1)}, z_n^{i-1}) + d(z_n^{i-1}, x_n)] \\
& \le (1 + L_i) d(x_n, z_n^{(i-1)}) + L_i d(y_n^{(i-1)}, z_n^{i-1}).
\end{aligned}
$$

Therefore, by (3.6) and (3.5) we have $\lim_n \operatorname{dist}(x_n, T_i x_n) = 0$ for all $i \in \mathbb{N}$. □

Corollary 3.8. *Let X, D, $\{T_i\}$ and $\{x_n\}$ be as in Theorem* 3.7. *Suppose T_i is Δ-demiclosed at zero for each $i \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to a common fixed point of $\{T_i\}$.*

Proof. Using Lemma 3.6 in place of Lemma 3.1 and Theorem 3.7 in place of Theorem 3.2, the proof follows similar arguments as in the proof of Corollary 3.3. □

Corollary 3.9. *Let X, D, $\{T_i\}$ and $\{x_n\}$ be as in Theorem* 3.7. *Suppose D is compact. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}$.*

Proof. Using Lemma 3.6 in place of Lemma 3.1 and Theorem 3.7 in place of Theorem 3.2, the proof follows similar arguments as in the proof of Corollary 3.4. ☐

Theorem 3.10. *Let* X*,* D*,* $\{T_i\}$*,* $\mathcal{F}$ *and* $\{x_n\}$ *be as in Lemma* 3.6*. Then* $\{x_n\}$ *converges strongly to a point* $p \in \mathcal{F}$ *if and only if* $\liminf_n dist(x_n, \mathcal{F}) = 0$*.*

Proof. Using Lemma 3.6 in place of Lemma 3.1, the proof follows similar arguments as in the proof of Theorem 3.5. ☐

4. Conclusion

In this work we have been able to develop algorithms for fixed points of finite and countable families of demicontractive multi-valued maps. Our theorems concern more general maps than quasi-nonexpansive maps whose finite families were considered by Akbar and Eslamian [1] in the setting of $CAT(0)$ spaces. In addition, our work complements the work of Chidume et al. in [6].

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