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IMPROVED GENERALIZED M-ITERATION FOR QUASI-NONEXPANSIVE MULTIVALUED MAPPINGS WITH APPLICATION IN REAL HILBERT SPACES

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Abstract. In this paper, we present a modified (improved) generalized M-iteration with the inertial technique for three quasi-nonexpansive multivalued mappings in a real Hilbert space. In addition, we obtain a weak convergence result under suitable conditions and the strong convergence result is achieved using the hybrid projection method with our modified generalized M-iteration. Finally, we apply our convergence results to certain optimization problem, and present some numerical experiments to show the efficiency and applicability of the proposed method in comparison with other improved iterative methods (modified SP-iterative scheme) in the literature. The results obtained in this paper extend, generalize and improve several results in this direction.

1. Introduction

A fixed point problem for a nonlinear mapping $T : C \to C$ is; find $x \in C$ such that

$$Tx = x. \tag{1.1}$$

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Fixed point theory has become an invaluable area of study in mathematics as many problems in mathematical sciences, engineering, physics, economics, game theory, etc can be transformed into a fixed point problem. It is well-known that solving a fixed point problem analytically is very difficult or almost impossible and thus the need to consider approximate solution for fixed point problems arises. Over the years researchers have developed several iterative schemes for solving fixed point problems for different operators but the research is still on going in order to develop a faster and more efficient iterative algorithms.

The Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{1.2}$$

is one of the earliest iterative process used to approximate equation (1.1), whenever T is a contraction mapping. It has been established that the Picard iterative process fails to approximate (1.1) in as much as T is a nonexpansive mapping even when the existence of the fixed point is guaranteed or known. The author in [10] showed that the class of nonexpansive self mappings on a closed and bounded subset of a uniformly convex Banach space has a fixed point. Thereafter, researchers in this area have developed different iterative processes to approximate fixed points of nonexpansive mappings and a host of other nonlinear mappings. One of the pressing and important concept in this area of research is developing a faster, efficient and reliable iterative algorithms for approximating fixed points of nonlinear mappings. The following are some well-known iterative algorithm in literature for approximating fixed points of nonlinear mappings. Among many others, are; Mann [22], Ishikawa [17], Krasnosel'skii [21], Agarwal [4], Noor [27], Jungck AM [24] and so on. There are numerous papers dealing with the approximation of fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces (for example, see [3, 4] and the references therein).

In 2011, Phuengrattana and Suantai [29] introduced SP-iterative process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n T y_n, \ n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. In 2017, Karakaya et al., in [20] introduce new iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence

$\{x_n\}$ in C is defined by

$$\begin{cases} z_n = Tx_n, \\ y_n = (1-\alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = Ty_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., process and some existing one in literature.

In 2018, Ullah et al., in [33] introduce new iteration process called M-iteration process, as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = Tz_n, \\ x_{n+1} = Ty_n, \ n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. They proved that their iterative process converges faster than all of Picard, Mann, Ishikawa, Noor, Abass et al., SP, CR, Normal-S process, the above listed iterative process and some existing one in literature.

Remark 1.1. It was established in [1] that the iterative processes (1.4) and (1.5) have the same rate of convergence.

In 2020, the authors in [13] introduced a generalized M-iteration in the framework of hyperbolic spaces. We will give the corresponding definition of generalized M-iteration as follows; Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = \beta_n z_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)Ty_n, \ n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. They established some fixed point results in the framework of hyperbolic spaces. They also stated it clearly that for $\beta_n = \gamma_n = 0$, then iterative process (1.6) becomes (1.5). More so, they claim the the generalized M-iteration converges faster than the M-iteration. They gave a numerical example to justify this claim.

Remark 1.2. (1) If $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$, then iterative processes (1.6) and (1.3) are the same.

(2) If $\alpha = \beta = \gamma = 1$, the SP iteration becomes the M-iteration.

Question 1: It is natural to ask if one can construct an iterative scheme that converges and approximates better than existing iterative schemes in the literature.

Question 2: It is natural to ask if one can modify iterative process (1.6) and obtain strong convergence for common fixed point of nonlinear mappings.

Question 3: It is natural to ask if one can modify iterative process (1.6) to approximate certain optimization problem.

Over the years researchers have developed several iterative schemes for solving fixed point problems for different operators but the research is still on going in order to develop a faster and more efficient iterative algorithms. In this regard, the inertial extrapolation method has proven to be an effective way for accelerating the rate of convergence of iterative algorithms. The technique was introduced in 1964 and is based on a discrete version of a second order dissipative dynamical system [26, 28]. The inertial type algorithms use its two previous iterates to obtain its next iterate [5]. For details on the inertia extrapolation see [6, 7, 8] and the references therein.

In 2001, Alvarez and Attouch [5] employed the inertial technique for maximal monotone operators by the proximal point algorithm. This scheme is called the inertial proximal point algorithm, it is define as follows: For each $x_0, x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \delta_n B)^{-1} y_n, \ n \geq 1, \end{cases} \tag{1.7}$$

where I is the identity mapping. They also established that if $\{\delta_n\}$ is nondecreasing and $\theta_n \subset [0, 1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty, \tag{1.8}$$

the algorithm (1.7) converges weakly to a zero of B. In addition, condition (1.8) holds for $\theta_n < \frac{1}{3}$. It is easy to see that in Algorithm (1.8) that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ needs to be computed in every iteration and this will definitely affect the effectiveness of the scheme.

Motivated by the research in this direction, we introduce a modified generalized M-iterative scheme with the inertial technique for finding a common fixed point of three quasi-nonexpansive multivalued mappings in the framework of real Hilbert spaces. In addition, we obtain a weak convergence using our proposed modified generalized M-iterative method, we adopt the hybrid projection methods with the modified generalized M-iteration to obtain strong convergence results. Finally, we apply our convergence results to certain optimization problem, and present some numerical experiments to show the efficiency and applicability of the proposed method in comparison with other existing methods (modified SP-iterative scheme) in the literature. The results obtained in this paper extend, generalize and improve several results in this direction.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel.

Let H be a real Hilbert space. The set of fixed point of a single valued mapping $T : C \to C$ (resp. multivalued mapping $T : C \to CB(C)$) will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$ (*resp.* $x \in Tx$). We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Lemma 2.1. ([15]) *Let H be a real Hilbert space. Then for $x, y \in H$ and $\alpha_n \in [0,1]$,*

(1) $\langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2) = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2)$.
(2) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.
(3) $\|x-y\|^2 \le \|x\|^2 + 2\langle y,\ x-y\rangle$.
(4) *If $\{x_n\}$ is a sequence in H, such that $x_n \rightharpoonup x^*$, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x^*\|^2 + \limsup_{n\to\infty} \|x^* - y\|^2 \qquad (2.1)$$

A subset $C \subset H$ is said to be proximinal if for each $x \in H$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C)$, $K(C)$ and $P(C)$ denote the families of nonempty closed bounded, compact and proximinal bounded subset of C, respectively. The Haudorff metric on $CB(C)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for all $A, B \in CB(C)$, where $d(x, B) = \inf\{\|x - b\|\}$.

A multivalued mapping $T : C \to CB(C)$is said to be nonexpansive if

$$H(Tx, Ty) \le \|x - y\|$$

for all $x, y \in C$. If the fixed point of T is nonempty and

$$H(Tx, Tp) \le \|x - p\|$$

for all $x \in C$ and $p \in F(T)$, then T is said to be a quasi-nonexpansive mapping.

Condition (A). Let H be a Hilbert space and C be a subset of H. A multivalued mapping $T : C \to CB(C)$ is said to satisfy Condition (A) if $\|x - p\| = d(x, Tp)$ for all $x \in H$ and $p \in F(T)$.

Lemma 2.2. ([12]) *Let H be a real Hilbert space. Let $T : H \to CB(H)$ be a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then, $F(T)$ is closed, and if T satisfies Condition (A), then $F(T)$ is convex.*

Lemma 2.3. ([31]) *Let X be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$, $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{n_m}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

Lemma 2.4. ([12]) *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x^*$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then, $x^* \in Tx^*$.*

Lemma 2.5. ([12]) *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping with $F(T) \neq \emptyset$. Then $F(T)$ is closed.*

Lemma 2.6. ([23]) *Let C be a nonempty closed convex subset of a real Hilbert space H. For each $x, y \in H$ and $b \in R$, the set*

$$C = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + b\}$$

is closed and convex.

Lemma 2.7. ([25]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then*

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2$$

for all $x \in H$ and $y \in C$.

Lemma 2.8. ([5]) *Let $\{a_n\}, \{\delta_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ such that*

$$a_{n+1} \leq a_n + \beta_n(a_n - a_{n-1}) + \delta_n$$

for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \delta_n < \infty$ and there exists a real number β with $0 \leq \beta_n < \beta$ for all $n \in \mathbb{N}$. Then the following hold:

(1) *there exists $a^* \in [0, \infty)$ such that $\lim_{n\to\infty} a_n = a^*$,*
(2) *$\sum_{n\in\mathbb{N}}(a_n - a_{n-1}) < \infty$, where $[t]_+ = \max\{t, 0\}$.*

3. Main results

In this section, we prove a weak convergence theorem for a modified generalized M-iterative scheme with the inertial technique term for three quasi-nonexpansive multivalued mappings. In addition, we establish strong convergence result using the hybrid projection method with our modified generalized M-iteration.

Assumption 3.1. Suppose that the following conditions hold:

(1) The set C is a nonempty, closed and convex subset of the real Hilbert space H.
(2) $P, Q, R : C \to CB(C)$ are quasi-nonexpansive multivalued mappings with $F(P) \cap F(Q) \cap F(R) \neq \emptyset$ and $I - Q, I - P, I - R$ are demiclosed at zero.
(3) P, Q, R satisfy condition (A).
(4) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
(5) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
(6) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 3.2. Initialization: Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0, 1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$ be arbitrary give points.

Iterative step:
Step 1. Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n < \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases}, \tag{3.1}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Step 2. Set

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1 - \alpha_n)w_n + \alpha_n P w_n, \\ y_n &\in \beta_n z_n + (1 - \beta_n) Q z_n, \\ x_{n+1} &\in \gamma_n y_n + (1 - \gamma_n) R y_n, \ n \geq 1. \end{aligned} \tag{3.2}$$

From step (3.1), it is easy to see that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. Indeed, we have that $\theta_n\|x_n - x_{n-1}\| \le \epsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

Theorem 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then, under the Assumptions 3.1 and the Opial's condition, $\{x_n\}$ converges weakly to a common fixed point of P, Q and R.*

Proof. Let $p \in F(P) \cap F(Q) \cap F(R)$, $a_n \in Pw_n, b_n \in Qz_n, c_n \in Ry_n$ and using Algorithm 3.2, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.3}$$

Also, using Algorithm 3.2 and (3.3), we have

$$\begin{aligned} \|z_n - p\| &\le (1-\alpha_n)\|w_n - p\| + \alpha_n\|a_n - p\| \\ &= (1-\alpha_n)\|w_n - p\| + \alpha_n d(a_n, Pp) \\ &\le (1-\alpha_n)\|w_n - p\| + \alpha_n H(Pw_n, Pp) \\ &\le (1-\alpha_n)\|w_n - p\| + \alpha_n\|w_n - p\| \\ &= \|w_n - p\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.4}$$

In addition, using Algorithm 3.2, (3.3) and (3.4), we have

$$\begin{aligned} \|y_n - p\| &\le \beta_n\|z_n - p\| + (1-\beta_n)\|b_n - p\| \\ &= \beta_n\|z_n - p\| + (1-\beta_n)d(b_n, Qp) \\ &\le \beta_n\|z_n - p\| + (1-\beta_n)H(Qz_n, Qp) \\ &\le \beta_n\|z_n - p\| + (1-\beta_n)\|z_n - p\| \\ &= \|z_n - p\| \\ &\le \|w_n - p\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.5}$$

Lastly, using Algorithm 3.2, (3.3),(3.4) and (3.5), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \gamma_n\|y_n-p\|+(1-\gamma_n)\|c_n-p\| \\
&= \gamma_n\|y_n-p\|+(1-\gamma_n)d(c_n,Rp) \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)H(Ry_n,Rp) \\
&\leq \gamma_n\|y_n-p\|+(1-\gamma_n)\|y_n-p\| \\
&= \|y_n-p\| \\
&\leq \|z_n-p\| \\
&\leq \|w_n-p\| \\
&\leq \|x_n-p\|+\theta_n\|x_n-x_{n-1}\|. \qquad (3.6)
\end{aligned}$$

It follows from Lemma 2.8 that $\lim_{n\to\infty}\|x_n-p\|$ exists and thus $\{x_n\}$ is bounded. Further more, using Algorithm 3.2 and Lemma 2.1 (1), we have

$$\begin{aligned}
\|w_n-p\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-p\|^2 \\
&= \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2. \qquad (3.7)
\end{aligned}$$

In addition, using Algorithm 3.2 and Lemma 2.1 (2), we obtain that

$$\begin{aligned}
\|z_n-p\|^2 &= (1-\alpha_n)\|w_n-p\|^2+\alpha_n\|a_n-p\|-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&= (1-\alpha_n)\|w_n-p\|^2+\alpha_n d(a_n,Pp)^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&\leq (1-\alpha_n)\|w_n-p\|^2+\alpha_n H(Pw_n,Pp)^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&\leq (1-\alpha_n)\|w_n-p\|^2+\alpha_n\|w_n-p\|^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&= \|w_n-p\|^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2 \\
&\leq \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle \\
&\quad +\theta_n^2\|x_n-x_{n-1}\|^2-\alpha_n(1-\alpha_n)\|w_n-a_n\|^2. \qquad (3.8)
\end{aligned}$$

Using Lemma 2.1 (2) and (3.8), we obtain

$$\begin{aligned}
\|y_n-p\|^2 &= \beta_n\|z_n-p\|^2+(1-\beta_n)\|b_n-p\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&= \beta_n\|z_n-p\|^2+(1-\beta_n)d(b_n,Qp)^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&\leq \beta_n\|z_n-p\|^2+(1-\beta_n)H(Qz_n,Qp)^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&\leq \beta_n\|z_n-p\|^2+(1-\beta_n)\|z_n-p\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&= \|z_n-p\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2 \\
&\leq \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|w_n-a_n\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2. \qquad (3.9)
\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \gamma_n\|y_n-p\|^2+(1-\gamma_n)\|c_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\
&= \gamma_n\|y_n-p\|^2+(1-\gamma_n)d(c_n,Rp)^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\
&\le \gamma_n\|y_n-p\|^2+(1-\gamma_n)H(Ry_n,Rp)^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\
&\le \gamma_n\|y_n-p\|^2+(1-\gamma_n)\|y_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\
&= \|y_n-p\|^2-\gamma_n(1-\gamma_n)\|y_n-c_n\|^2\\
&\le \|x_n-p\|^2+2\theta_n\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\
&\quad -\alpha_n(1-\alpha_n)\|w_n-a_n\|^2-\beta_n(1-\beta_n)\|z_n-b_n\|^2\\
&\quad -\gamma_n(1-\gamma_n)\|y_n-c_n\|^2. \qquad (3.10)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\gamma_n(1-\gamma_n)\|y_n-c_n\|^2+\beta_n(1-\beta_n)\|z_n-b_n\|^2+\alpha_n(1-\alpha_n)\|w_n-a_n\|^2\\
&\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\frac{\theta_n}{\alpha_n}2\alpha_n\langle x_n-p,x_n-x_{n-1}\rangle\\
&\quad +\frac{\theta_n}{\alpha_n}\theta_n\alpha_n\|x_n-x_{n-1}\|^2\to 0 \text{ as } n\to\infty, \qquad (3.11)
\end{aligned}$$

using our assumptions and the fact that $\lim_{n\to\infty}\|x_n-p\|$ exists, we obtain

$$\lim_{n\to\infty}\|y_n-c_n\|=\lim_{n\to\infty}\|z_n-b_n\|=\lim_{n\to\infty}\|w_n-a_n\|=0. \qquad (3.12)$$

Using (3.12), we have

$$\begin{aligned}
\|w_n-x_n\| &= \|x_n+\theta_n(x_n-x_{n-1})-x_n\|\\
&= \frac{\theta_n}{\alpha_n}\alpha_n\|x_n-x_{n-1}\|\to 0 \text{ as } n\to\infty, \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
\|z_n-w_n\| &= \|(1-\alpha_n)w_n+\alpha_n a_n-w_n\|\\
&= \alpha_n\|w_n-a_n\|\to 0 \text{ as } n\to\infty, \qquad (3.14)
\end{aligned}$$

$$\|z_n-x_n\|=\|z_n-w_n\|+\|w_n-x_n\|\to 0 \text{ as } n\to\infty, \qquad (3.15)$$

$$\begin{aligned}
\|y_n-z_n\| &= \|\beta_n z_n+(1-\beta_n)b_n-z_n\|\\
&\le \|b_n-z_n\|+\beta_n\|z_n-b_n\|\to 0 \text{ as } n\to\infty \qquad (3.16)
\end{aligned}$$

and

$$\|y_n-x_n\|=\|y_n-z_n\|+\|z_n-x_n\|\to 0 \text{ as } n\to\infty. \qquad (3.17)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ for some $x^*\in C$. By using (3.13), we obtain that $w_{n_k}\rightharpoonup x^*$ and since $I-P$ is demiclosed at 0 and using (3.12), we have that $x^*\in Px^*$. In addition, using (3.15), we obtain that $z_{n_k} \rightharpoonup x^*$ and since $I - Q$ is demiclosed at 0 and using (3.12), we have that $x^* \in Qx^*$. Lastly, using (3.17), we obtain that $y_{n_k} \rightharpoonup x^*$ and since $I - P$ is demiclosed at 0 and using (3.12), we have that $x^* \in Rx^*$. Thus, we have that

$$x^* \in F(P) \cap F(Q) \cap F(R).$$

Furthermore, suppose that $\{x_n\}$ converges weakly to some y^* and let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ converging weakly to some $y^* \in F(P) \cap F(Q) \cap F(R)$. Now, suppose that $x^* \neq y^*$, then by Opial's condition and Lemma 2.3, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \lim_{k\to\infty} \|x_{n_k} - y^*\| \\
&= \lim_{n\to\infty} \|x_n - y^*\| \\
&= \lim_{j\to\infty} \|x_{n_j} - y\| \\
&< \lim_{j\to\infty} \|x_{n_j} - x\| \\
&= \lim_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

This is a contradiction. So $x^* = y^*$. Hence, $\{x_n\}$ converges weakly to a common fixed point of P, Q, and R. □

In what follows, we present an algorithm for the strong convergence of our modified iteration.

Assumption 3.4. Suppose that the following conditions hold:

(1) The set C is a nonempty, closed and convex subset of a real Hilbert space H.
(2) $P, Q, R : C \to CB(C)$ a quasi-nonexpansive multivalued mappings with $F(P) \cap F(Q) \cap F(R) \neq \emptyset$ and $I - Q, I - P, I - R$ are demiclosed at 0.
(3) P, Q, R satisfy condition (A).
(4) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
(5) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
(6) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 3.5. Initialization: Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0,1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$, be arbitrary and $C = C_1$.

Iterative step:
Step 1. Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that

$0 \leq \theta_n < \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \theta, & \text{otherwise,} \end{cases} \tag{3.18}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Step 2. Set

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n P w_n, \\ y_n &\in \beta_n z_n + (1-\beta_n) Q z_n, \\ q_n &\in \gamma_n y_n + (1-\gamma_n) R y_n, \\ C_{n+1} &= \{z \in C_n : \|q_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\qquad - 2\theta_n \langle x_n - z, x_{n-1} - x_n \rangle\} \\ x_{n+1} &= P_{C_{n+1}} x_1, \quad \forall\, n \geq 1. \end{aligned} \tag{3.19}$$

Theorem 3.6. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.5. Then, under the Assumptions 3.4, $\{x_n\}$ converges strongly to a common fixed point of P, Q, and R.*

Proof. For clarity, we divide our proofs into 4 steps.

Step 1. We will establish that $\{x_n\}$ is well defined.
Let $a_n \in P w_n, b_n \in Q z_n$ and $c_n \in R y_n$. Since P, Q and R satisfy condition (A), using Lemma 2.2, we obtain that $F(P) \cap F(Q) \cap F(R)$ is closed and convex. In addition, using usual routine, it is easy to show that C_n is closed and convex. More so, using the definition of C_{n+1} and Lemma 2.6, we obtain that C_{n+1} is also closed and convex. Thus, C_n is closed and convex for all $n \in \mathbb{N}$.

Now, for all $p \in F(P) \cap F(Q) \cap F(R)$, we have that

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &\leq \|x_n - p\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_n - p, x_{n-1} - x_n \rangle. \end{aligned} \tag{3.20}$$

More so, using (3.20), we have

$$\begin{aligned}\|z_n - p\|^2 &= (1-\alpha_n)\|w_n - p\|^2 + \alpha_n\|a_n - p\| - \alpha_n(1-\alpha_n)\|w_n - a_n\|^2 \\ &\leq (1-\alpha_n)\|w_n - p\|^2 + \alpha_n d(a_n, Pp)^2 \\ &\leq (1-\alpha_n)\|w_n - p\|^2 + \alpha_n H(Pw_n, Pp)^2 \\ &\leq (1-\alpha_n)\|w_n - p\|^2 + \alpha_n\|w_n - p\|^2 \\ &= \|w_n - p\|^2 \\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle. \quad (3.21)\end{aligned}$$

Again, using (3.21), we obtain

$$\begin{aligned}\|y_n - p\|^2 &= \beta_n\|z_n - p\|^2 + (1-\beta_n)\|b_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - b_n\|^2 \\ &\leq \beta_n\|z_n - p\|^2 + (1-\beta_n)d(b_n, Qp)^2 \\ &\leq \beta_n\|z_n - p\|^2 + (1-\beta_n)H(Qz_n, Qp)^2 \\ &\leq \beta_n\|z_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\ &= \|z_n - p\|^2 \\ &\leq \|w_n - p\|^2 \quad (3.22) \\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle.\end{aligned}$$

Lastly, using (3.22), we have that

$$\begin{aligned}\|q_n - p\|^2 &= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|c_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - c_n\|^2 \\ &= \gamma_n\|y_n - p\|^2 + (1-\gamma_n)d(c_n, Rp)^2 \\ &\leq \gamma_n\|y_n - p\|^2 + (1-\gamma_n)H(Ry_n, Rp)^2 \\ &\leq \gamma_n\|y_n - p\|^2 + (1-\gamma_n)\|y_n - p\|^2 \\ &= \|y_n - p\|^2 \\ &\leq \|x_n - p\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - p, x_{n-1} - x_n\rangle. \quad (3.23)\end{aligned}$$

Thus, using (3.23), we have that $p \in C_n$ for all $n \in \mathbb{N}$. It follows that

$$F(P) \cap F(Q) \cap F(R) \subseteq C_n,$$

for all $n \in \mathbb{N}$ as such $C_n \neq \emptyset$. Hence, $\{x_n\}$ is well defined.

Step 2. We will establish that $\{x_n\}$ is a Cauchy sequence in C and that $x \to x^* \in C$ as $n \to \infty$.
Since $x_n \in P_{C_n}x_1$, $C_{n+1} \subseteq C_n$ and $x_{n+1} \in C_n$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\| \quad (3.24)$$

for all $n \in \mathbb{N}$. In addition, since $F(P) \cap F(Q) \cap F(R) \subseteq C_n$, we have that

$$\|x_n - x_1\| \leq \|z - x_1\| \tag{3.25}$$

for all $n \in \mathbb{N}$ and $z \in F(P) \cap F(Q) \cap F(R)$. It follows from (3.24) and (3.25) that $\{\|x_n - x_1\|\}$ is bounded and nondecreasing. Hence, we obtain that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. More so, for $m > n$ and by the definition of C_n, we have that $x_m \in P_{C_m} x_1 \in C_m \subseteq C_n$. Using Lemma 2.7, we have that

$$\|x_n - x_m\|^2 + \|x_n - x_1\|^2 \leq \|x_m - x_1\|^2. \tag{3.26}$$

It follows from (3.26) that $\lim_{n\to\infty} \|x_n - x_m\| = 0$, since $\lim_{n\to\infty} \|x_n - x_1\|$ exists. As such, we have that $\{x_n\}$ is a Cauchy sequence in C, hence $x_n \to x^* \in C$ as $n \to \infty$.

Step 3. We will establish that

$$\lim_{n\to\infty} \|y_n - c_n\| = \lim_{n\to\infty} \|z_n - b_n\| = \lim_{n\to\infty} \|w_n - a_n\| = 0.$$

From Step 2, it is easy to see that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in C_n$, using the fact that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have that

$$\begin{aligned}
\|q_n - x_n\| &= \|q_n - x_{n+1} + x_{n+1} - x_n\| \\
&\leq \|q_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\leq \sqrt{\|x_n - x_{n+1}\|^2 + 2\theta_n \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_n - x_{n+1}, x_{n-1} - x_n\rangle} \\
&\quad + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.27}$$

Since R satisfies condition (A) and using (3.8) and (3.9), we have that

$$\begin{aligned}
\|q_n - p\|^2 &= \gamma_n \|y_n - p\|^2 + (1-\gamma_n)\|c_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - c_n\|^2 \\
&\leq \|y_n - p\|^2 - \gamma_n(1-\gamma_n)\|y_n - c_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n \langle x_n - x_{n-1}, w_n - p\rangle - (1-\alpha_n)\alpha_n \|w_n - a_n\|^2 \\
&\quad - \beta_n(1-\beta_n)\|z_n - b_n\| - \gamma(1-\gamma_n))\|y_n - c_n\|^2,
\end{aligned} \tag{3.28}$$

it implies

$$\begin{aligned}
&(1-\alpha_n)\alpha_n \|w_n - a_n\|^2 + \beta_n(1-\beta_n)\|z_n - b_n\| + \gamma(1-\gamma_n))\|y_n - c_n\|^2 \\
&\leq \|x_n - p\|^2 - \|q_n - p\|^2 + 2\frac{\theta_n}{\alpha_n}\alpha_n \langle x_n - x_{n-1}, w_n - p\rangle.
\end{aligned} \tag{3.29}$$

Using (3.27) and our assumption, we have that

$$\begin{aligned}
\lim_{n\to\infty} \|y_n - c_n\| &= \lim_{n\to\infty} \|z_n - b_n\| \\
&= \lim_{n\to\infty} \|w_n - a_n\| = 0.
\end{aligned} \tag{3.30}$$

Using (3.30), we have

$$\begin{aligned}\|w_n - x_n\| &= \|x_n + \theta_n(x_n - x_{n-1}) - x_n\| \\ &= \frac{\theta_n}{\alpha_n}\alpha_n\|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.31}$$

$$\begin{aligned}\|z_n - w_n\| &= \|(1-\alpha_n)w_n + \alpha_n a_n - w_n\| \\ &= \alpha_n\|w_n - a_n\| \to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.32}$$

$$\|z_n - x_n\| = \|z_n - w_n\| + \|w_n - x_n\| \to 0 \text{ as } n \to \infty, \tag{3.33}$$

$$\begin{aligned}\|y_n - z_n\| &= \|\beta_n z_n + (1-\beta_n)b_n - z_n\| \\ &\le \|b_n - z_n\| + \beta_n\|z_n - b_n\| \to 0 \text{ as } n \to \infty \end{aligned} \tag{3.34}$$

and

$$\|y_n - x_n\| = \|y_n - z_n\| + \|z_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.35}$$

We have established that $x_n \to x^* \in C$, it follows from (3.31), we obtain that $w_{n_k} \to x^*$ and since $I - P$ is demiclosed at 0 and using (3.30), we have that $x^* \in Px^*$. In addition, using similar approach, we obtain that $x^* \in F(Q)$ and $x^* \in F(R)$. Thus, we have that

$$x^* \in F(P) \cap F(Q) \cap F(R).$$

Step 4. Finally, we have to show that $x^* \in P_{F(P)\cap F(Q)\cap F(R)}x_1$.
It follows from (3.25), we have that

$$\|x^* - x_1\| \le \|z - x_1\|$$

for all $z \in F(P) \cap F(Q) \cap F(R)$. Thus, by the definition of projection operator (P_C) we have that

$$x^* = P_{F(P)\cap F(Q)\cap F(R)}x_1.$$

Thus, the proof is complete. □

4. Application and numerical examples

In this section, we present an application and a numerical example in finite dimensional Hilbert spaces and compare our proposed Algorithm 3.2 and Algorithm 3.5 with modified NOOR and modified SP-iteration (see that appendix for these algorithms).

4.1. **Application to Common Inclusion Problem.** In this section, we apply our results to common inclusion problem.

The common inclusion problem is one of the interesting problems in this area of research. This problem has received great attention over the years due to its fruitful applications in almost all areas of sciences. In particular, it is applied to some problems in image processing, machine learning, signal processing and linear inverse problem. The inclusion problem is defined as find $x \in H$ such that

$$0 \in Ax + Bx, \tag{4.1}$$

where $A : H \to H$ is an α-inverse strongly monotone operator and $B : H \to 2^H$ is a maximal monotone operator. It is well known that the resolvent $J_\lambda^B(I - \lambda A)$ is nonexpansive if $\lambda \in (0, 2\alpha)$. As such our algorithms take the form:

Assumption 4.1. Suppose that the following conditions hold:

(1) The set C is a nonempty, closed and convex subset of a real Hilbert space H.
(2) Let $A_i : H \to H$ is an α-inverse strongly monotone operator and $B_i : H \to 2^H$ is a maximal monotone operator, where $i = 1, 2, 3$.
(3) The solution set $\Omega = \{\cap_{i=1}^3 (A_i + B_i)^{-1}(0)\} \neq \emptyset$.
(4) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
(5) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
(6) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 4.2. Initialization: Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0, 1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$, be arbitrary and $C = C_1$.

Iterative step:
Step 1. Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n < \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \theta, & \text{otherwise}, \end{cases} \tag{4.2}$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.

Step 2. Set

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$
\begin{aligned}
z_n &= (1-\alpha_n)w_n + \alpha_n J_\lambda^{B_1}(I-\lambda A_1)w_n,\\
y_n &= \beta_n z_n + (1-\beta_n)J_\lambda^{B_2}(I-\lambda A_2)z_n,\\
q_n &= \gamma_n y_n + (1-\gamma_n)J_\lambda^{B_3}(I-\lambda A_3)y_n,\\
C_{n+1} &= \{z \in C_n : \|q_n - z\|^2 \le \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2\\
&\qquad - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\}\\
x_{n+1} &= P_{C_{n+1}}x_1, \ \ \forall\, n \ge 1.
\end{aligned}
\tag{4.3}
$$

Theorem 4.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 4.2. Then, under the Assumptions 4.1, $\{x_n\}$ converges strongly to a point in Ω.*

5. Appendix

Modified SP-Iterative Scheme

Assumption 5.1. Suppose that the following conditions hold:

(1) The set C is a nonempty, closed and convex subset of a real Hilbert space H.
(2) $T_1, T_2, T_3 : C \to CB(C)$ a quasi-nonexpansive multivalued mappings with $F(T_1)\cap F(T_2)\cap F(T_3) \neq \emptyset$ and $I-T_1, I-T_2, I-T_3$ are demiclosed at 0.
(3) T_1, T_2, T_3 satisfy condition (A).
(4) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
(5) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
(6) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 5.2. Initialization: Given $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\epsilon_n\} \subset (0,1)$ for all $n \in \mathbb{N}$. Let $x_0, x_1 \in C$ be arbitrary.

Iterative step:
Step 1. Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \le \theta_n < \bar{\theta}_n$, where

$$
\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \theta, & \text{otherwise} \end{cases}, \tag{5.1}
$$

where $\theta > 0$ and $\{\epsilon_n\}$ is a positive sequence such that $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.
Step 2. Set

$$
w_n = x_n + \theta_n(x_n - x_{n-1}).
$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n T_1 w_n, \\ y_n &\in (1-\beta_n)z_n + \beta_n T_2 z_n, \\ x_{n+1} &\in (1-\gamma_n)y_n + \gamma_n T_3 y_n, \ n \geq 1. \end{aligned} \tag{5.2}$$

Assumption 5.3. Suppose that the following conditions hold:

(1) The set C is a nonempty, closed and convex subset of a real Hilbert space H.
(2) $T_1, T_2, T_3 : C \to CB(C)$ a quasi-nonexpansive multivalued mappings with $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ and $I - T_1, I - T_2, I - T_3$ are demiclosed at 0.
(3) T_1, T_2, T_3 satisfy condition (A).
(4) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
(5) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.
(6) $0 < \liminf_{n\to\infty} \gamma_n < \limsup_{n\to\infty} \gamma_n < 1$.

Algorithm 5.4. Initialization: *Given* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\epsilon_n\} \subset (0, 1)$ *for all* $n \in \mathbb{N}$. *Let* $x_0, x_1 \in C$, *be arbitrary and* $C = C_1$.

Iterative step:
Step 1. *Given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$, *choose* θ_n *such that* $0 \leq \theta_n < \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \theta, & \text{otherwise}, \end{cases} \tag{5.3}$$

where $\theta > 0$ *and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\alpha_n) \Rightarrow \lim\limits_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.
Step 2. *Set*

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Then, compute

$$\begin{aligned} z_n &\in (1-\alpha_n)w_n + \alpha_n T_1 w_n, \\ y_n &\in (1-\beta_n)z_n + \beta_n T_2 z_n, \\ q_n &\in (1-\gamma_n)y_n + \gamma_n T_3 y_n, \\ C_{n+1} &= \{z \in C_n : \|q_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\qquad - 2\theta_n \langle x_n - z, x_{n-1} - x_n \rangle\} \\ x_{n+1} &= P_{C_{n+1}} x_1, \ \ \forall\, n \geq 1. \end{aligned} \tag{5.4}$$

5.1. **Numerical Example.**

Example 5.5. Define a mapping $P, Q, R : [0,1] \to [0,1]$ as

$$Px = \begin{cases} [0, \frac{x}{2}] \text{ if } x \leq 0.5, \\ \{1\} \text{ if } x > 0.5, \end{cases} \tag{5.5}$$

$$Qx = \begin{cases} [0, \frac{x}{4}] \text{ if } x \leq 0.5, \\ \{1\} \text{ if } x > 0.5, \end{cases} \tag{5.6}$$

and

$$Rx = \begin{cases} [0, \frac{x}{10}] \text{ if } x \leq 0.5, \\ \{1\} \text{ if } x > 0.5. \end{cases} \tag{5.7}$$

Then it is easy to see that P, Q and R are quasi-nonexpansive and satisfies condition (A), and $F(P) \cap F(Q) \cap F(R) = \{0, 1\}$. We choose the following parameter $\theta = 0.01, \epsilon_n = \frac{1}{(n+1)^2}, \alpha_n = \frac{4n+2}{5n+2}, \beta_n = \frac{n+1}{5n+4}, \gamma_n = \frac{2n}{3n+5}$. We make different choices of the initial values x_0 and x_1 as follows:
Ex 4.4a: $x_0 = 0.5, x_1 = 0.3$;
Ex 4.4b: $x_0 = 0.9, x_1 = 0.4$;
Ex 4.4c: $x_0 = 0.75, x_1 = 0.12$;
Ex 4.4d: $x_0 = 0.29, x_1 = 0.49$.

Table 1. Numerical results.

		Alg. 3.2	Alg 5.2
Ex 4.4a	CPU time (sec)	0.0012	0.0016
	No of Iter.	10	15
Ex 4.4b	CPU time (sec)	0.0013	0.0019
	No of Iter.	11	20
Ex 4.4c	CPU time (sec)	0.0011	0.0012
	No of Iter.	9	18
Ex 4.4d	CPU time (sec)	0.0011	0.0012
	No of Iter.	10	16

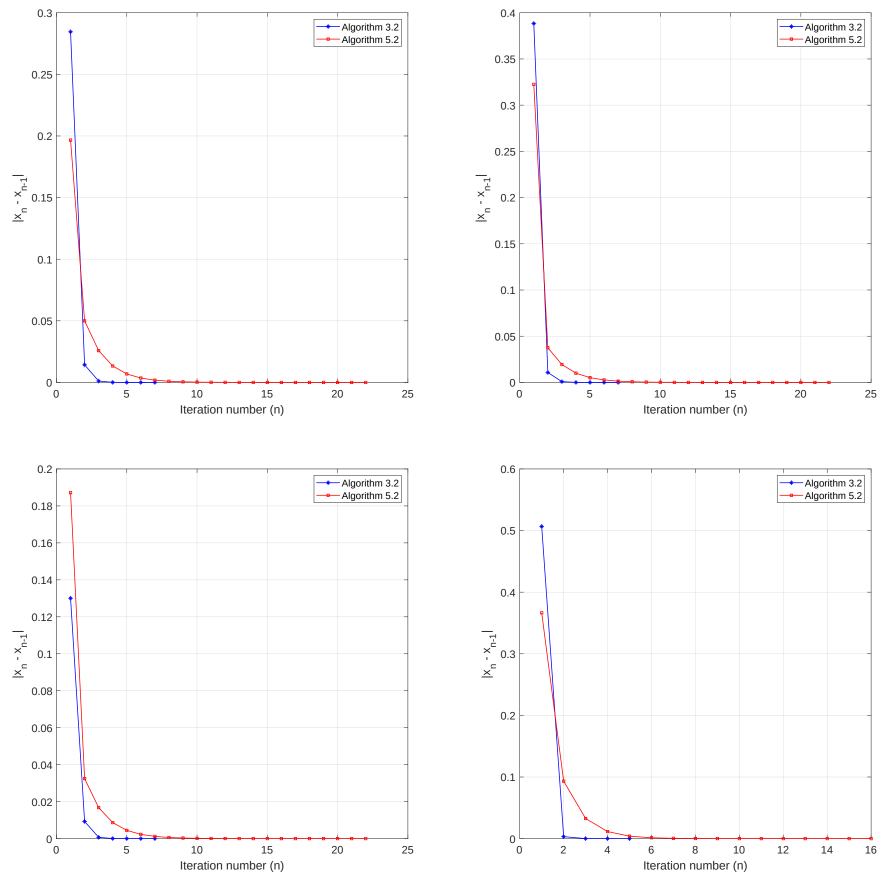

FIGURE 1. Example 5.5, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

Example 5.6. Let $H = \mathbb{R}^3, C = [3,6]^3$ and $C_1 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{(x_1-6)^2 + (x_2-6)^2 + (x_3-6)^2} \leq 3\}$. We defined $P, Q, R : \mathbb{R}^3 \to CB(\mathbb{R}^3)$ as :

$$Px = \begin{cases} (6,6,6), & \text{if } x_1 \in C_1 \\ \{y = (y_1, y_2, y_3) \\ \in C : \sqrt{(y_1-6)^2 + (y_2-6)^2 + (y_3-6)^2} \leq \frac{1}{\|x\|_1}\}, & \text{otherwise} \end{cases}, \tag{5.8}$$

$$Qx = \begin{cases} (6,6,6), & \text{if } x_1 \in C_1 \\ \\ \{y = (6,y,6) \in C : y \in [(x_2+6)(\frac{\arcsin(19x_2-76)}{2}) + x_2, 6]\}, & \text{otherwise} \end{cases}, \tag{5.9}$$

and

$$Rx = \begin{cases} (6,6,6), & \text{if } x_1 \in C_1 \\ \\ \{y = (6,6,y) \in C : y \in [(x_2-6)(\frac{\arccos(15x_2-60)}{5}) + x_2, 6]\}, & \text{otherwise} \end{cases}. \tag{5.10}$$

Choose $\theta = 0.001, \alpha_n = \frac{1}{n+1}, \epsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{3n}{3n+5}, \beta_n = \frac{2}{n^2+5}$. It is easy to verify that all hypothesis of Theorem 3.6 and Theorem 3.3 are satisfied and $F(P) \cap F(Q) \cap F(R) = (6,6,6) \neq \emptyset$. We use different choices of x_0, x_1 and test the convergence of our algorithm with $\|x_{n+1} - x_n\| < 10^{-7}$ as stopping criterion. We choose the following parameter $\theta = 0.01, \epsilon_n = \frac{1}{(n+1)^2}, \alpha_n = \frac{4n+2}{5n+2}, \beta_n = \frac{n+1}{5n+4}, \gamma_n = \frac{2n}{3n+5}$ We make different choices of the initial values x_0 and x_1 as follows:

Ex 4.5a: $x_0 = (4.1, 4.7, 5), x_1 = (4.893, 5.77, 5)$.
Ex 4.5b: $x_0 = (4.98, 4.3, 4), x_1 = (4.33, 4.42, 4.42)$.
Ex 4.5c: $x_0 = (4.2, 4.3, 4.2), x_1 = (5.3, 5.2, 5.42)$.
Ex 4.5d: $x_0 = (4.59, 5.23, 4.89), x_1 = (5.98, 5, 5.24)$.

Table 2. Numerical results.

		Alg. 3.5	Alg 5.4
Ex 4.4a	CPU time (sec)	0.0035	0.0040
	No of Iter.	23	25
Ex 4.4b	CPU time (sec)	0.0048	0.0056
	No of Iter.	21	27
Ex 4.4c	CPU time (sec)	0.0045	0.0060
	No of Iter.	20	26
Ex 4.4d	CPU time (sec)	0.0045	0.0062
	No of Iter.	21	29

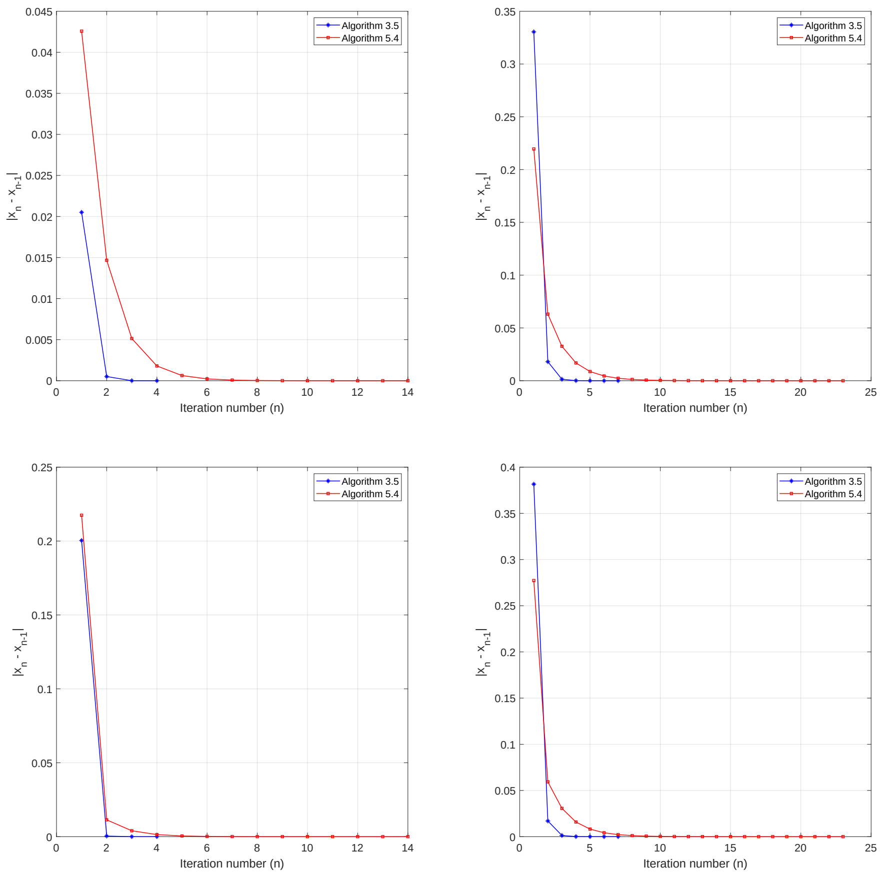

Figure 2. Example 5.6, Top Left: Case I; Top Right: Case II; Bottom Left: Case III; Bottom Right: Case IV.

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