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FIXED POINT THEOREMS FOR CONDENSING MULTIMAPS ON ABSTRACT CONVEX UNIFORM SPACES

A. Amini-Harandi[1], A. P. Farajzadeh[2], D. O'Regan[3] and R. P. Agarwal[4]

[1]Department of Mathematics, University of Shahrekord
Shahrekord, 88186-34141, Iran
e-mail: aminih_a@yahoo.com

[2]Department of Mathematics, Razi University
Kermanshah, 67149, Iran
e-mail: ali-ff@sci.razi.ac.ir

[3]Department of Mathematics, National University of Ireland
Galway, Ireland
e-mail: donal.oregan@nuigalway.ie

[4]Department of Mathematical Sciences, Florida Institute of Technology
Melbourne, FL 32901, USA.
e-mail: agarwal@fit.edu

Abstract. In this paper we present fixed point theory for condensing mulitimaps on abstract convex uniform spaces. Also we obtain a nonlinear alternative of Leray-Schauder type for Mönch type maps. Our main results unify and improve some well-know results in the literature.

1. Introduction

Throughout the paper, $\langle X \rangle$ denotes the family of all nonempty finite subsets of nonempty set X. Let X and Y be topological spaces with $A \subseteq X$ and $B \subseteq Y$. Let $F : X \multimap Y$ be a multimap with nonempty values. The image of A under F is the set $F(A) = \bigcup_{x \in A} F(x)$; and the inverse image of B under F is $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. F is said to be:

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(i) closed if its graph, $Gr(F) = \{(x, y) \in X \times Y \ : y \in F(x)\}$ is a closed set in the product space $X \times Y$,
(ii) upper semicontinuous, if for each closed set $B \subseteq Y$, $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is closed in X,
(iii) compact if $\overline{F(X)}$ is compact subset of Y.

It is well-known that if Y is compact Hausdorff and $F(x)$ is closed for each $x \in X$, then F is upper semicontinuous if and only if F is closed. For the remainder of this paper we assume all topological spaces are Hausdorff.

Definition 1.1. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values. When $D \subseteq E$, the space is denoted by $(E \supseteq D; \Gamma)$. In such a case, a subset X of E is said to be Γ-convex if, for any $A \in \langle X \cap D \rangle$, we have $\Gamma(A) \subseteq X$. For a nonempty subset Q of E, the Γ-convex hull of Q, denoted by $co_\Gamma(Q)$, is defined by

$$co_\Gamma(Q) = \bigcap \{C : Q \subseteq C \subseteq E, C \text{ is } \Gamma\text{-convex}\},$$

and the closed Γ-convex hull of Q, denoted by $\overline{co}_\Gamma(Q)$, is defined by $\overline{co}_\Gamma(Q) = \overline{co_\Gamma(Q)}$. In the case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.
An abstract convex space with any topology is called an *abstract convex topological space.*

Examples 1.2. A convexity space $(E, \mathcal{C})$ in the classical sense [3], G-convex spaces, C-spaces, convex spaces and almost convex spaces [4] are the main examples of abstract convex spaces.

Definition 1.3. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma(A)) \subseteq G(A), \text{ for all } A \in \langle X \rangle,$$

then G is called a KKM map with respect to F. A KKM map $G : D \multimap Z$ is a KKM map with respect to the identity map 1_E. A multimap $F : E \multimap Z$ is said to have the KKM property if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{\overline{G(x)}\}_{x \in X}$ has the finite intersection property. We denote

$$\text{KKM (E,Z)} := \{F : E \multimap Z : F \text{ has the KKM property}\}.$$

Definition 1.4. For an abstract convex space $(E \supseteq D; \Gamma)$, let X be a Γ-convex subset of E, and D' a nonempty subset of $X \cap D$. Let $\Gamma' : \langle D' \rangle \multimap X$ be a map defined by

$$\Gamma'(A) = \Gamma(A) \subseteq X \text{ for } A \in D'.$$

Then $(X \supseteq D'; \Gamma')$ itself is an abstract convex space called a subspace.

Lemma 1.5. (Park [6]) *Let $(E \supseteq D; \Gamma)$ be an abstract convex space, $(X \supseteq D'; \Gamma')$ a subspace, Z a set. If $F \in KKM(E, Z)$, then $F|_X \in KKM(X, Z)$.*

Definition 1.6. For a given abstract convex space $(E, D; \Gamma)$ and a topological space X, a map $H : X \multimap E$ is called a Φ-map if there exists a map $G : X \multimap D$ such that

(i) for each $x \in X$, $A \in \langle G(x) \rangle$ implies $\Gamma(A) \subseteq H(x)$; and
(ii) $X = \bigcup\{Int\ G^{-}(y) : y \in D\}$.

If $(E,\mathcal{U})$ is a uniform space, then its topology is induced by the uniformity $\mathcal{U}$ is the family of all subsets G of E such that for each in $x \in G$, there is $U \in \mathcal{U}$ such that $U[x] \subseteq G$, where $U[x] = \{y \in X : (x, y) \in U\}$. If $K \subseteq E$ and $U \in \mathcal{U}$, then $U[K] = \bigcup_{x \in K} U[x]$. A subset S of a uniform space E is said to be precompact if, for any entourage V, there is a finite subset N of E such that $S \subseteq V[N]$.

Definition 1.7. An *abstract convex uniform space* $(E, D; \Gamma; \mathcal{U})$ is an abstract convex space so that its topology is induced by the uniformity $\mathcal{U}$. In section 2, we shall assume that the convex structure Γ and the base $\mathcal{B}$ of $\mathcal{U}$ satisfy the following conditions:

$$V[K] \text{ is } \Gamma-\text{convex whenever } K \text{ is a } \Gamma-\text{convex subset of } E \text{ and } V \in \mathcal{B}; \quad (1.1)$$

$$co_{\Gamma}(A) \text{ is precompact whenever } A \text{ is precompact.} \quad (1.2)$$

In $(E, D; \Gamma; \mathcal{U})$, a subset Z of E is called a Φ-*set* if, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : Z \multimap E$ such that $Gr(T) \subseteq U$. If E itself is a Φ-set, then it is called a Φ-*space.*

Note part (i) of the following lemma was proved in [5], in the setting of locally G-convex spaces.

Lemma 1.8. *Let* $(E, \Gamma; \mathcal{U})$ *be an abstract convex space. Then the following statements hold:*

(i) *if* K *is a* Γ*-convex subset* E*, then its closure* $\overline{K}$ *is also* Γ*-convex,*
(ii) $\overline{co}_{\Gamma}(Q) = \bigcap\{C : Q \subseteq C \subseteq E, C$ *is closed and* Γ*-convex*$\}$,
(iii) *if* $X \subseteq E$ *be a* Φ*-set and* $Y \subseteq X$ *then* Y *is also a* Φ*-set.*

Proof. (i) Let $\mathcal{B}$ be a base of $\mathcal{U}$ as described above, $V \in \mathcal{B}$, and $A = \{a_1, ...a_n\} \subseteq \langle \overline{K} \rangle$. Choose $\{x_1, ...x_n\} \subseteq \langle K \rangle$ with $x_i \in V[a_i] \cap K$ for any $i = 1, ..., n$. Since $V[K]$ is Γ-convex and $a_i \in V[x_i] \subseteq V[K]$ for any $i = 1, ..., n$, we infer that $\Gamma(A) \subseteq V[K]$ for all $V \in \mathcal{B}$, and hence $\Gamma(A) \subseteq \bigcap_{V \in \mathcal{B}} V[K] = \overline{K}$. This shows that $\overline{K}$ is Γ-convex.
(ii) By (i), $\overline{co}_{\Gamma}(Q)$ is a closed, Γ-convex set which contains Q, so

$$\bigcap\{C : Q \subseteq C \subseteq E, C \text{ is closed and } \Gamma\text{-convex}\} \subseteq \overline{co}_{\Gamma}(Q).$$

On the other hand, we have

$$\bigcap\{C : Q \subseteq C \subseteq E, C \text{ is closed and } \Gamma\text{-convex}\}$$

$$= \overline{\bigcap\{C : Q \subseteq C \subseteq E, C \text{ is closed and } \Gamma\text{-convex}\}}$$

$$\supseteq \overline{\bigcap\{C : Q \subseteq C \subseteq E, C \text{ is } \Gamma\text{-convex}\}}$$

$$= \overline{co_\Gamma(Q)} = \overline{co}_\Gamma(Q).$$

(iii) Since X is a Φ-set, for any entourage $U \in \mathcal{U}$, there exists a Φ-map $H : X \multimap E$ such that $Gr(H) \subseteq U$. Since $H : X \multimap E$ is a Φ-map there exists a map $G : X \multimap E$ such that for each $x \in X$, $A \in \langle G(x)\rangle$ implies $\Gamma(A) \subseteq H(x)$, and $X = \bigcup\{Int\ G^-(y) : y \in E\}$. As a result, for each $x \in Y \subseteq X$, $A \in \langle G_{|Y}(x)\rangle$ implies $\Gamma(A) \subseteq H(x) = H_{|Y}(x)$ and

$$Y = Y \cap X = Y \cap \bigcup\{Int\ G^-(y) : y \in E\}$$

$$= \bigcup\{Y \cap Int\ G^-(y) : y \in E\} = \bigcup\{Int\ G^-_{|Y}(y) : y \in E\}.$$

Thus, $H_{|Y} : Y \multimap E$ is a Φ-map and $Gr(H_{|Y}) \subseteq Gr(H) \subseteq U$. □

Theorem 1.9. ([6,8]) *Let $(E, D; \Gamma; \mathcal{U})$ be an abstract convex uniform space, and $F \in KKM(E, E)$ be a closed compact map. If $\overline{F(E)}$ is a Φ-set, then F has a fixed point.*

Examples 1.10. (i) Any locally convex subset of a topological vector space E is a Φ-set in E.

(ii) Any subset of the Zima type in a G-convex uniform space $(E \supseteq D; \Gamma; \mathcal{U})$ such that every singleton is Γ-convex is a Φ-set.

(iii) For a locally G-convex space $(E \supseteq D; \Gamma; \mathcal{U})$, any nonempty subset X of E is a Φ-set. A locally G-convex space $(E \supseteq D; \Gamma; \mathcal{U})$ is a Φ-set.

(iv) A metric G-convex space $(E \supseteq D; \Gamma)$ is a Φ-space whenever D is dense in X and every open ball is Γ-convex.

2. Fixed point theory

A slight modifications of the proof of Lemma 3.4 in [5], yields the following lemma.

Lemma 2.1. *Let X be a nonempty closed, Γ-convex subset of an abstract convex uniform space $(E, \Gamma; \mathcal{U})$ and $F : X \multimap X$. If $\emptyset \neq Q \subseteq X$, then there exists a closed, Γ-convex set $K = K(F, Q)$ with $Q \subseteq K \subseteq X$ and $K = \overline{co}_\Gamma(F(K) \cup Q)$.*

Proof. Let

$$\mathcal{F} = \{A \subseteq E : \ A \text{ is closed, } \Gamma\text{-convex and } \overline{co}_\Gamma(F(X \cap A) \cup Q) \subseteq A\}.$$

Since $E \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. Define a partial order by inverse inclusion, that is, for A, $B \in \mathcal{F}$, $A \leq B \Leftrightarrow B \subseteq A$. Let $\mathcal{C}$ be any chain in $\mathcal{F}$. Put $M = \bigcap_{A \in \mathcal{C}} A$. Since each $A \in \mathcal{C}$ is closed, Γ-convex and contain Q, we infer that M is closed, Γ-convex and contain Q, for all $A \in \mathcal{C}$, it follows from

$$F(M \cap X) \cup Q \subseteq F(A \cap X) \cup Q$$

that $\overline{co}_\Gamma(F(M \cap X) \cup Q) \subseteq \overline{co}_\Gamma(F(A \cap X) \cup Q) \subseteq A$ and so

$$\overline{co}_\Gamma(F(M \cap X) \cup Q) \subseteq \bigcap_{A \in \mathcal{C}} A = M.$$

Thus $M \in \mathcal{F}$ and M is an upper bound of $\mathcal{C}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element, say K. We claim that $\overline{co}_\Gamma(F(K \cap X) \cup Q) = K$. In fact, put $K_0 = \overline{co}_\Gamma(F(K \cap X) \cup Q)$. It is obvious that K_0 is closed, Γ-convex and contain Q. Furthermore, since $\overline{co}_\Gamma(F(K_0 \cap X) \cup Q) \subseteq \overline{co}_\Gamma(F(K \cap X) \cup Q) = K_0$, we have $K_0 \in \mathcal{F}$ and $K_0 \geq K$. By the maximality of K, we conclude that $K = K_0$, that is

$$\overline{co}_\Gamma(F(K \cap X) \cup Q) = K.$$

Finally, since $F(X) \subseteq X$, it follows since X is closed that $K \subseteq X$ and $\overline{co}_\Gamma(F(K) \cup Q) = K$. □

Theorem 2.2. *Let X be a nonempty closed, Γ-convex subset of an abstract convex uniform Φ-space $(E, \Gamma; \mathcal{U})$. Let $F \in KKM(X, X)$ be a closed map, then F has a fixed point provided the following condition hold:*

$$\begin{aligned}&\textit{for any } x_0 \in X, \textit{ and } A \subseteq X \textit{ with } A = \overline{co}_\Gamma(\{x_0\} \cup F(A))\\ &\textit{we have that } A \textit{ is compact.}\end{aligned} \tag{2.1}$$

Proof. Putting $Q = \{x_0\}$ in Lemma 2.1, we obtain that there exists a closed, Γ-convex set $K \subseteq X$ with $K = \overline{co}_\Gamma(F(K) \cup \{x_0\})$. Since (2.1) holds, then K is compact. Now $F : K \multimap K$ is closed and compact and $F_{|K} \in KKM(K, K)$ by Lemma 1.5. Since E is a Φ-space, then $F(K)$ is a Φ-set and so by Theorem 1.9, F has a fixed point. □

Theorem 2.3. *Let X be a nonempty complete, Γ-convex subset of an abstract convex uniform Φ-space $(E, \Gamma; \mathcal{U})$. Let $F \in KKM(X, X)$ be a closed map, then F has a fixed point provided that the following condition hold:*

$$\begin{aligned}&\textit{whenever } x_0 \in X, \ \ A \subseteq X, \ F(A) \subseteq A \textit{ and } A \setminus \overline{co}_\Gamma(F(A)) \subseteq \{x_0\}\\ &\textit{we have that } A \textit{ is a precompact subset of } X.\end{aligned} \tag{2.2}$$

Proof. Choose $x_0 \in X$ and let $A = \bigcup_{i \geq 0} F^i(x_0)$ where $F^0(x_0) = \{x_0\}$. Then $F(A) \subseteq A$ and $A \setminus \overline{co}_\Gamma(F(A)) \subseteq \{x_0\}$, so A is precompact, and hence $\overline{A}$ is compact since it is a precompact and closed subset of the complete set X. Define $G : \overline{A} \multimap \overline{A}$ by $G(x) = F(x) \cap \overline{A}$. Since F is closed and $\overline{A}$ is compact, it is easy to see that $G(x) \neq \emptyset$, $\forall\, x \in \overline{A}$. Put

$$\mathcal{A} = \{Y :\ Y \text{ is a nonempty closed subset of } \overline{A} \text{ and } \ G(Y) \subseteq Y\}.$$

Since $\overline{A} \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Define a partial order $\leq$ on $\mathcal{A}$ by $A \leq B \Leftrightarrow B \subseteq A$. Let $\mathcal{C}$ be any chain in $\mathcal{A}$ and put $M = \bigcap_{L \in \mathcal{C}}$. Now M is an upper bound of $\mathcal{C}$ and so, by Zorn's Lemma, $\mathcal{A}$ has a maximal element, say Q. Since F is closed, so is G, and this with the compactness of $\overline{A}$ guarantees that G is upper semicontinuous. Therfore, $G(Q)$ is compact. Putting $Y = G(Q)$ and noting that $G(Y) = G(G(Q)) \subseteq G(Q) = Y$, the maximality of Q gives us that $Q = Y$. Thus

$$Q = G(Q) = F(Q) \cap \overline{A} \subseteq F(Q).$$

Let $K = K(F, Q)$ be the subset of X described in Lemma 2.1, so $K = \overline{co}_\Gamma(F(K) \cup Q)$. Since $Q \subseteq F(Q) \subseteq K$, we have $K = \overline{co}_\Gamma(F(K))$ and so we have shown that there exists a closed, Γ-convex subset $K \subseteq X$ such that $K = \overline{co}_\Gamma(F(K)$. Now (2.2) implies that K is precompact subset of X and note in fact that it is compact. Thus by Theorem 1.9, $F : K \multimap K$ has a fixed point. □

Now, we present a Mönch type result for KKM multimaps.

Theorem 2.4. *Let X be a nonempty complete, Γ-convex subset of an abstract convex uniform Φ-space $(E, \Gamma; \mathcal{U})$. Suppose $F \in KKM(X, X)$ is closed, and satisfies the following properties:*

$$F \text{ maps compact sets into relatively compact sets}; \tag{2.3}$$

$$\begin{aligned} &A = co_\Gamma(\{x_0\} \cup F(A)) \text{ with } \overline{A} = \overline{C} \text{ and } C \subseteq A \text{ countable},\\ &\text{implies } \overline{A} \text{ is compact}; \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\text{for any relatively compact subset } A \text{ of } X\\ &\text{there exists a countable subset } B \subseteq A \text{ with } \overline{B} = \overline{A}; \end{aligned} \tag{2.5}$$

$$F(\overline{A}) \subseteq \overline{F(A)} \text{ for any realatively compact subset } A \text{ of } X. \tag{2.6}$$

Then F has a fixed point.

Proof. We follow the proof of Theorem 2.5 in [2]. Let $D_0 = co_\Gamma(\{x_0\})$, $D_{n+1} = co_\Gamma(\{x_0\} \cup F(D_n))$, $n = 0, 1, 2, \ldots$ and

$$D_\infty = \bigcup_{n=0}^{\infty} D_n.$$

Now for $n = 0, 1, 2, ...$ notice D_n is Γ-convex. Also by induction, we see that

$$D_0 \subseteq D_1 \subseteq ... \subseteq D_{n-1} \subseteq D_n \subseteq ...$$

Consequently, D_∞ is Γ-convex. Since for each n,

$$co_\Gamma(\{x_0\} \cup F(D_n)) \subseteq co_\Gamma(\{x_0\} \cup F(D_\infty)),$$

so

$$D_\infty = \bigcup_{n=0}^{\infty} co_\Gamma(\{x_0\} \cup F(D_n)) \subseteq co_\Gamma(\{x_0\} \cup F(D_\infty)).$$

On the other hand, D_∞ is a Γ-convex set which contains $\bigcup_{n=0}^{\infty} F(D_n) = F(D_\infty)$, consequently $co_\Gamma(\{x_0\} \cup F(D_\infty)) \subseteq D_\infty$. Thus

$$D_\infty = co_\Gamma(\{x_0\} \cup F(D_\infty)). \tag{2.7}$$

It is easy to see (use induction with (1.2), note X is complete, and (2.3)) that D_n is relatively compact for $n = 0, 1, 2, ...$ Now (2.5) implies that for each $n = 0, 1, 2, ...$ there exists C_n with C_n countable, $C_n \subseteq D_n$, and $\overline{C_n} = \overline{D_n}$. Let $C_\infty = \bigcup_{n=0}^{\infty} C_n$. Now since

$$\bigcup_{n=0}^{\infty} D_n \subseteq \bigcup_{n=0}^{\infty} \overline{D_n} \subseteq \overline{\bigcup_{n=0}^{\infty} D_n},$$

we have

$$\overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} D_n} = \overline{D_\infty} \text{ and } \overline{\bigcup_{n=0}^{\infty} \overline{D_n}} = \overline{\bigcup_{n=0}^{\infty} \overline{C_n}} = \overline{\bigcup_{n=0}^{\infty} C_n} = \overline{C_\infty}.$$

Thus $\overline{D_\infty} = \overline{C_\infty}$. This together with (2.4) and (2.7) implies that $\overline{D_\infty}$ is compact. From (2.7) we have $F(D_\infty) \subseteq D_\infty$, and this together with (2.6) yields

$$F(\overline{D_\infty}) \subseteq \overline{F(D_\infty)} \subseteq \overline{D_\infty}.$$

Also notice $F_{|\overline{D_\infty}} \in KKM(\overline{D_\infty}, \overline{D_\infty})$ is closed. Now apply Theorem 1.9. $\square$

Corollary 2.5. *Let X a nonempty complete, Γ-convex subset of a metrizable abstract convex uniform space (E, Γ) such that every open ball is convex and $1_X \in KKM(X, X)$. Suppose that $F : X \to X$ is a continuous map with the property (2.4). Then F has a fixed point.*

Proof. Since every open ball is convex, then (E, Γ) is an abstract convex uniform space from part (iv) of Examples 1.10. Since $1_X \in KKM(X, X)$ and $F : X \to X$ is continuous, then $F \in KKM(X, X)$ and (2.3) holds. Also (2.5) follows, since compact metric spaces are separable. Apply Theorem 2.4. $\square$

Now we extend the concept of measure of noncompactness and condensing multimaps on locally G-convex in Huang et al. [5] to abstract convex uniform spaces.

Definition 2.6. For a subset A of abstract convex uniform spaces $(E, \Gamma; \mathcal{U})$, define

$$\Psi(A) = \{V \in \mathcal{B} : A \subseteq V[S], \text{ for some precompact subset } S \text{ of } E\}.$$

We call the set $\Psi(A)$ a measure of noncompactness of A.

Essentially the same reasoning as in [5, Proposition 3.2] guarantees the following result.

Theorem 2.7. *Let A and B be subsets of $(E, \Gamma; \mathcal{U})$. Then,*

(i) *A is precompact if and only if $\Psi(A) = \mathcal{B}$;*
(ii) *$\Psi(A) \supseteq \Psi(B)$ if $A \subseteq B$;*
(iii) *$\Psi(co_\Gamma(A)) = \Psi(A)$;*
(iv) *$\Psi(A \cup B) = \Psi(A) \cap \Psi(B)$.*

Definition 2.8. Suppose that X be a nonempty subset of an abstract convex uniform space $(E, \Gamma; \mathcal{U})$ and Ψ is the measure of noncompactness in Definition 2.6. A multimap $F : X \multimap E$ is called *condensing* provided that if $A \subseteq X$ and $\Psi(F(A)) \subseteq \Psi(A)$, then A is precompact. F is called *generalized condensing* if, whenever $A \subseteq X$, $F(A) \subseteq A$ and $A \setminus \overline{co}_\Gamma(F(A))$ is precompact, Then A is precompact.
It is obvious that every compact map or every map defined on a compact set is condensing. Also, every condensing map is generalized condensing.

Remark 2.9. Every condensing (respectively generalized condensing) map $F : X \multimap X$ satisfies condition (2.1) (respectively (2.2)). Thus, by Theorems 2.2 and 2.3, we get the following.

Corollary 2.10. *Let X be a nonempty complete, Γ-convex subset of an abstract convex uniform Φ-space $(E, \Gamma; \mathcal{U})$. If $F \in KKM(X, X)$ is either condensing or generalized condensing and closed, then F has a fixed point.*

3. Essentialty for Mönch type mappings

Let $(E, \Gamma; \mathcal{U})$ be an abstract convex uniform space, U an open subset of E and $x_0 \in U$.

Definition 3.1. We let $M(\overline{U}, E)$ denotes the set of all continuous maps $F : \overline{U} \to E$ which satisfy Mönch's condition (i.e., if $C \subseteq co_\Gamma(\{x_0\} \cup F(C))$ and $C \subseteq \overline{U}$ is countable, then $\overline{C}$ is compact).

Definition 3.2. We let $F \in M_{\partial U}(\overline{U}, E)$ if $F \in M(\overline{U}, E)$ with $x \neq F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 3.3. A map $F \in M_{\partial U}(\overline{U}, E)$ is essential in $M_{\partial U}(\overline{U}, E)$ if for every $G \in M_{\partial U}(\overline{U}, E)$ with $G_{|\partial U} = F_{|\partial U}$ there exists $x \in U$ with $x = G(x)$

Definition 3.4. Let $(E, \Gamma; \mathcal{U})$ be an abstract convex uniform space. Throughout this section, we will assume that the convex structure Γ satisfies the following conditions:

$$\text{for each } a \in E, \text{ the multimap } x \multimap co_\Gamma\{a, x\} \text{ is closed.} \tag{3.1}$$

$$\begin{aligned} &\text{for each } a \in E, \text{ there exists a continuous map} \\ &\omega(t, x) : [0,1] \times E \to co_\Gamma\{a, x\} \text{ such that } \omega(0, x) = a \text{ and } \omega(1, x) = x. \end{aligned} \tag{3.2}$$

Let E be a topological vector space, $A \subseteq E$, $a \in E$, and $\Gamma(A) = \operatorname{conv}(A)$. Let $\omega(t, x) = (1-t)a + tx$, and it is easy to show that (E, Γ) satisfies (3.1) and (3.2).

Definition 3.5. A metric space (M, d) is said be a hyperconvex metric space if for any collection of points x_α of M and any collection r_α of non-negative real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have

$$\bigcap_\alpha B(x_\alpha, r_\alpha) \neq \emptyset.$$

For each $A \subseteq M$, Set

$$\Gamma(A) = co(A) = \bigcap\{B \subseteq M : B \text{ is a closed ball in } M \text{ such that } A \subset B\}.$$

It is well known that for any hyperconvex metric space M there exist an index set I and a natural isometric embedding from M into $l_\infty(I)$, and a nonexpansive retraction $r : l_\infty(I) \to M$. For each $a, b \in M$ we have

$$r(\operatorname{conv}(a, b))$$

$$\subseteq r\Big(\bigcap\{B \subseteq l_\infty : B \text{ is a closed ball in } l_\infty \text{ such that } \operatorname{conv}(a, b) \subset B\}\Big)$$

$$= \bigcap\{B \subseteq M : B \text{ is a closed ball in } M \text{ such that } \{a, b\} \subset B\} = co(a, b).$$

Thus,

$$r(\operatorname{conv}(a, b)) \subseteq co(a, b).$$

Let $\omega(t, x) = r((1-t)a + tx)$. Then ω is continuous and $\omega(0, x) = a$ and $\omega(1, x) = x$. Also, it is easy to see that the map $x \to co(a, x)$ is closed. Thus, (M, Γ) satisfies (3.1) and (3.2).

The proof of the following theorem follows the lines of Theorem 2.1 in [1].

Theorem 3.6. *Let $(E, \Gamma; \mathcal{U})$ be an abstract convex uniform space which satisfies (3.1) and (3.2), U an open subset of E and $x_0 \in U$. Suppose $F \in M(\overline{U}, E)$ with*

$$\text{the constant map } x_0 \text{ is essential in } M_{\partial U}(\overline{U}, E) \tag{3.3}$$

and

$$x \notin co_\Gamma(\{x_0\} \cup F(x)) \text{ for any } x \in \partial U \tag{3.4}$$

holding. Also assume that

$$\begin{array}{l} E \text{ is such that any closed subset is compact} \\ \text{iff it is sequentially compact.} \end{array} \tag{3.5}$$

Then F is essential in $M_{\partial U}(\overline{U}, E)$.

Proof. Let $H \in M_{\partial U}(\overline{U}, E)$ with $H_{|\partial U} = F_{|\partial U}$. We must show H has a fixed point in U. Consider

$$B = \{x \in \overline{U} \ : \ x \in co_\Gamma(\{x_0\} \cup H(x))\}.$$

Now $B \neq \emptyset$ since $x_0 \in U$. Let $x_\alpha \in B$ be a convergent net with $x_\alpha \to x \in \overline{U}$. Since H is continuous, so the multimap $x \multimap co_\Gamma\{a, H(x)\}$ is closed from (3.1). This together with $x_\alpha \in co_\Gamma(\{x_0\} \cup H(x_\alpha))$ implies that $x \in co_\Gamma(\{x_0\} \cup H(x))$, which shows that B is closed. In addition, $B \cap \partial U = \emptyset$ since (3.4) holds and $H_{|\partial U} = F_{|\partial U}$. We now claim that there exists a continuous $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(B) = 1$. Since uniform topological spaces are completely regular, the claim will be true if we show B is compact. To see this let $C = \{x_n\}_{n=1}^{\infty}$ be any sequence in B. We have $C \subseteq co_\Gamma(\{x_0\} \cup H(C))$. Since $H \in M_{\partial U}(\overline{U}, E)$, we have that $\overline{C}$ is compact and so is sequentially compact by (3.5). Without loss of generality, we may assume $x_n \to x \in \overline{C}$. Now since B is closed, we get $x \in B = \overline{B}$. Consequently $B = \overline{B}$ is sequentially compact, so is compact from (3.5). By (3.2), there exists a continuous map $\omega(t, x) : [0, 1] \times E \to co_\Gamma\{x_0, x\}$ such that $\omega(0, x) = x_0$ and $\omega(1, x) = x$. Define a map $R_\mu : \overline{U} \to E$ by

$$R_\mu(x) = \omega(\mu(x), H(x)).$$

We first show R_μ satisfies the Mönch condition. Let $C \subseteq \overline{U}$ be countable and

$$C \subseteq co_\Gamma(\{x_0\} \cup R_\mu(C)).$$

Now since $R_\mu(x) = \omega(\mu(x), H(x)) \subseteq co_\Gamma(\{x_0\} \cup H(x))\}$ we have $R_\mu(C) \subseteq co_\Gamma(\{x_0\} \cup H(C))$. Thus

$$C \subseteq co_\Gamma(\{x_0\} \cup R_\mu(C)) \subseteq co_\Gamma(\{x_0\} \cup H(C)).$$

Since $H \in M_{\partial U}(\overline{U}, E)$ we have that $\overline{C}$ is compact. Thus $R_\mu \in M_{\partial U}(\overline{U}, E)$ with $(R_\mu)_{|\partial U} = x_0$. Now since the constant map x_0 is essential in $M_{\partial U}(\overline{U}, E)$

there exists $x \in U$ with $x = R_\mu(x)$. Consequently, $x \in B$ and so $\mu(x) = 1$ and $R_\mu(x) = \omega(1, H(x)) = H(x)$. Thus $x = H(x)$. □

We give now an example of a constant essential map in $M_{\partial U}(\overline{U}, E)$.

Theorem 3.7. *Let $(E, \Gamma; \mathcal{U})$ be a complete metrizable, abstract convex uniform space such that every open ball is Γ-convex, $1_E \in KKM(E, E)$, and as in section 2 we assume (1.1), (1.2) hold. Let U be an open subset of E with $x_0 \in U$. Then the constant map x_0 is essential in $M_{\partial U}(\overline{U}, E)$.*

Proof. We follow the proof of Theorem 2.5 in [1]. Let $\theta \in M_{\partial U}(\overline{U}, E)$ with $\theta_{|\partial U} = x_0$. We must show that there exists $x \in U$ with $\theta(x) = x$. Let $Q = \overline{co}_\Gamma(\theta(\overline{U}))$ and let $F : Q \to Q$ be given by

$$F(x) = \begin{cases} \theta(x) & x \in \overline{U}, \\ x_0 & \text{otherwise.} \end{cases}$$

Now $x_0 \in Q$, $F : Q \to Q$ is continuous and satisfies the Mönch condition. To see this let $C \subseteq Q$ be countable with $C \subseteq co_\Gamma(\{x_0\} \cup F(C))$. Then

$$C \subseteq co_\Gamma(\{x_0\} \cup \theta(\overline{U} \cap C)).$$

Notice $C \cap \overline{U} \subseteq Q$ is countable and $C \cap \overline{U} \subseteq co_\Gamma(\{x_0\} \cup \theta(\overline{U} \cap C))$. Now since $\theta \in M_{\partial U}(\overline{U}, E)$ we have $\overline{C \cap \overline{U}}$ is compact. Then since θ is continuous, $\theta(\overline{C \cap \overline{U}})$ is compact, and now since $(E, \Gamma; \mathcal{U})$ satisfies (1.2) and is complete, $\overline{co}_\Gamma(\{x_0\} \cup \theta(\overline{U} \cap C))$ is compact. Thus since

$$C \subseteq \overline{co}_\Gamma(\{x_0\} \cup \theta(\overline{U} \cap C))$$

we have that $\overline{C}$ is compact. Corollary 2.5 guarantees that there exists $x \in Q$ with $F(x) = x$. Now if $x \notin U$ we have $x_0 = F(x) = x$, which is a contradiction since $x_0 \in U$. Thus $x \in U$ so $x = F(x) = \theta(x)$. □

Combining Theorem 3.6 and Theorem 3.7 gives the following nonlinear alternative of Leray-Schauder type for Mönch type maps.

Theorem 3.8. *Let $(E, \Gamma; \mathcal{U})$ be a complete metrizable abstract convex uniform space such that every open ball is Γ-convex, $1_E \in KKM(E, E)$, and as in section 2 we assume (1.1), (1.2) hold. Let U be an open subset of E with $x_0 \in U$. Suppose $F \in M_{\partial U}(\overline{U}, E)$ satisfies (3.4). Then F is essential in $M_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

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