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ON SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH STRUVE FUNCTIONS

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Abstract. The main object of this paper is to provide necessary and sufficient conditions for the generalized Struve functions of first kind to be in the classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$. Furthermore, we give conditions for the integral operator $\mathcal{L}(m,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathcal{C}^*(k,\lambda)$. Several corollaries and consequences of the main results are also considered.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathcal{U}=\{z:|z|<1\}$ with conditions $f(0)=0$ and $f'(0)=1$ having the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (z\in\mathcal{U}). \tag{1.1}$$

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We denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0). \tag{1.2}$$

Let $\mathcal{S}(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{T}$ consisting of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, respectively [39].

A function f of the form (1.1) is in $\mathcal{S}(k, \lambda)$ if it satisfies the condition

$$\left| \frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - 1}{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} + 1} \right| < k \quad (0 < k \leq 1,\ 0 \leq \lambda < 1, z \in \mathcal{U})$$

and $f \in \mathcal{C}(k, \lambda)$ if and only if $zf' \in \mathcal{S}(k, \lambda)$. We note that $\mathcal{S}(k, 0) = \mathcal{S}(k)$ and $\mathcal{C}(k, 0) = \mathcal{C}(k)$, where the classes $\mathcal{S}(k)$ and $\mathcal{C}(k)$ were introduced and studied by Padmanabhan [35] (see also, [30] and [34]). Also, let $\mathcal{S}^*(k, \lambda) = \mathcal{S}(k, \lambda) \cap \mathcal{T}$ and $\mathcal{C}^*(k, \lambda) = \mathcal{C}(k, \lambda) \cap \mathcal{T}$. The classes $\mathcal{S}^*(k, \lambda)$ and $\mathcal{C}^*(k, \lambda)$ were introduced by Frasin et al. [19].

A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{R}^\tau(A, B)$, $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathcal{U}).$$

The class $\mathcal{R}^\tau(A, B)$ was introduced by Dixit and Pal [11].

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [6, 9, 20, 23, 24, 25, 26, 27, 31, 37, 38] and the Bessel functions [2, 3, 4, 5, 18, 21, 28, 29, 36].

We recall here the Struve function of order p (see [41]), denoted by $\mathcal{H}_p$, is given by

$$\mathcal{H}_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2})\ \Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \ \ \forall\, z \in \mathbb{C}, \tag{1.3}$$

which is the particular solution of the second order non-homogeneous differential equation

$$z^2\omega''(z) + z\omega'(z) + (z^2 - p^2)\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + \frac{1}{2})}, \tag{1.4}$$

where p is unrestricted real (or complex) number. The solution of the non-homogeneous differential equation

$$z^2\omega''(z)+z\omega'(z)-(z^2+p^2)\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{1.5}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathcal{L}_p(z)=-e^{-ip\pi/2}\mathcal{H}_p(iz)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{3}{2})}\left(\frac{z}{2}\right)^{2n+p+1},\ \ \forall\, z\in\mathbb{C}. \tag{1.6}$$

Consider the second order non-homogeneous linear differential equation [41],

$$z^2\omega''(z)+bz\omega'(z)[cz^2-p^2+(1-b)p]\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{b}{2})}, \tag{1.7}$$

where $b,p,c\in\mathbb{C}$ which is natural generalization of Struve equation. Note that when $b=c=1$, then we get the Struve function (1.3) and for $c=-1,b=1$ we get the modified Struve function (1.6). This permit us to study Struve and modified Struve functions.

Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{b+2}{2})}\left(\frac{z}{2}\right)^{2n+p+1},\ \ \forall\, z\in\mathbb{C}, \tag{1.8}$$

which is the particular solution of the differential equation (1.7). Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathcal{U}$. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z)=2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)\ z^{\frac{-p-1}{2}}\ \omega_{p,b,c}\ (\sqrt{z}).$$

By using well-known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1 & (n=0),\\ a(a+1)(a+2)\cdots(a+n-1) & (n\in\mathbb{N}=\{1,2,3,\cdots\}),\end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned}u_{p,b,c}(z)&=\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(3/2)_n\ (m)_n}\ z^n \qquad (1.9)\\ &=b_0+b_1z+b_2z^2+\cdots+b_nz^n+\cdots\ ,\end{aligned}$$

where $m = \left(p + \frac{b+2}{2}\right) \neq 0, -1, -2, \cdots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2u''(z) + 2(2p+b+3)zu'(z) + (cz+2p+b)u(z) = 2p+b.$$

For convenience, throughout in the sequel, we use the following notations

$$u_{p,b,c}(z) = u_p(z), \qquad m = p + \frac{b+2}{2},$$

and for $c < 0, m > 0$, let

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n = z + \sum_{n=2}^{\infty} b_{n-1} z^n \tag{1.10}$$

and

$$\Psi(z) = z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n. \tag{1.11}$$

Now, we consider the linear operator

$$\mathcal{I}(c,m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(c,m)f(z) = zu_p(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} a_n\ z^n.$$

Recently, Orhan and Yagmur [41] have determined various sufficient conditions for the parameters p, b and c such that the functions $u_p(z)$ or $z \to zu_p(z)$ to be univalent, starlike, convex and close to convex in the open unit disk.

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [6, 8, 9, 23, 24, 25, 26, 27, 31, 37, 38]), Struve functions (see [1, 10, 22]), Poisson distribution series (see [12, 14, 16, 17, 32, 33]) and Pascal distribution series (see [7, 13, 15]), we obtain sufficient conditions for the function $h_\mu(z)$, given by

$$h_\mu(z) = (1-\mu)zu_p(z) + \mu z{u_p}'(z) = z + \sum_{n=2}^{\infty} (1-\mu+n\mu) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n, \tag{1.12}$$

where $0 \leq \mu \leq 1$, to be in the classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$, and also we prove that those sufficient conditions are necessary for functions of the form (1.11) to be in the classes $\mathcal{S}^*(k,\lambda)$ and $\mathcal{C}^*(k,\lambda)$. Furthermore, we give necessary and sufficient conditions for $\mathcal{I}(c,m)f$ to be in $\mathcal{C}(k,\lambda)$ provided that the function f is in the class $\mathcal{R}^\tau(A,B)$. Finally, we give conditions for the integral operator $\mathcal{L}(m,c,z) = \int_0^z (2 - u_p(t))dt$ to be in the class $\mathcal{C}^*(k,\lambda)$.

To establish our main results, we need the following lemmas.

Lemma 1.1. ([19]) *A sufficient condition for a function f of the form* (1.1) *to be in the class $\mathcal{S}(k,\lambda)$ is*

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n| \leq 2k \quad (0<k\leq 1; 0\leq\lambda<1) \tag{1.13}$$

and a necessary and sufficient condition for a function f of the form (1.2) *to be in the class $\mathcal{S}^*(k,\lambda)$ is that the condition* (1.13) *is satisfied.*

Lemma 1.2. ([19]) *A sufficient condition for a function f of the form* (1.1) *to be in the class $\mathcal{C}(k,\lambda)$ is*

$$\sum_{n=2}^{\infty} n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n| \leq 2k \quad (0<k\leq 1; 0\leq\lambda<1) \tag{1.14}$$

and a necessary and sufficient condition for a function f of the form (1.2) *to be in the class $\mathcal{C}^*(k,\lambda)$ is that the condition* (1.14) *is satisfied.*

Lemma 1.3. ([11]) *If $f \in \mathcal{R}^{\tau}(A,B)$ is of the form* (1.2)*, then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \qquad n\in\mathbb{N}\backslash\{1\}.$$

The result is sharp for the function

$$f(z)=\int_0^z\left(1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}}\right)dt \qquad (z\in\mathcal{U}; n\in\mathbb{N}\backslash\{1\}).$$

2. Main results

Theorem 2.1. *Let $c<0$ and $m>0$. Then $h_\mu(z)\in\mathcal{S}(k,\lambda)$ if*

$$\begin{aligned}&\mu\left[(1-\lambda)+k(1+\lambda)\right]u_p''(1)+[(2\mu+1)((1-\lambda)+k(1+\lambda))\\ &-\mu(1-\lambda)(1-k)]u_p'(1)+2ku_p(1)\leq 4k.\end{aligned} \tag{2.1}$$

Proof. From (1.10)

$$zu_p(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}z^n,$$

we have

$$u_p(1)-1=\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{2.2}$$

Differentiating $zu_p(z)$ with respect to z and taking $z = 1$, we get

$$u_p'(1) + u_p(1) - 1 = \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{2.3}$$

Also, differentiating $zu_p'(z) + u_p(z)$ with respect to z and taking $z = 1$, we have

$$u_p''(1) + 2u_p'(1) = \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{2.4}$$

Since $h_\mu(z) \in \mathcal{S}(k, \lambda)$, by virtue of Lemma 1.1 and (1.12), it suffices to show that

$$\begin{aligned} F(k,\lambda) = &\sum_{n=2}^{\infty}(1-\mu+n\mu)[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)] \\ &\times \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\ \le\ & 2k. \end{aligned} \tag{2.5}$$

Now,

$$\begin{aligned} F(k,\lambda) = \mu((1-\lambda)+k(1+\lambda)) &\sum_{n=2}^{\infty} n^2 \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\ + [(1-\mu)\,[(1-\lambda)+k(1+\lambda)] &- \mu(1-\lambda)(1-k)] \\ \times \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}& \\ - (1-\mu)(1-\lambda)(1-k) &\sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}. \end{aligned}$$

Writing $n^2 = n(n-1) + n$, we get

$$\begin{aligned} F(k,\lambda) = \mu((1-\lambda)+k(1+\lambda)) &\sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\ + [(1-\lambda)+k(1+\lambda)-\mu(1-\lambda)(1-k)] &\sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\ - (1-\mu)(1-\lambda)(1-k) &\sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}. \end{aligned}$$

From (2.2), (2.3) and (2.4), we immediately have

$$\begin{aligned} F(k,\lambda) &= \mu((1-\lambda)+k(1+\lambda))[u_p''(1)+2u_p'(1)] \\ &\quad + [(1-\lambda)+k(1+\lambda)-\mu(1-\lambda)(1-k)]\,[u_p'(1)+u_p(1)-1] \\ &\quad - (1-\mu)(1-\lambda)(1-k)[u_p(1)-1] \\ &= \mu((1-\lambda)+k(1+\lambda))u_p''(1) \\ &\quad + [(2\mu+1)((1-\lambda)+k(1+\lambda))-\mu(1-\lambda)(1-k)]u_p'(1) \\ &\quad + 2k[u_p(1)-1]. \end{aligned}$$

Therefore, we see that the last expression is bounded above by $2k$ if and only if (2.1) holds. □

Theorem 2.2. *Let $c<0$ and $m>0$. Then $zu_p(z)\in\mathcal{S}(k,\lambda)$ if*

$$[(1-\lambda)+k(1+\lambda)]u_p'(1)+2ku_p(1)\leq 4k. \tag{2.6}$$

Proof. By virtue of (1.13), it suffices to show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right)\leq 2k. \tag{2.7}$$

Since $h_0(z)=zu_p(z)$, by taking $\mu=0$ in (2.5) we get the inequality(2.7). Hence, by taking $\mu=0$ in the Theorem 2.1, we get the desired result given in (2.6). □

Theorem 2.3. *Let $c<0$ and $m>0$. Then $zu_p(z)\in\mathcal{C}(k,\lambda)$ if*

$$\begin{aligned} &[(1-\lambda)+k(1+\lambda)]\,u_p''(1)+[3((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]u_p'(1) \\ &\quad + 2ku_p(1)\leq 4k. \end{aligned} \tag{2.8}$$

Proof. In view of (1.14), it suffices to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right)\leq 2k.$$

By definition, $u_p(z)\in\mathcal{C}(k,\lambda)$ if and only if $zu_p'(z)\in\mathcal{S}(k,\lambda)$. That is, by taking $\mu=1$ we have $h_1(z)=zu_p'(z)\in\mathcal{S}(k,\lambda)$. Hence, by taking $\mu=1$ in the Theorem 2.1, we get the desired result given in (2.8). □

Remark 2.4. The above conditions (2.1) and (2.8) are also necessary for functions $\Psi(z)$ given by (1.11) and has the form

$$\begin{aligned} h_\mu^*(z) &= (1-\mu)\Psi(z)+\mu z\Psi'(z) \\ &= z-\sum_{n=2}^{\infty}(1-\mu+n\mu)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n, \end{aligned}$$

to be in the classes $\mathcal{S}^*(k,\lambda)$ and $\mathcal{C}^*(k,\lambda)$, respectively.

Putting $\lambda = 0$ in Theorems 2.1-2.3, we obtain the following corollaries.

Corollary 2.5. *Let $c < 0$ and $m > 0$. Then $h_\mu(z) \in \mathcal{S}(k)$ if*

$$\mu(k+1)u_p''(1) + [\mu(3k+1) + k + 1]u_p'(1) + 2ku_p(1) \leq 4k. \tag{2.9}$$

Corollary 2.6. *Let $c < 0$ and $m > 0$. Then $zu_p(z) \in \mathcal{S}(k)$ if*

$$(1+k)u_p'(1) + 2ku_p(1) \leq 4k. \tag{2.10}$$

Corollary 2.7. *Let $c < 0$ and $m > 0$. Then $zu_p(z) \in \mathcal{C}(k)$ if*

$$(1+k)u_p''(1) + 2(2k+1)u_p'(1) + 2ku_p(1) \leq 4k. \tag{2.11}$$

3. Inclusion properties

Making use of Lemma 1.2, we will study the action of the Struve function on the class $\mathcal{C}(k,\lambda)$.

Theorem 3.1. *Let $c < 0, m > 0$. If for $f \in \mathcal{R}^\tau(A,B)$, the inequality*

$$(A-B)\,|\tau|\,[((1-\lambda) + k(1+\lambda))u_p'(1) + 2k(u_p(1) - 1)] \leq 2k \tag{3.1}$$

is satisfied, then $\mathcal{I}(c,m)f \in \mathcal{C}(k,\lambda)$.

Proof. Let f be of the form (1.2) belong to the class $\mathcal{R}^\tau(A,B)$. By virtue of (1.14), it suffices to show that

$$\sum_{n=2}^{\infty} n[n((1-\lambda) + k(1+\lambda)) - (1-\lambda)(1-k)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \leq 2k.$$

Since $f \in \mathcal{R}^\tau(A,B)$, by Lemma 1.3, we have

$$|a_n| \leq (A-B)\frac{|\tau|}{n}.$$

Hence,

$$\begin{aligned}\Phi(k,\lambda) &= \sum_{n=2}^{\infty} n[n((1-\lambda) + k(1+\lambda)) - (1-\lambda)(1-k)]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \\ &\leq (A-B)\,|\tau| \sum_{n=2}^{\infty}[n((1-\lambda) + k(1+\lambda)) - (1-\lambda)(1-k)] \\ &\quad \times \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}.\end{aligned}$$

Further, proceeding as in Theorem 2.1 we get

$$\Phi(k,\lambda) \leq (A-B)\,|\tau|\,[((1-\lambda)+k(1+\lambda))u_p'(1)+2k(u_p(1)-1)].$$

But this last expression is bounded above by $2k$ if and only if (3.1) holds. □

Theorem 3.2. *Let $c<0$, $m>0$. Then*

$$\mathcal{L}(m,c,z)=\int_0^z (2-u_p(t))dt \tag{3.2}$$

is in the class $\mathcal{C}^*(k,\lambda)$ if and only if the condition (2.6) *is satisfied.*

Proof. Since

$$\mathcal{L}(m,c,z)=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\frac{z^n}{n},$$

from (1.14), we need only to show that

$$\Lambda(k,\lambda)=\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right)\leq 2k.$$

Proceeding as in Theorem 2.1, we get

$$\Lambda(k,\lambda)=((1-\lambda)+k(1+\lambda))u_p'(1)+2k(u_p(1)-1),$$

which is bounded above by $2k$ if and only if (2.6) holds. □

Putting $\lambda=0$ in Theorems 3.1 and 3.2, we obtain the following corollaries.

Corollary 3.3. *Let $c<0, m>0$. If for $f\in\mathcal{R}^\tau(A,B)$, the inequality*

$$(A-B)\,|\tau|\,[(1+k)u_p'(1)+2k(u_p(1)-1)]\leq 2k$$

is satisfied, then $\mathcal{I}(c,m)f\in\mathcal{C}(k,\lambda)$.

Corollary 3.4. *Let $c<0, m>0$. Then the integral operator $\mathcal{L}(m,c,z)$ defined by* (3.2) *is in $\mathcal{C}^*(k,\lambda)$ if and only if the condition* (2.10) *is satisfied.*

4. Concluding remarks

Our present investigation was motivated essentially by several works dealing with the interesting problem of finding necessary and sufficient conditions for the generalized Struve functions of first kind to be in certain subclasses of analytic functions defined in the open unit disk $\mathcal{U}$ (see, for example, [1], [10], [22] and [40]). In our study, we derive the necessary and sufficient conditions for the generalized Struve functions to be in the classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$. Furthermore, we give conditions for the integral operator $\mathcal{L}(m,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathcal{C}^*(k,\lambda)$. Several corollaries and consequences of the main results are also considered.

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