Nonlinear Functional Analysis and Applications
Vol. 27, No. 1 (2022), pp. 141-148
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.01.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME INEQUALITIES ON POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then Govil proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|,$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maximal at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

In this paper, we extend the above inequality to polar derivative of a polynomial. Further, we also prove an improved version of above inequality into polar derivative.

1. Introduction

If $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

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[0]Received May 20, 2021. Revised December 5, 2021. Accepted December 8, 2021.
[0]2020 Mathematics Subject Classification: 30C10, 30C15.
[0]Keywords: Inequalities, polynomial, zeros, maximum modulus, polar derivative.
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The above inequality is the well-known Bernstein inequality [5]. Inequality (1.1) is best possible and equality holds for the polynomial $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequality (1.1) can be sharpened. In fact, Erdös conjectured and later Lax [12] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

Inequality (1.2) is sharp for polynomials having their zeros on $|z| = 1$.

The polar derivative of a polynomial $p(z)$ of degree n with respect to a real or complex number α, denoted by $D_\alpha p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Aziz and Shah [4] extended (1.1) to polar derivative by proving that if $p(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq n|\alpha| \max_{|z|=1} |p(z)|. \tag{1.3}$$

Further, Aziz [1] extended inequality (1.2) to polar derivative and proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every real or complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |p(z)|. \tag{1.4}$$

It was asked by Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large can

$$\left\{ \max_{|z|=1} |p'(z)| \Big/ \max_{|z|=1} |p(z)| \right\} \text{ be ?}$$

A partial answer to this problem was given by Malik [13], who proved for the case $k \geq 1$ that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.5}$$

Equality in (1.5) holds for $p(z) = (z + k)^n$.

For the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise estimate for maximum of $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable.

For quite some time, it was believed that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then the inequality analogous to (1.5) should be

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \tag{1.6}$$

until E.B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

There are many extensions of inequality (1.5)(see Dewan and Bidkham [7], Dewan and Mir [8] and Chan and Malik [6]).

However, for the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, Govil [9] proved inequality (1.6) with extra condition.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \tag{1.7}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}. \tag{1.8}$$

2. Lemmas

For the proofs of the theorems, we will use the following lemmas. The first lemma is a special case of a result due to Govil and Rahman [11].

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$,*

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{2.1}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

The next lemma was proved by Aziz [1, Lemma 2] in more general form. However, we present a simple proof of this lemma which we think is new, simply by using the definition of polar derivative of a polynomial and Lemma 2.1 due to Govil and Rahman [11].

Lemma 2.2. *If $p(z)$ is a polynomial of degree n and α is any real or complex number, then on $|z| = 1$,*

$$|D_\alpha p(z)| + |D_\alpha q(z)| \le n(|\alpha| + 1) \max_{|z|=1} |p(z)|, \tag{2.2}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Proof. Let $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$. Then it is easy to verify that on $|z| = 1$,

$$|q'(z)| = |np(z) - zp'(z)|. \tag{2.3}$$

Now, for every real or complex number α, the polar derivative of $p(z)$ with respect to α is

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{2.4}$$

This implies on $|z| = 1$,

$$|D_\alpha p(z)| \le |np(z) - zp'(z)| + |\alpha||p'(z)|. \tag{2.5}$$

Using (2.3) in (2.5), on $|z| = 1$, we have

$$|D_\alpha p(z)| \le |q'(z)| + |\alpha||p'(z)|. \tag{2.6}$$

Similarly on $|z| = 1$,

$$|D_\alpha q(z)| \le |p'(z)| + |\alpha||q'(z)|. \tag{2.7}$$

Adding (2.6) and (2.7), we have

$$|D_\alpha p(z)| + |D_\alpha q(z)| \le (|\alpha| + 1) \left\{|p'(z)| + |q'(z)|\right\}. \tag{2.8}$$

Using Lemma 2.1 in (2.8), we get

$$|D_\alpha p(z)| + |D_\alpha q(z)| \le n(|\alpha| + 1) \max_{|z|=1} |p(z)|, \tag{2.9}$$

which completes the proof of Lemma 2.2. □

The next lemma is due to Aziz and Rather [3].

Lemma 2.3. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge k$,*

$$\max_{|z|=1} |D_\alpha p(z)| \ge n\left(\frac{|\alpha| - k}{1 + k^n}\right) \max_{|z|=1} |p(z)|. \tag{2.10}$$

The following lemma was obtained by Govil and McTume [10].

Lemma 2.4. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every real or complex number α with $|\alpha| \ge 1 + k + k^n$,*

$$\max_{|z|=1} |D_\alpha p(z)| \ge \frac{n(|\alpha| - k)}{1 + k^n} \max_{|z|=1} |p(z)| + n\left\{\frac{|\alpha| - (1 + k + k^n)}{1 + k^n}\right\} \min_{|z|=k} |p(z)|. \tag{2.11}$$

3. Main results

In this paper, we first prove the following result which extends Theorem 1.1 to polar derivative of $p(z)$.

Theorem 3.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge \frac{1}{k}$,*

$$\max_{|z|=1} |D_\alpha p(z)| \le \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)|, \tag{3.1}$$

provided $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maximal at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Proof. The proof of this theorem follows on the same lines as that of next theorem but instead of applying Lemma 2.4, we apply Lemma 2.3 and we omit it. □

Remark 3.2. From the hypotheses of Theorem 3.1, $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maximal at the same point on $|z| = 1$. Further, if they are divided by $|\alpha|$ and considering limit as $\alpha \to \infty$, then they become $|p'(z)|$ and $|q'(z)|$ respectively which attain their maximal at the same point on $|z| = 1$. Hence, dividing both sides of inequality (3.1) as well as the quantities $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ by $|\alpha|$ and taking respective limit as $|\alpha| \to \infty$, we readily get inequality (1.7) of Theorem 1.1 along with the agreement that $|p'(z)|$ and $|q'(z)|$ attain their maximal at the same point on the circle $|z| = 1$.

Next, under the same hypotheses, we further prove the following improved result which sharpens Theorem 3.1.

Theorem 3.3. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \le 1$, then for every real or complex number α with $|\alpha| \ge 1 + \frac{1}{k} + \frac{1}{k^n}$,*

$$\max_{|z|=1} |D_\alpha p(z)| \le \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)| - n\left\{\frac{k^n|\alpha| - (k^n + k^{n-1} + 1)}{k^n(1 + k^n)}\right\} \min_{|z|=k} |p(z)|, \tag{3.2}$$

provided $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maximal at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Proof. Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \le 1$. In other words, $p(z)$ has all its zeros in $|z| \ge k$, $k \le 1$ and hence all the zeros of $q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$ lie in $|z| \le 1/k$, $1/k \ge 1$. Applying Lemma 2.4 on $q(z)$, for $|\alpha| \ge 1 + \frac{1}{k} + \frac{1}{k^n}$, we have

$$\max_{|z|=1} |D_\alpha q(z)| \ge \frac{n(|\alpha| - \frac{1}{k})}{1 + \frac{1}{k^n}} \max_{|z|=1} |q(z)| + n\left\{\frac{|\alpha| - (1 + \frac{1}{k} + \frac{1}{k^n})}{1 + \frac{1}{k^n}}\right\} m',$$

where $m' = \min_{|z|=\frac{1}{k}} |q(z)|$, which is equivalent to

$$\max_{|z|=1} |D_\alpha q(z)| \ge \frac{nk^{n-1}(k|\alpha| - 1)}{1 + k^n} \max_{|z|=1} |q(z)| + n\left\{\frac{k^n|\alpha| - (k^n + k^{n-1} + 1)}{1 + k^n}\right\} m'. \tag{3.3}$$

Now

$$m' = \min_{|z|=\frac{1}{k}} |q(z)| = \frac{1}{k^n} \min_{|z|=k} |p(z)|, = \frac{m}{k^n}, \tag{3.4}$$

where $m = \min_{|z|=k} |p(z)|$. Using (3.4) in (3.3), we have

$$\max_{|z|=1} |D_\alpha q(z)| \ge \frac{nk^{n-1}(k|\alpha| - 1)}{1 + k^n} \max_{|z|=1} |q(z)| + n\left\{\frac{k^n|\alpha| - (k^n + k^{n-1} + 1)}{k^n(1 + k^n)}\right\} m. \tag{3.5}$$

Also, since on $|z| = 1$, $|p(z)| = |q(z)|$, inequality (3.5) can be written as

$$\max_{|z|=1} |D_\alpha q(z)| \geq \frac{nk^{n-1}(k|\alpha| - 1)}{1 + k^n} \max_{|z|=1} |p(z)| + n \left\{ \frac{k^n|\alpha| - (k^n + k^{n-1} + 1)}{k^n(1 + k^n)} \right\} m. \tag{3.6}$$

By Lemma 2.2, on $|z| = 1$,

$$|D_\alpha p(z)| + |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)|. \tag{3.7}$$

Let z_0 be a point on $|z| = 1$ such that $\max_{|z|=1} |D_\alpha q(z)| = |D_\alpha q(z_0)|$. Since $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maximal at the same point on $|z| = 1$ with $|\alpha| \geq 1 + \frac{1}{k} + \frac{1}{k^n}$, we have

$$\max_{|z|=1} |D_\alpha p(z)| = |D_\alpha p(z_0)|.$$

Thus, in particular (3.7) gives

$$\max_{|z|=1} |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)|. \tag{3.8}$$

Combining (3.6) and (3.8), we have

$$n(|\alpha| + 1) \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)| \geq \frac{nk^{n-1}(k|\alpha| - 1)}{1 + k^n} \max_{|z|=1} |p(z)| + n \left\{ \frac{k^n|\alpha| - (k^n + k^{n-1} + 1)}{k^n(1 + k^n)} \right\} m, \tag{3.9}$$

which on simplification gives

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n(k^n + |\alpha| + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)| - n \left\{ \frac{k^n|\alpha| - (k^n + k^{n-1} + 1)}{k^n(1 + k^n)} \right\} m. \tag{3.10}$$

This completes the proof. □

Remark 3.4. If we adopt the similar arguments of Remark 3.2 in Theorem 3.3, we have the following result proved by Aziz and Ahmad [2, Theorem 3].

Corollary 3.5. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1 + k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\}, \tag{3.11}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain their maximal at the same point on the circle* $|z| = 1$, *where*

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

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