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ITERATIVE METHOD FOR SOLVING FINITE FAMILIES OF VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS OF CERTAIN MULTI-VALUED MAPPINGS

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Abstract. In this paper, we propose a viscosity iterative algorithm for approximating a common solution of finite family of variational inequality problem and fixed point problem for finite family of multi-valued type-one demicontractive mappings in real Hilbert spaces. A strong convergence result of the aforementioned problems were proved and some consequences of our result was also displayed. In addition, we discuss an application of our main result to convex minimization problem. The result presented in this article complements and extends many recent results in literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $CB(C)$ and $K(C)$ denote the family of nonempty, closed and bounded subset and nonempty compact subset of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$\mathcal{H}(A,B) = \max\Big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\Big\} \text{ for } A,B \in CB(C),$$

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where $d(x, C) = \inf\{||x - y|| : y \in C\}$.

Let $T : C \to CB(C)$ be a multi-valued mapping. Then $P_T x = \{u \in Tx : ||x - u|| = d(x, Tx)\}$. A point $x \in C$ is called a fixed point of T if $x \in Tx$. However, if $Tx = \{x\}$, then x is called a strict point of T. We denote the set of fixed points of T by $F(T)$. A multi-valued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\mathcal{H}(Tx, Ty) \leq L||x - y||, \ x, y \in C. \tag{1.1}$$

In (1.1), if $L \in (0, 1)$, then T is called a contraction while T is called nonexpansive if $L = 1$.

A mapping $T : C \to CB(C)$ is said to be

(i) *of type-one*, if

$$||u - v|| \leq \mathcal{H}(Tx, Ty), \ \forall x, y \in C, \ u \in P_T x, \ v \in P_T y,$$

(ii) *λ-hybrid* (see [45]), if there exists $\lambda \in \mathbb{R}$ such that

$$(1 + \lambda)\mathcal{H}(Tx, Ty)^2 \leq (1 - \lambda)||x - y||^2 + \lambda d(y, Tx)^2 + \lambda d(x, Ty)^2, \ \forall x, y \in C,$$

(iii) *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, Ty) \leq ||x - y||, \ \forall x \in C, y \in F(T),$$

(iv) *demicontractive-type* in the sense of [24] if $F(T) \neq \emptyset$ and

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + kd^2(x, Tx), \ x \in C, \ y \in F(T) \text{ and } k \in (0, 1).$$

Remark 1.1. Clearly, every multi-valued quasi-nonexpansive mapping is a multi-valued demicontractive-type mapping. However, the following example shows that the converse of this statement is not always true.

Let C be a nonempty, closed and convex subset of a real Hilbert space H, the variational inequality problem (VIP) is to find $x \in C$ such that

$$\langle A(x), y - x\rangle \geq 0, \ \forall y \in C, \tag{1.2}$$

where $A : C \to H$ is a nonlinear mapping. We denote by VI(C,A) the solution set of (1.2).

Variational inequality theory introduced by Stampacchia and Fichera [19, 43] independently, in early sixties in mechanics and potential theory respectively provides the natural, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems arising in elasticity, economics, transportation, optimization, control theory and engineering sciences (see [1, 5, 2, 3, 4, 10, 11, 20, 22, 28, 33, 34, 36, 37, 39]).

The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. The first aspect reveals the fundamental facts on the qualitative behavior of solutions to important classes of problems. On the other hand, it allows us to develop highly efficient and powerful numerical methods to solve, for instance, obstacle, unilateral, free and moving boundary value problems.

In 1985, Pang [38] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a VIP.

In 1976, Korpelevich [29] proposed the following extragradient method for solving VIP (1.2), when A is monotone and Lipschitz continuous in the finite-dimensional Euclidean space $\mathbb{R}^N$.

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases}$$

for each $n \in \mathbb{N}$ under some suitable conditions, the sequence $\{x_n\}$ and $\{y_n\}$ converge to the some point $z \in VI(C, A)$.

In 2012, Censor et al. [14] introduced the general common solutions to variational inequality problem (CSVIP), which consists of finding common solutions to unrelated variational inequalities for finite number of sets. That is, find $x^* \in \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \cdots, N$,

$$\langle A_i(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C_i, \; i = 1, 2, \cdots, N, \tag{1.3}$$

where $A_i : H \to H$ is a nonlinear operator for each $i = 1, 2, \cdots, N$ and C_i is a nonempty, closed and convex subset of H. They proved a weak convergence theorem for approximating a solution of (1.3) using the following algorithm

$$\begin{cases} x_0 \in H, \\ x_{k+1} = \Pi_{i=1}^N (P_{C_i}(I - \lambda A_i))(x_k). \end{cases}$$

For more information on research output on variational inequality problem, (see [1, 25, 46, 47] and the references contained in).

In 2017, Ming Tian and Bing-Nan Jiang [47] proposed an iterative method for finding an element to solve a class of split variational inequality problems under weaker conditions and get a weak convergence theorem, Let H_1 be real Hilbert space. Let C be a nonempty closed convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0, f : C \to H_1$ be a monotone and k- Lipschitz continuous mapping and $T : H_2 \to H_2$ be a nonexpansive mapping. Setting $\Gamma := \{z \in VI(C, f) : Az \in Fix(T)\}$, assume that $\Gamma \neq \emptyset$, Let the sequences $\{x_n\}, \{y_n\}$ and t_n be generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = P_C(x_n - \gamma_n A^*(I-T)Ax_n), \\ t_n = P_C(y_n - \lambda_n f(y_n)), \\ x_(n+1) = P_C(y_n - \lambda_n f(t_n)). \end{cases}$$

Recently, Chinedu Izuchukwu [25] introduced a new iterative algorithm for approximating a common solution of certain class of multiplesets split variational inequality problems. Let H_1 and H_2 be real Hilbert spaces, and for each $i = 1, 2, \cdots, N$, let C_i be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $f_i : H_1 \to H_1$ be an α_i-inverse strongly monotone mapping and $S : H_2 \to H_2$ be $k-$strictly pseudo-contractive mapping. Assume that

$$\Gamma = \{z \in \bigcap_{i=1}^{N} VI(C_i, f_i) : Az \in F(S) \neq \emptyset\}$$

and the sequence $\{x_n\}$ is generated for arbitrary $x_1, u \in H_1$ by

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I-T_\beta)Au_n), \\ x_{n+1} = P_{C_N}(I-\lambda f_N) \circ \cdots \circ P_{C_1}(I-\lambda f_1)y_n, \quad n \geq 1. \end{cases}$$

Motivated by the aforementioned results, we introduced a viscosity iterative method for approximating a common solution of finite families of variational inequality problem and fixed point problem for finite family of multi-valued demicontractive-type mappings in real Hilbert spaces. We prove a strong convergence result to a common solution of the aforementioned problems and state some consequences of our main results. We also give an application of our main result. The result present in this paper extends and complements many related results in literature.

Our contributions are as follows:

(i) We were able to dispense for each $p \in \Gamma$, $S_i(p) = \{p\}$ for all $i \in \mathbb{N}$, see [3]. The type-one condition employed in this paper is weaker than the condition employed in [3].

(ii) By taking $g = u$ for some $u \in H$, the algorithm (3.1) becomes the Halpern-type algorithm.

(iii) We prove a strong convergence result without imposing a compactness condition, see [3]. The strong convergence result proved in this article is more desirable than the weak convergence result proved in [14, 47].

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$" respectively.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $M : C \to C$ is said to be

(i) monotone, if

$$\langle Mx - My, x - y\rangle \geq 0,\ \forall\ x, y \in C,$$

(ii) $\alpha-$inverse strongly monotone (ism), if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y\rangle \geq \alpha\|Mx - My\|^2,\ \forall\ x, y \in C,$$

(iii) firmly nonexpansive, if

$$\langle Mx - My, x - y\rangle \geq \|Mx - My\|^2,\ \forall\ x, y \in C,$$

(iv) Lipschitz, if there exists a constant $L > 0$ such that

$$\|Mx - My\| \leq L\|x - y\|,\ \forall\ x, y \in C.$$

Remark 2.1. It is generally known that every α- ism mapping is $\frac{1}{\alpha}$ Lipschitz continuous (see [9]).

If M is a multi-valued mapping, that is, $M : H \to 2^H$, then M is called monotone, if

$$\langle x - y, u - v\rangle \geq 0,\quad \forall\ x, y \in H, u \in M(x), v \in M(y)$$

and M is maximal monotone, if the graph $G(M)$ of M defined by

$$G(M) := \{(x, y) \in H \times H : y \in M(x)\}$$

is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v\rangle \geq 0$ for all $(y, u) \in G(M)$ implies $u \in M(x)$. A mapping $T : C \to C$ is said to be averaged nonexpansive if for all $x, y \in C, T = (1-\beta)I + \beta S$ holds for a nonexpansive operator $S : C \to C$ and $\beta \in (0, 1)$. The term "averaged mapping" was first developed by Baillon *et al.* [8]. Recall that a mapping T is firmly nonexpansive if and only if T can be expressed as $T = \frac{1}{2}(I + s)$, where S is nonexpansive (see [35]). Thus, we make the following remark which can be easily verified.

Remark 2.2. In a real Hilbert space, T is firmly nonexpansive if and only if it is averaged with $\beta = \frac{1}{2}$.

The metric projection P_C is a map defined on H onto C which assign to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_c x, z - P_C x\rangle \leq 0$, for all $z \in C$ and P_C is a firmly non-expansive mapping. We also know that if f is β-inverse strongly monotone mapping with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is averaged nonexpansive (see [14], Lemma 2.9). Hence, from Remark 2.2 we obtain the following.

Remark 2.3. In a real Hilbert space, if f is β-inverse strongly monotone with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is firmly nonexpansive.

For more information on metric projections, (see [14, 21]) and the references therein. Recall that the normal cone of C at the point $z \in H$ is define as

$$N_C z = \begin{cases} \{d \in H : \langle d, y - z\rangle \leq 0 y \in C\}, & z \in C, \\ \\ \emptyset, & \text{otherwise}. \end{cases}$$

Definition 2.4. Let H be a real Hilbert space and $T : H \to CB(H)$ a multi-valued mapping. Then, T is said to be demiclosed at the origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $d(x_n, T(x_n)) \to 0$, we have $x^* \in Tx^*$.

Lemma 2.5. ([15]) *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in (0, 1)$, we have*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2$,
(ii) $||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2$,
(iii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$.

Lemma 2.6. ([18]) *Let H be a real Hilbert space and $x_i \in H$, $(1 \leq i \leq m)$ and $\{\alpha_i\}_{i=1}^m \subset (0, 1)$ such that $\sum_{i=1}^m \alpha_i = 1$, the following identity holds:*

$$||\sum_{i=1}^m \alpha_i x_i||^2 = \sum_{i=1}^m \alpha_i ||x||^2 - \sum_{i,j=1,\ i\neq j} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Lemma 2.7. ([42]) *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^\infty \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \ \ n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we propose a viscosity iterative algorithm for approximating a common solution of finite family of variational inequality problem and fixed point problem for finite family of multi-valued type-one demicontractive mappings in real Hilbert spaces. A strong convergence result of the aforementioned problems will be proved and some consequences of our result are also displayed.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \to CB(H)$ be a finite family of multi-valued type-one demicontractive type mappings with constant $k_i \in (0,1)$ such that $I - S_i$ is demiclosed at zero. Let $f_j : H \to H$ for $j = 1, 2, \cdots, N$ be an α_j-inverse strongly monotone mapping and $g : H \to H$ be a contractive mapping with constant $\theta \in (0,1)$. Suppose that*

$$\Omega := \{\bigcap_{i=1}^{m} F(S_i) \bigcap \bigcap_{j=1}^{N} VI(C, f_j)\} \neq \emptyset.$$

For $x_1 \in H$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \gamma_n g(x_n) + (1-\gamma_n)x_n, \\ u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i} z_n^i, \ n \geq 1, \\ x_{n+1} = P_C(I - \lambda f_N) \circ P_C(I - \lambda f_{N-1}) \circ \cdots \circ P_C(I - \lambda f_1)u_n, \end{cases} \tag{3.1}$$

for $n \in \mathbb{N}$, where $z_n^i \in P_{S_i w_n}$, $P_{S_i w_n} = \{z_n^i \in S_i w_n : ||z_n^i - w_n|| = d(w_n, S_i w_n)\}$ and $\lambda \in (0, 2\alpha)$, $\alpha = \min\{\alpha_j, \ j = 1, 2, \cdots, N\}$, and the sequences $\{\beta_{n,i}\}_{n=1}^{\infty}$ for all $i \geq 0$ and $\{\gamma_n\}_{n=1}^{\infty}$ satisfy the following conditions:

(i) $\{\beta_{n,0}\} \in (k, 1)$, $\{\beta_{n,i}\} \in (0,1)$ *such that* $\sum_{i=0}^m \beta_{n,i} = 1, \ k < a \leq \beta_{n,i} \leq b < 1, \ i = 1, 2, ..., m, \ k := \sup_{i\geq 1}\{k_i\} < 1$;

(ii) $\gamma_n \in (0,1)$, $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to an element in Ω.

Proof. Let $z \in \Omega$, $\Phi^N = P_C(I - \lambda f_N) \circ P_C(I - \lambda f_{N-1}) \circ \cdots P_C(I - \lambda f_1)$, where $\Phi^0 = I$. Then from (3.1), Lemma 2.5 and Lemma 2.6, and the fact that S_i is of type-one demicontractive-type mapping for each i, we obtain that

$$\begin{aligned}
||x_{n+1}-x^*||^2 &= ||P_C(I-\lambda f_N)(\Phi^{N-1}u_n)-z||^2\\
&\leq ||\Phi^{N-1}u_n-z||^2\\
&\vdots\\
&\leq ||u_n-z||^2\\
&= ||\beta_{n,0}w_n+\sum_{i=1}^{m}\beta_{n,i}z_n^i-z||^2\\
&= ||\beta_{n,0}(w_n-z)+\sum_{i=1}^{m}\beta_{n,i}(z_n^i-z)||^2\\
&= \beta_{n,0}||w_n-z||^2+\sum_{i=1}^{m}\beta_{n,i}||z_n^i-z||^2-\sum_{i=1}^{m}\beta_{n,0}\beta_{n,i}||w_n-z_n^i||^2\\
&\leq \beta_{n,0}||w_n-z||^2+\sum_{i=1}^{m}\beta_{n,i}\mathcal{H}^2(S_iw_n,S_iz)\\
&\quad -\sum_{i=1}^{m}\beta_{n,0}\beta_{n,i}||w_n-z_n^i||^2\\
&\leq \beta_{n,0}||w_n-z||^2+\sum_{i=1}^{m}\beta_{n,i}\big[||w_n-z||^2+k||w_n-z_n^i||^2\big]\\
&\quad -\sum_{i=1}^{m}\beta_{n,0}\beta_{n,i}||w_n-z_n^i||^2\\
&= ||w_n-z||^2+(k-\beta_{n,0})\sum_{i=1}^{m}\beta_{n,i}||w_n-z_n^i||^2\\
&\leq ||w_n-z||^2. \qquad (3.2)
\end{aligned}$$

This implies that

$$\begin{aligned}
||x_{n+1}-z|| &\leq ||w_n-z||\\
&= ||\gamma_n(g(x_n)-z)+(1-\gamma_n)(x_n-z)||\\
&\leq \gamma_n||g(x_n)-z||+(1-\gamma_n)||x_n-z||\\
&\leq \gamma_n\big(||g(x_n)-z||+||g(z)-z||\big)+(1-\gamma_n)||x_n-z||\\
&\leq \gamma_n\big(\theta||x_n-z||+||g(z)-z||\big)+(1-\gamma_n)||x_n-z||
\end{aligned}$$

$$= (1-\gamma_n(1-\theta))||x_n - z|| + \gamma_n(1-\theta)\frac{||g(z)-z||}{1-\theta}$$
$$\leq \max\{||x_n - z||, \frac{||g(z)-z||}{1-\theta}\}.$$

By continuous taking this process, we obtain that

$$||x_n - z|| \leq \max\{||x_1 - z||, \frac{||g(z)-z||}{1-\theta}\},$$

for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ and $\{w_n\}$ are bounded.

Now, from (3.2) and Lemma 2.5, we obtain that

$$\begin{aligned}||x_{n+1} - z||^2 &\leq ||u_n - z||^2 \\ &= ||w_n - z||^2 + (k - \beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2 \\ &= ||\gamma_n g(x_n) + (1-\gamma_n)x_n - z||^2 + (k - \beta_{(n),0})\sum_{i=1}^{m}\beta_{n,i}||w_n - z_n^i||^2 \\ &\leq (1-\gamma_n)^2||x_n - z||^2 + 2\gamma_n\langle x_{n+1} - z, g(x_n) - z\rangle \\ &\quad + (k - \beta_{n,0})\sum_{i=1}^{m}\beta_{n,i}||w_n - z_n^i||^2 \\ &\leq (1-\gamma_n)||x_n - z||^2 + \gamma_n(2\langle x_{n+1} - z, g(x_n) - z\rangle). \end{aligned} \tag{3.3}$$

On substituting $d_n = \langle x_{n+1} - z, g(x_n) - z\rangle$ in view of Lemma 2.7, we need to prove that $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every $\{||x_{n_k} - z||\}$ of $\{||x_n - z||\}$ satisfying the condition

$$\lim_{k\to\infty}\{||x_{n_{k+1}} - z|| - ||x_{n_k} - z||\} \geq 0. \tag{3.4}$$

To show this, suppose that $\{||x_{n_k} - z||\}$ is a subsequence of $\{||x_n - z||\}$ such that (3.4) holds. Then

$$\begin{aligned}&\liminf_{k\to\infty}\left(||x_{n_{k+1}} - z||^2 - ||x_{n_k} - z||^2\right) \\ &= \lim_{k\to\infty}\Big((||x_{k+1} - z|| - ||x_{n_k} - z||)(||x_{n_{k+1}} - z|| + ||x_{n_k} - z||)\Big) \\ &\geq 0.\end{aligned}$$

Now, using (3.3), we have that

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\sum_{i=1}^{m}\beta_{n_k,i}(\beta_{n_k,0}-k)||w_{n_k}-z^i_{n_k}||^2\right)\\
&\le \limsup_{k\to\infty}\Big((1-\gamma_{n_k})||x_{n_k}-z||^2-||x_{n_{k+1}}-z||^2\\
&\quad+\gamma_{n_k}\big(2\langle x_{n_{k+1}}-z, g(x_{n_k}-z\rangle)\Big)\\
&\le \limsup_{k\to\infty}\Big(||x_{n_k}-z||^2-||x_{n_{k+1}}-z||^2\Big)\\
&\quad+\limsup_{k\to\infty}\Big(\gamma_{n_k}(2\langle x_{n_{k+1}}-z, g(x_{n_k})-z\rangle\rangle\Big)\\
&= -\liminf_{k\to\infty}\Big(||x_{n_{k+1}}-z||^2-||x_{n_k}-z||^2\Big)\\
&\le 0.
\end{aligned} \tag{3.5}$$

Using condition (i) and (ii) of (3.1), we obtain that

$$\lim_{k\to\infty}||w_{n_k}-z^i_{n_k}||=0.,\ i=1,2,\cdots,m.$$

Hence, we have that

$$\lim_{k\to\infty}d(w_{n_k},S_iw_{n_k})=\lim_{k\to\infty}||w_{n_k}-z^i_{n_k}||=0,\ i=1,2,\cdots,m. \tag{3.6}$$

From (3.1), we obtain that

$$||w_{n_k}-x_{n_k}||=\gamma_{n_k}||g(x_{n_k})-x_{n_k}||\to 0 \quad \text{as} \quad k\to\infty. \tag{3.7}$$

Also, from (3.1) and (3.6), we obtain that

$$||u_{n_k}-w_{n_k}||\le\sum_{i=1}^{m}\beta_{n_k,i}||z^i_{n_k}-w_{n_k}||\to 0 \quad \text{as} \quad k\to\infty. \tag{3.8}$$

Using (3.7) and (3.8), we get that

$$\lim_{k\to\infty}||u_{n_k}-x_{n_k}||=0. \tag{3.9}$$

Applying the firmly nonexpansivity of $P_C(I-\lambda f_N)$ (See Remark 2.3), we obtain that

$$\begin{aligned}||x_{n+1}-z||^2 &= ||P_C(I-\lambda f_N)\Phi^{N-1}u_n - z||^2\\ &\le \langle x_{n+1}-z, \Phi^{N-1}u_n - z\rangle\\ &= \frac{1}{2}\Bigg(||x_{n+1}-z||^2 + ||\Phi^{N-1}u_n - z||^2\\ &\quad - ||x_{n+1}-\Phi^{N-1}u_n||^2\Bigg), \end{aligned} \tag{3.10}$$

which implies from (3.10) that

$$\begin{aligned}&\limsup_{k\to\infty}\left(||x_{n_{k+1}}-\Phi^{N_{k-1}}u_{n_k}||^2\right)\\ &\le \limsup_{k\to\infty}\left(||\Phi^{N_{k-1}}u_{n_k}-z||^2 - ||x_{n_{k+1}}-z||^2\right)\\ &\vdots\\ &\le \limsup_{k\to\infty}\left(||u_{n_k}-z||^2 - ||x_{n_{k+1}}-z||^2\right)\\ &\le \limsup_{k\to\infty}\Bigg(||u_{n_k}-x_{n_k}||^2 + 2||u_{n_k}-x_{n_k}||\ ||x_{n_k}-z||^2\\ &\quad + ||x_{n_k}-z||^2 - ||x_{n_{k+1}}-z||^2\Bigg)\\ &\le \limsup_{k\to\infty}\left(||u_{n_k}-x_{n_k}||^2 + 2||u_{n_k}-x_{n_k}||\ ||x_{n_k}-z||^2\right)\\ &\quad + \limsup_{k\to\infty}\left(||x_{n_k}-z||^2 - ||x_{n_{k+1}}-z||^2\right)\\ &= -\liminf_{k\to\infty}\left(||x_{n_{k+1}}-z||^2 - ||x_{n_k}-z||^2\right)\\ &\le 0. \end{aligned} \tag{3.11}$$

Hence,

$$\lim_{k\to\infty}||x_{n_{k+1}}-\Phi^{N_{k-1}}u_{n_k}|| = 0. \tag{3.12}$$

By a similar argument as in (3.11) and applying (3.8), we obtain that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left(||\Phi^{N_{k-1}}u_{n_k}-\Phi^{N_{k-2}}u_{n_k}||^2\right)\\
&\le \limsup_{k\to\infty}\left(||\Phi^{N_{k-2}}u_{n_k}-z||^2-\Phi^{N_{k-1}}u_{n_k}-z||^2\right)\\
&\vdots\\
&\le \limsup_{k\to\infty}\left(||u_{n_k}-z||^2-||\Phi^{N_{k-1}}u_{n_k}-z||^2\right)\\
&\le \limsup_{k\to\infty}\left(||u_{n_k}-z||^2-||x_{n_{k+1}}-z||^2\right)\\
&\le \limsup_{k\to\infty}\Big(||u_{n_k}-x_{n_k}||^2+2||u_{n_k}-x_{n_k}||\;||x_{n_k}-z||^2\\
&\quad +||x_{n_k}-z||^2-||x_{n_{k+1}}-z||^2\Big)\\
&\le \limsup_{k\to\infty}\left(||u_{n_k}-x_{n_k}||^2+2||u_{n_k}-x_{n_k}||\;||x_{n_k}-z||^2\right)\\
&\quad +\limsup_{k\to\infty}\left(||x_{n_k}-z||^2-||x_{n_{k+1}}-z||^2\right)\\
&=-\liminf_{k\to\infty}\left(||x_{n_{k+1}}-z||^2-||x_{n_k}-z||^2\right)\\
&\le 0. \qquad (3.13)
\end{aligned}
$$

Hence,

$$\lim_{k\to\infty}||\Phi^{N_{k-1}}u_{n_k}-\Phi^{N_{k-2}}u_{n_k}||=0. \qquad (3.14)$$

Continuing in the same manner, we obtain that

$$
\begin{aligned}
\lim_{k\to\infty}||\Phi^{N_{k-2}}u_{n_k}-\Phi^{N_{k-3}}u_{n_k}|| = \cdots &= \lim_{k\to\infty}||\Phi^{2}u_{n_k}-\Phi^{1}u_{n_k}||\\
&= \lim_{k\to\infty}||\Phi^{1}u_{n_k}-u_{n_k}||=0. \qquad (3.15)
\end{aligned}
$$

From (3.12), (3.14) and (3.15), we conclude that

$$\lim_{k\to\infty}||\Phi^{j}u_{n_k}-\Phi^{j-1}u_{n_k}||=0,\ j=1,2,\cdots,N. \qquad (3.16)$$

By Remark 2.1, we have that f_j is Lipschitz continuous for each $j=1,2,\cdots,N$. Thus,

$$\lim_{k\to\infty}||f_j\Phi^{j}u_{n_k}-f_j\Phi^{j-1}u_{n_k}||=0,\quad j=1,2,\cdots,N. \qquad (3.17)$$

Also,

$$||x_{n_{k+1}} - u_{n_k}|| \leq ||\Phi^N u_{n_k} - \Phi^{N-1} u_{n_k}|| + ||\Phi^{N-1} u_{n_k} - \Phi^{N-2} u_{n_k}|| + \cdots + ||\Phi^1 u_{n_k} - u_{n_k}||, \tag{3.18}$$

which implies from (3.16) that

$$\lim_{k\to\infty} ||x_{n_{k+1}} - u_{n_k}|| = 0. \tag{3.19}$$

By applying (3.9) and (3.19), we have that

$$\lim_{k\to\infty} ||x_{n_{k+1}} - x_{n_k}|| = 0. \tag{3.20}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_t}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_t}}\}$ converges weakly to p. By (3.7) and (3.9), we have that there exist subsequences $\{w_{n_{k_t}}\}$ of $\{w_{n_k}\}$ and $\{u_{n_{k_t}}\}$ of $\{u_{n_k}\}$ that converges weakly to p respectively. Thus, by the demi-closedness of S_i at zero and (3.6), we obtain that $p \in F(S_i)$ for each $i = 1, 2, \cdots, m$.

We next show that $p \in \cap_{j=1}^N VI(C, f_j)$.

Let

$$B_j v = \begin{cases} f_j(v) + N_C v, & \forall\ v \in C; \\ \phi, & \forall\ v \notin C. \end{cases}$$

Then, B_j is maximal monotone for each $j = 1, 2, \cdots, N$. Let $(v, w) \in G(B_j)$. Then we have

$$w \in B_j v = f_j(v) + N_C v.$$

Hence

$$w - f_j(v) \in N_C v.$$

For $\Phi^j u_{n_{k_t}} \in C$, we obtain

$$\langle v - \Phi^j u_{n_{k_t}}, w - f_j v\rangle \geq 0,\ j = 1, 2, \cdots, N. \tag{3.21}$$

From $\Phi^j u_{n_{k_t}} = P_C(I - \lambda f_j)\Phi^{j-1} u_{n_{k_t}}$, we have

$$\langle v - \Phi^j u_{n_{k_t}}, \Phi^j u_{n_{k_t}} - (\Phi^{j-1} u_{n_{k_t}} - \lambda f_j \Phi^{j-1} u_{n_{k_t}})\rangle \geq 0,\ j = 1, 2, \cdots, N,$$

which implies that

$$\langle v - \Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}} - \Phi^{j-1} u_{n_{k_t}}}{\lambda} + f_j \Phi^{j-1} u_{n_{k_t}}\rangle \geq 0,\ \text{for each } j = 1, 2, \cdots, N.$$

From (3.21), we have

$$\begin{aligned}
&\langle v-\Phi^j u_{n_{k_t}}, w\rangle \\
&\geq \langle v-\Phi^j u_{n_{k_t}}, f_j v\rangle \\
&\geq \langle v-\Phi^j u_{n_{k_t}}, f_j v\rangle - \langle v-\Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}}-\Phi^{j-1}u_{n_{k_t}}}{\lambda}+f_j\Phi^{j-1}u_{n_{k_t}}\rangle \\
&= \langle v-\Phi^j u_{n_{k_t}}, f_j z - f_j\Phi^{j-1}u_{n_{k_t}} - \frac{\Phi^j u_{n_{k_t}}-\Phi^{j-1}u_{n_{k_t}}}{\lambda}\rangle \\
&= \langle v-\Phi^j u_{n_{k_t}}, f_j z - f_j\Phi^{j-1}u_{n_{k_t}}\rangle + \langle v-\Phi^j u_{n_{k_t}}, f_j\Phi^j u_{n_{k_t}} - f_j\Phi^{j-1}u_{n_{k_t}}\rangle \\
&\quad - \langle v-\Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}}-\Phi^{j-1}u_{n_{k_t}}}{\lambda}\rangle \\
&\geq \langle v-\Phi^j u_{n_{k_t}}, f_j\Phi^j u_{n_{k_t}} - f_j\Phi^{j-1}u_{n_{k_t}}\rangle \\
&\quad - \langle v-\Phi^j u_{n_{k_t}}, \frac{\Phi^j u_{n_{k_t}}-\Phi^{j-1}u_{n_{k_t}}}{\lambda}\rangle. \qquad (3.22)
\end{aligned}$$

By applying (3.16), (3.17) and (3.19) on (3.22), we obtain that

$$\langle v-p, w\rangle \geq 0.$$

Now, we conclude that since $B_j, j = 1, 2, \cdots, N$ is maximal monotone, thus $p \in B_j^{-1}(0)$, which implies that $0 \in B_j(p)$. Hence $p \in \cap_{j=1}^{\infty} VI(C, f_j)$. Therefore, we conclude that $p \in \Omega$.

We next show that $\limsup_{k\to\infty}\langle x_{n_{k+1}}-z, g(x_{n_k})-z\rangle \leq 0$. Indeed, let $\{x_{n_{k_t}}\}$ be a sequence such that $\{x_{n_k}\}$ converges weakly to p and

$$\limsup_{k\to\infty}\langle x_{n_{k+1}}-z, g(x_{n_k})-z\rangle = \lim_{t\to\infty}\langle x_{n_{k_t}+1}-z, g(x_{n_{k_t}})-z\rangle.$$

Using (3.20), we obtain that

$$\begin{aligned}
\limsup_{k\to\infty}\langle x_{n_{k+1}}-z, g(x_{n_k})-z\rangle &= \lim_{t\to\infty}\langle x_{n_{k_t+1}}-z, g(x_{n_{k_t}})-z\rangle \\
&\leq \langle p-z, g(p)-z\rangle \\
&\leq 0. \qquad (3.23)
\end{aligned}$$

On substituting (3.23) in (3.3) and applying Lemma 2.7, we obtain that $\{x_{n_k}\}$ converges strongly to p. This completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \to CB(H)$ be a finite family of multi-valued type-one demicontractive-type mappings with constant $k_i \in (0,1)$ such that $I-S_i$ is demiclosed at zero. Let $f : H \to H$ be an α-inverse strongly monotone* *mapping and* $g : H \to H$ *be a contractive mapping with constant* $\theta \in (0,1)$. *Suppose that*

$$\Omega := \Big\{ \bigcap_{i=1}^{m} F(S_i) \bigcap VI(C,f) \Big\} \neq \emptyset.$$

For $x_1 \in H$, *let the sequence* $\{x_n\}$ *be defined by*

$$\begin{cases} w_n = \gamma_n g(x_n) + (1-\gamma_n)x_n, \\ u_n = \beta_{n,0} w_n + \sum_{i=1}^{m} \beta_{n,i} z_n^i, \ n \geq 1, \\ x_{n+1} = P_C(I - \lambda f)u_n, \ n \in \mathbb{N}, \end{cases} \tag{3.24}$$

where $z_n^i \in P_{S_i w_n}$, $P_{S_i w_n} = \{z_n^i \in S_i w_n : ||z_n^i - w_n|| = d(w_n, S_i w_n)\}$ *and* $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $\lambda \in (0, 2\alpha)$, *and the sequences* $\{\beta_{n,i}\}_{n=1}^{\infty}$ *for all* $i \geq 0$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *satisfy the following conditions:*

(i) $\{\beta_{n,0}\} \in (k,1)$, $\{\beta_{n,i}\} \in (0,1)$ *such that* $\sum_{i=0}^{m} \beta_{n,i} = 1$, $k < a \leq \beta_{n,i} \leq b < 1$, $i = 1,2,...,m$, $k := \sup_{i \geq 1}\{k_i\} < 1$;

(ii) $\gamma_n \in (0,1)$, $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Then $\{x_n\}$ *converges strongly to an element in* Ω.

Corollary 3.3. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *and* $\{S_i\}_{i=1}^{m} : H \to CB(H)$ *be a finite family of multi-valued quasi-nonexpansive mappings. Let* $f : H \to H$ *be an* α*-inverse strongly monotone mapping and* $g : H \to H$ *be a contractive mapping with constant* $\theta \in (0,1)$. *Suppose that*

$$\Omega := \Big\{ \bigcap_{i=1}^{m} F(S_i) \bigcap VI(C,f) \Big\} \neq \emptyset.$$

For $x_1 \in H$, *let the sequence* $\{x_n\}$ *be defined by*

$$\begin{cases} w_n = \gamma_n g(x_n) + (1-\gamma_n)x_n, \\ u_n = \beta_{n,0} w_n + \sum_{i=1}^{m} \beta_{n,i} z_n^i, \ n \geq 1, \\ x_{n+1} = P_C(I - \lambda f)u_n, \ n \in \mathbb{N}, \end{cases} \tag{3.25}$$

where $z_n^i \in S_i u_n$, $\lambda \in (0, 2\alpha)$, *and the sequences* $\{\beta_{n,i}\}_{n=1}^{\infty}$ *for all* $i \geq 0$ *and* $\{\gamma_n\}_{n=1}^{\infty}$ *satisfy the following conditions:*

(i) $\{\beta_{n,0}\} \in (0,1)$, $\{\beta_{n,i}\} \in (0,1)$ *such that* $\sum_{i=0}^{m} \beta_{n,i} = 1$;

(ii) $\gamma_n \in (0,1)$, $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Then $\{x_n\}$ *converges strongly to an element in* Ω.

4. Application to convex minimization problem

Let $F : C \to \mathbb{R}$ be a convex and differentiable function. We know that if $\triangledown F$ is $\frac{1}{\alpha}$-Lipschitz continuous, then it is α-inverse strongly monotone, where $\triangledown F$ is the gradient of F. Moreover,

$$p = argmin_{x\in C}F(x) \quad \Leftrightarrow \quad p \in VI(C, \triangledown F). \tag{4.1}$$

Suppose the solution set of (4.1) is Γ. Then the setting $f_j = \triangledown F_j$ for each $j = 1, 2, \cdots, N$ in (3.1), we obtain the following result.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $\{S_i\}_{i=1}^m : H \to CB(H)$ be a finite family of multi-valued type-one demicontractive-type mappings with constant $k_i \in (0,1)$ such that $I - S_i$ is demiclosed at zero. Let $F_j : H \to \mathbb{R}$ for $j = 1, 2, \cdots, N$ be an α_j-inverse strongly monotone mapping and $g : H \to H$ be a contractive mapping with constant $\theta \in (0,1)$. Suppose that*

$$\Omega := \{ \bigcap_{i=1}^{m} F(S_i) \bigcap \bigcap_{j=1}^{N} VI(C, \triangledown F_j)\} \neq \emptyset.$$

For $x_1 \in H$, let the sequence $\{x_n\}$ be defined by

$$\begin{cases} w_n = \gamma_n g(x_n) + (1-\gamma_n)x_n, \\ u_n = \beta_{n,0}w_n + \sum_{i=1}^m \beta_{n,i} z_n^i, \ n \geq 1, \\ x_{n+1} = P_C(I-\lambda \triangledown F_N) \circ P_C(I-\lambda \triangledown F_{N-1}) \circ \cdots \circ P_C(I-\lambda \triangledown F_1)u_n, \end{cases} \tag{4.2}$$

for all $n \in \mathbb{N}$, where $z_n^i \in P_{S_i w_n}$, $P_{S_i w_n} = \{z_n^i \in S_i w_n : ||z_n^i - w_n|| = d(w_n, S_i w_n)\}$ and $\lambda \in (0, 2\alpha)$, $\alpha = \min\{\alpha_j, \ j = 1, 2, \cdots, N\}$, and the sequences $\{\beta_{n,i}\}_{n=1}^\infty$ for all $i \geq 0$ and $\{\gamma_n\}_{n=1}^\infty$ satisfy the following conditions:

(i) $\{\beta_{n,0}\} \in (k, 1)$, $\{\beta_{n,i}\} \in (0,1)$ *such that* $\sum_{i=0}^m \beta_{n,i} = 1$, $k < a \leq \beta_{n,i} \leq b < 1$, $i = 1, 2, ..., m$, $k := \sup_{i\geq 1}\{k_i\} < 1$;

(ii) $\gamma_n \in (0,1)$, $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^\infty \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to an element in Ω.

References

[1] H.A. Abass, K.O. Aremu, L.O. Jolaoso and O.T. Mewomo, *An inertial forward-backward splitting method for approximating solutions of certain optimization problem,* J. Nonlinear Funct. Anal., **2020** (2020), Article ID 6.

[2] H.A. Abass, C. Izuchukwu and K.O. Aremu, *A common solution of family of minimization and fixed point problem for multi-valued type-one demicontractive-type mappings,* Adv. Nonlinear Var. Inequal., **21**(2) (2018), 94–108.

[3] H.A. Abass, C.C. Okeke and O.T. Mewomo, *On split equality mixed equilibrium and fixed point problems of generalized k_i-strictly pseudo-contractive multivalued mappings,* Dyn. Contin. Discrete Impuls. Syst., Series B: Applications and Algorithms, **25**(6) (2018), 369–395.

[4] F. Akutsah, H.A. Abass, A.A. Mebawondu and O.K. Narain, *On split generalized mixed equilibrium and fixed point problems of an infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces*, Asian European J. Math., (2021), 1–20.

[5] P.N. Anh, H.T.C. Thach and J.K. Kim, *Proximal-like subgradient methods for solving multi-valued variational inequalities,* Nonlinear Funct. Anal. Appl., **25**(3) (2020), 437-451, doi.org/10.22771/nfaa.2020.25.03.03.

[6] K.O. Aremu, H.A. Abass, A.A. Mebawondu and O.K. Oyewole, *An inertial iterative method for split generalized vector mixed equilibrium and xed point problems.* The Journal of Analysis, **29**(2) (2021), DOI:10.1007/s41478-021-00312-x.

[7] K.O. Aremu, C. Izuchukwu, H.A. Abass, O.T. Mewomo, *On a Viscosity Iterative Method for Solving Variational Inequality Problems in Hadamard Spaces,* Axioms, **9**(4) (2020), 143; https://doi.org/10.3390/axioms9040143.

[8] J.B. Baillon, R.E. Bruck and S. Reich, *On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces,* Houston J. Math., **4** (1978), 1–9.

[9] J.B. Baillon and G. Haddad, *Quelques proprietes des operateurs angle bornes et cycliquement monotones,* Isr. J. Math., **26** (1977), 137–150.

[10] C. Baiocchi and A. Capello, *Variational and Quasi-variational inequalities,* Wiley, New York, 1984.

[11] A. Bensoussan and J.L. Lions, *Applications of variational inequalities to stochastic control,* North-Holland, Amsterdam, 1982.

[12] O.A. Boikanyo and G. Morosanu, *A proximal point algorithm converging strongly for general errors,* Optim. Lett., **4** (2010), 635–641.

[13] S.S. Chang, D.P. Wu, L. Wang and G. Wang, *Proximal Point Algorithms Involving Fixed Point of Nonspreading-type multivalued mappings in Hilbert space,* J. Nonlinear Sci. Appl., **9** (2016), 5561–5569.

[14] Y. Censor, A. Gibali and S. Reich, *A von Neumann alternating method for finding common solutions to variational inequalities,* Nonlinear Anal., **75** (2012), 4596-4603.

[15] C.E. Chidume, *Geometric properties of Banach spaces and nonlinear spaces and nonlinear iterations,* Springer Verlag Series, Lecture Notes in Mathematics, ISBN 978-84882-189-7, (2009).

[16] C.E. Chidume and S. Maruster, *Iterative methods for the computation of fixed points of demicontractive mappings,* J. Comput. Appl. Math., **234** (2010), 861–882.

[17] C.E. Chidume, P. Ndambomve, A.U. Bello and M.E. Okpala, *The multiple-sets split equality fixed point problem for countable families of multivalued demicontractive mappings,* Int. J. Math. Anal., **9** (2015), 453–467.

[18] C.E. Chidume and M.E. Okpala, *Fixed point iteration for a countable family of multivalued strictly pseudocontractive-type mappings,* SpringerPlus, **4**(506) (2015). https://doi.org/10.1186/s40064-015-1280-4.

[19] G. Fichera, *Problemi elasstostatici con vincoli unilaterali: II problema di signorini ambigue condizione al contorno,* Atti Accad. Naz. Lincei. Mem. Cl. Sci. Nat. Sez. Ia, **7**(8) (1963), 91–140.

[20] R. Glowinski, *Numerical methods for nonlinear variational problems,* Springer, Berlin, 1984.

[21] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry and nonexpansive mappings,* Marcel Dekker, New York, 1984.

[22] F. Grannessi and A. Mangeri, *Variatioanal inequalities and Network equilibrium problems,* Plenum Press, New York, 1995.

[23] F.O. Isiogugu, *Demiclosedness principle and approximation theorems for certain classes of multivalued mappings in Hilbert spaces,* Fixed Point Theory Appl., **61** (2013). https://doi.org/10.1186/1687-1812-2013-61.

[24] F.O. Isiogugu and M.O. Osilike, *Convergence theorems for new classes of multivalued hemicontractive-type mappings,* Fixed Point Theory Appl., **93** (2014). https://doi.org/10.1186/1687-1812-2014-93.

[25] C. Izuchukwu, *Strong convergence theorem for a class of multiple-sets split variatioanl inequality problems in Hilbert spaces,* Int. J. Nonlinear Anal. Appl., **9**(1) (2018), 27–40.

[26] L.O. Jolaoso, K.O. Oyewole, C.C. Okeke and O.T. Mewomo, *A unified algorithm for solving split generalized mixed equilibrium problem and fixed point of nonspreading mapping in Hilbert space,* Demonstr. Math., **51** (2018), 211–232.

[27] G. Kassay, *The Proximal Point Algorithm for Reflexive Banach spaces,* Studia Univ. Babes Bolyai Math., **30** (1985), 9–17.

[28] J.K. Kim, A.H. Dar and Salahuddin, *Existence theorems for the generalized relaxed pseudomonotone variational inequalities*, Nonlinear Funct. Anal. Appl., **25**(1) (2020), 25–34, doi.org/10.22771/nfaa.2020.25.01.03.

[29] G.M. Korpelevich, *An extragradient method for finding saddle points and for other problems,* Ekon Mat. Metody, **12** (1976), 747–756.

[30] W. Laowang and B. Panyanak, *Strong and* Δ*-convergence theorems for multivalued mappings in CAT(0) spaces,* J. Ineq. Appl., **2009** (2009), Art. Id 730132, 16 pages.

[31] K. Lerkchaiyaphum and W. Phuengrattana, *Proximal Point Algorithms for Numerical Reckoning Fixed Points of Hybrid-Type Multivalued Mappings in Hilbert Spaces,* Khayyan J. Math., **3** (2017), 81–89.

[32] B. Martinet, *Regularisation d' inequations varaiationnelles par approximations successives,* Rev. Fr. Inform. Rec. Oper., **4** (1970), 154–158.

[33] A.A. Mebawondu, *A proximal point algorithms for finding common fixed points of a finite family of nonexpansive multivalued mappings in real Hilbert spaces,* Khayyam J. Math., **5**(2) (2019), 113-123.

[34] A.A. Mebawondu, L.O. Jolaoso, H.A. Abass and O.K. Narain, *Generalized relaxed inertial method with regularization for solving split feasibility problems in real Hilbert spaces,* Asian European J. Math., (2021), 1–36.

[35] A. Moudafi, *Split monotone variational inclusions,* J. Optim. Theory Appl., **150** (2011), 275–283.

[36] K. Muangchoo, *A viscosity type projection method for solving pseudomonotone variational inequalities,* Nonlinear Funct. Anal. Appl., **26**(2) (2021), 347–371, doi.org/10.22771/nfaa.2021.26.02.08.

[37] O.K. Oyewole, H.A. Abass and O.T. Mewomo, *A strong convergence algorithm for a fixed point constrained split null point problem,* Rend. Circ. Mat. Palermo, II. **70** (2021) 389-408. https://doi.org/10.1007/s12215-020-00505-6.

[38] J.S. Pang, *"Asymmetric variational inequalities over product of sets" Applications and iterative methods,* Math. Program, **31** (1985), 206–219.

[39] T. Ram, J.K. Kim and R. Kour, *On optimal solutions of well-posed problems and variational inequalities*, Nonlinear Funct. Anal. Appl., **26**(4) (2021), 781-792, doi.org/10.22771/nfaa.2021.26.04.08.

[40] R.T. Rockafellar, *Monotone Operators and the proximal point algorithm,* SIAM J. Control Optim., **14** (1976), 877–898.

[41] R.T. Rockafellar and R. Wets, *Variational analysis*, Volume 317 of Grundlehren der Mathematischen Wissenschafte, Berlin, Springer, 1998.

[42] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces,* Nonlinear Anal., **75**(2) (2012), 742–750.

[43] G. Stampacchia, *Formes bilinearieres coercitivities sur les ensembles convexes,* C. R. Acad. Sci. Paris, **258** (1964), 4413–4416.

[44] S. Suantai, P. Cholamjiak, Y.J. Cho and W. Cholamjiak, *On solving split equilibrium problems and fixed point problems of nonspreading multivalued mappings in Hilbert spaces,* Fixed Point Theory Appl., **35** (2016), 1–16.

[45] S. Suantai and W. Phuengrattan, *Existence and convergence theorems for λ- hybrid mappings in Hilbert spaces,* Dyn. Continuous, Disc. Impul. Sys. Serie A: Math. Anal., **22** (2015), 177–188.

[46] J. Tang, J. Zhu, S.S. Chang, M. Liu and X. Li, *A new modified proximal point algorithm for a finite family of minimization problem and fixed point for a finite family of demicontractive mappings in Hadamard spaces,* Nonlinear Funct. Anal. Appl., **25**(3) (2020), 563-577, doi.org/10.22771/nfaa.2020.25.03.11.

[47] M. Tian and B.N. Jiang, *Weak convergence result theorem for a class of split variational inequality problems and applications in Hilbert spaces,* J. Ineq. Appl., **2017**(123) (2017). https://doi.org/10.1186/s13660-017-1397-9.

[48] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc., **2** (2002), 240–252.