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# HYBRID INERTIAL CONTRACTION PROJECTION METHODS EXTENDED TO VARIATIONAL INEQUALITY PROBLEMS

**N. D. Truong[1], J. K. Kim[2] and T. H. H. Anh[3]**

[1]Department of Mathematics, Hai Phong University, Haiphong, Vietnam
e-mail: truongnd@dhhp.edu.vn

[2]Department of Mathematics Education, Kyungnam University
Changwon, 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

[3]Department of Mathematics, Hai Phong University, Haiphong, Vietnam
e-mail: anhtth@dhhp.edu.vn

**Abstract.** In this paper, we introduce new hybrid inertial contraction projection algorithms for solving variational inequality problems over the intersection of the fixed point sets of demicontractive mappings in a real Hilbert space. The proposed algorithms are based on the hybrid steepest-descent method for variational inequality problems and the inertial techniques for finding fixed points of nonexpansive mappings. Strong convergence of the iterative algorithms is proved. Several fundamental experiments are provided to illustrate computational efficiency of the given algorithm and comparison with other known algorithms

## 1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, $C$ be a nonempty, closed and convex subset of $\mathcal{H}$, and $A : \mathcal{H} \to \mathcal{H}$ be an operator. The variational inequality problem, in short VI$(C, A)$, is to

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[0]Corresponding author: J. K. Kim(jongkyuk@kyungnam.ac.kr).

find a point $x^* \in C$ such that

$$\langle A(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

The variational inequality problem $\mathrm{VI}(C, A)$ was introduced first by Kinderlehrer and Stampacchia in [21] and has been applied to solve practical problems from various fields such as partial differential equations, optimal control, economics, circuits in electronics and others, see, for examples, [20, 22, 27, 30, 33, 35].

Thanks to this, many authors have constructed a large number of numerical methods for solving the problem $\mathrm{VI}(C, A)$. Among the algorithms for solving the problem $\mathrm{VI}(C, A)$, the projection method is one of the most popular and attractive methods in $\mathcal{H}$. It has been shown that the projection method, in general, is not convergent for the monotone problem $\mathrm{VI}(C, A)$. In order to obtain convergent projection algorithms, the extragradient algorithms have been proposed first by Korpelevich in [23] in the Euclidean space $\mathcal{R}^n$ for solving the problem $\mathrm{VI}(C, A)$, where the cost mapping $A$ is monotone and Lipschitz continuous, see also [6].

To enhance convergence of double projection algorithms, recently hybrid projection-cutting, linesearch projection algorithms and other have been proposed for the pseudomonotone problem $\mathrm{VI}(C, A)$ without Lipschitz continuous assumptions of $A$ [19]. However, the projection of a point onto the domain $C$ may not be easy to compute due to the complexity of the convex set $C$. To order to reduce the complexity probably caused by the projection, Yamada [36, 37] proposed hybrid steepest-descent algorithms for solving the problem $\mathrm{VI}(C, A)$. By using a nonexpansive mapping $S : \mathcal{H} \to \mathcal{H}$, i.e, $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$ via its fixed point set $C := \{x \in \mathcal{H} : Tx = x\}$, the iterative sequence is defined by the starting point $x^0 \in \mathcal{H}$ and

$$x^{k+1} = Sx^k - \lambda_{k+1} \mu A(Sx^k), \quad k \geq 0.$$

Under assumptions that $A$ is strongly monotone and Lipschitz continuous, the author proved convergence results of $\{x^k\}$ to the unique solution of the problem $\mathrm{VI}(C, A)$ under certain conditions onto parameters $\lambda_k$ and $\mu$.

In recent years, much studies have been given to develop efficient the hybrid steepest-descent algorithms. Some popular algorithms for solving the problem $\mathrm{VI}(C, A)$ are found such as the modified methods of Zeng, Wong and Yao [41, 42], the relaxed methods of Zeng et al. [43], the implicit methods of Ceng et al. [15, 16] and other [3, 28, 34].

For each $i \in I := \{1, 2, ...\}$, let $S_i : \mathcal{H} \to \mathcal{H}$ be a mapping. The set of all fixed points of $S_i$ is denoted by $\mathrm{Fix}(S_i)$ and assumed to be $\Omega := \bigcap_{i \in I} \mathrm{Fix}(S_i) \neq \emptyset$.

In this paper, we consider the variational inequality problem with the fixed point constraint VIF$(\Omega, A)$, which consists of the following:

$$\text{Find } x^* \in \Omega \quad \text{such that} \quad \langle A(x^*), x - x^* \rangle \geq 0, \quad x \in \Omega.$$

When $S_i (i \in I)$ is the identity mapping, the problem VIF$(\Omega, A)$ is formulated in the form of the problem VI$(C, A)$. This problem has closely related to many other, for example, as the fixed point problem when $A = 0$, the lexicographic variational inequality problem when $S_i x := Pr_C[x - \lambda G(x)] (i \in I)$, where $\lambda > 0, G : C \to \mathcal{H}$, $Pr_C$ is the metric projection on $C$, and other [8, 5, 19, 32]. In the case $I = \{1, 2, ..., n\}$ and $S_i$ is nonexpansive for each $i \in I$, Yamada [36] introduced the following iteration algorithm:

$$x^{k+1} = S_{[k+1]} x^k - \lambda_{k+1} \mu A(S_{[k+1]} x^k), \quad k \geq 0, \tag{1.1}$$

where $S_{[k]} := S_{k \text{mod} n}$ for $k \in \mathcal{N}$ with the mod function taking values in the set $I$. Under the conditions that $A$ is $\beta-$strongly monotone and $L-$Lipschitz continuous, $\mu \in (0, \frac{2\beta}{L^2}),\ \lambda_k \in (0, 1),\ \sum_{k=0}^{\infty} \lambda_k = \infty,\ \sum_{k=0}^{\infty} |\lambda_k - \lambda_{k+n}| < \infty$, $\lim_{k \to \infty} \lambda_k = 0$, the author proved the strong convergence of $\{x^k\}$ to the unique solution of the problem VIF$(\Omega, A)$.

Motivated and inspired by the Yamada algorithm (1.1), many interesting solution algorithms have been extended for a special class of the problem VIF$(\Omega, A)$ such as the parameter approximation algorithms of Zeng et al. in [42] and of Yao and Noor in [38], the three-step relaxed hybrid steepest-descent algorithms of Ding et al. [18] and Yao et al. in [39] and some other [36, 43].

One of the useful tools for solving the monotone problem VI$(C, A)$ is the inertial iteration technique. This technique was first used by Polyak [29] as an acceleration process in solving a smooth convex minimization problem. It includes two-step iterations, where one is defined by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm speeds up or accelerates the rate of convergence of the sequence generated by the algorithm.

Recently, there are growing interests in modified inertial techniques for solving a strongly monotone and Lipschitz continuous class of the problem VIF$(\Omega, A)$ [13, 14, 25], the minimization of the sum of two nonconvex functions [12], Ky Fan minimax inequalities [17], maximal monotone operators [1, 9] and other [11, 24].

The purpose of this paper is to propose new iteration algorithms by using the recent interest on the hybrid steepest-descent method (1.1) and inertial iteration techniques for solving the problem VIF$(\Omega, A)$, where the cost mapping $A$ is strongly monotone and Lipschitz continuous on $\mathcal{H}$. Furthermore, we will show strong convergence results of the proposed algorithms under the conditions on parameters.

The paper is organized as follows. In Section 2, we present some definitions and lemmas which will be used in the paper. Section 3 deals with an hybrid inertial subgradient algorithm for solving the problem $\mathrm{VIF}(\Omega, A)$ and proves its strong convergence in a real Hilbert space $\mathcal{H}$. In the final section, some numerical results are provided, which show the advantages of the proposed algorithm.

## 2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space. For each $i \in I := \{1, 2, ...\}$, let $S_i : \mathcal{H} \to \mathcal{H}$ be a mapping. The fixed point set of $S_i$ is denoted by $\mathrm{Fix}(S_i)$ and assumed to be nonempty.

**Definition 2.1.** A mapping $S_i : \mathcal{H} \to \mathcal{H}$ is said to be: .

(1) *quasinonexpansive* on $\mathcal{H}$, if

$$\|S_i x - \hat{x}\| \leq \|x - \hat{x}\|, \quad \forall (x, \hat{x}) \in \mathcal{H} \times \mathrm{Fix}(S_i);$$

(2) $\tau_i-$*strictly pseudocontractive* on $\mathcal{H}$, where $\tau_i \in [0, 1)$, if

$$\|S_i x - S_i y\|^2 \leq \|x - y\|^2 + \tau_i \|(x - y) - [S_i x - S_i y]\|^2, \quad \forall x, y \in \mathcal{H};$$

(3) $\beta_i-$*demicontractive* on $\mathcal{H}$ where $\beta_i \in [0, 1)$, if

$$\|S_i x - \hat{x}\|^2 \leq \|x - \hat{x}\|^2 + \beta_i \|x - S_i x\|^2, \quad \forall (x, \hat{x}) \in \mathcal{H} \times \mathrm{Fix}(S_i);$$

(4) *demiclosed*, if $\{x^k\}$ weakly converges to $\bar{x}$ and $\{(I - S_i)(x^k)\}$ strongly converges to 0, then $\bar{x} \in \mathrm{Fix}(S_i)$.

**Definition 2.2.** Let $S : \mathcal{H} \to \mathcal{H}$ be a mapping such that $\emptyset \neq \mathrm{Fix}(S) \subset \bigcap_{i \in I} \mathrm{Fix}(S_i)$. Then

(1) $\{S_i\}$ is said to satisfy *NST-condition (I)* with $S$ [9], if for each bounded sequence $\{x^i\} \subset \mathcal{H}$,

$$\lim_{i \to \infty} \|x^i - S_i x^i\| = 0 \quad \text{implies} \quad \lim_{i \to \infty} \|x^i - S x^i\| = 0.$$

(2) the sequence $\{S_i\}$ with a nonempty common fixed point set is said to satisfy the condition $(Z)$ [9, 10], if whenever $\{x^i\}$ is a bounded sequence in $\mathcal{H}$ such that

$$\lim_{i \to \infty} \|x^i - S_i x^i\| = 0,$$

It follows that every weak cluster point of $\{x^i\}$ belongs to $\bigcap_{i \in I} \mathrm{Fix}(S_i)$.

**Definition 2.3.** A mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be:

(1) *strongly monotone* with constant $\beta > 0$ (shortly $\beta$-strongly monotone), if

$$\langle A(x) - A(y), x - y \rangle \geq \beta \|y - x\|^2, \quad \forall x, y \in C;$$

(2) *Lipschitz continuous* with constant $L > 0$ (shortly $L$-Lipschitz continuous), if

$$\|A(x) - A(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(3) *contraction* with constant $L > 0$, if $A$ is $L$-Lipschitz continuous where $L < 1$;

(4) *nonexpansive*, if $A$ is 1-Lipschitz continuous on $\mathcal{H}$.

Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. For each $x \in \mathcal{H}$, there exists a unique point in $C$, denoted by $Pr_C(x)$ satisfying

$$\|x - Pr_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping $Pr_C$ is usually called the *metric projection* of $\mathcal{H}$ on $C$. An important property of $Pr_C$ is nonexpansive on $\mathcal{H}$.

Now we recall the following lemmas which are useful tools for proving our convergence results.

**Lemma 2.4.** ([26, Remark 4.2]) *Let $S : \mathcal{H} \to \mathcal{H}$ be a $\mathcal{K}$-demicontractive mapping, $Fix(S) \neq \emptyset$ and $\alpha \in [0, 1 - \mathcal{K}]$. Then,*

$$\|S_\alpha x - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 - \alpha(1 - \mathcal{K} - \alpha)\|Sx - x\|^2, \quad \forall \bar{x} \in \mathrm{Fix}(S), x \in \mathcal{H},$$

*where $S_\alpha = (1 - \alpha)I + \alpha S$ and $I$ is the identity mapping.*

**Lemma 2.5.** ([31, Lemma 2.6]) *Let $\{s_k\}$ be a sequence of nonnegative real numbers and $\{p_k\}$ be a sequence of real numbers. Let $\{\alpha_k\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$. Assume that*

$$s_{k+1} \leq (1 - \alpha_k)s_k + \alpha_k p_k, \quad k \in \mathcal{N}.$$

*If $\limsup_{i\to\infty} p_{k_i} \leq 0$ for every subsequence $\{s_{k_i}\}$ of $\{s_k\}$ satisfying*

$$\liminf_{i\to\infty} (s_{k_i+1} - s_{k_i}) \geq 0,$$

*then $\lim_{k\to\infty} s_k = 0$.*

## 3. Algorithm and its convergence

For solving the variational inequality problem VIF$(\Omega, A)$ over the fixed point set, we assume the mappings $A$ and parameters $S_k (k \in I)$ satisfy the following conditions:

$(A_1)$ The $A$ is $\beta$-strongly monotone and $L$-Lipschitz continuous;

$(A_2)$ For each $k \in I$, $S_k$ is $\xi_k-$demicontractive and satisfies the condition $(Z)$ with

$$\Omega := \bigcap_{k\in I} \mathrm{Fix}(S_k) \neq \emptyset;$$

$(A_3)$ For every $k \geq 0$, positive parameters $\beta_k, \gamma_k, \tau_k, \lambda_k$ and $\{\mu_k\}$ satisfy the following restrictions:

$$\begin{cases} 0 < c_1 \leq \beta_k \leq c_2 < 1, \mu_k \leq \eta, \\ \alpha_k \in (0, 1-\xi_k], \inf_k \alpha_k > 0, \\ 0 < \gamma_k < 1, \lim\limits_{k\to\infty} \gamma_k = 0, \sum_{k=1}^{\infty} \gamma_k = \infty, \\ \lim\limits_{k\to\infty} \frac{\tau_k}{\gamma_k} = 0, \lambda_k \in \left(\frac{\beta}{L^2}, \frac{2\beta}{L^2}\right), a \in (0,1), \sqrt{1 - 2\lambda_k\beta + \lambda_k^2 L^2} < 1 - a. \end{cases} \tag{3.1}$$

**Algorithm 3.1. (Hybrid inertial contraction projection algorithm)**

**Initialization:** Take $x^0, x^1 \in \mathcal{H}$ arbitrarily.

**Iterative steps:** $k = 1, 2, \ldots$.

**Step 1.** Compute an inertial parameter

$$\theta_k = \begin{cases} \min\left\{\mu_k, \dfrac{\tau_k}{\|x^k - x^{k-1}\|}\right\} & \text{if } \|x^k - x^{k-1}\| \neq 0, \\ \mu_k & \text{otherwise.} \end{cases} \tag{3.2}$$

**Step 2**. Compute

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ \bar{S}_k w^k = (1-\alpha_k)w^k + \alpha_k S_k w^k, \\ z^k = (1-\gamma_k)\bar{S}_k w^k + \gamma_k\left[w^k - \lambda_k A(w^k)\right], \\ \bar{S}_k z^k = (1-\alpha_k)z^k + \alpha_k S_k z^k, \\ x^{k+1} = (1-\beta_k)\bar{S}_k w^k + \beta_k \bar{S}_k z^k. \end{cases} \tag{3.3}$$

**Step 3.** Set $k := k+1$ and return to Step 1.

A strong convergence result is established by the following theorem.

**Theorem 3.2.** *Assume that the assumptions $(A_1)-(A_3)$ are satisfied. Then, the sequence $\{x^k\}$ generated by the Algorithm 3.1 converges strongly to a unique solution $x^*$ of the problem VIF$(\Omega, A)$.*

*Proof.* Since $A$ is $\beta$-strongly monotone and $L$-Lipschitz continuous on $\mathcal{H}$, for each $\lambda_k > 0$, we have

$$\begin{aligned}
&\|[w^k - \lambda_k A(w^k)] - [x^* - \lambda_k A(x^*)]\|^2 \\
&= \|w^k - x^*\|^2 - 2\lambda_k \langle A(w^k) - A(x^*), w^k - x^*\rangle \\
&\quad + \lambda_k^2 \|A(w^k) - A(x^*)\|^2 \\
&\le \|w^k - x^*\|^2 - 2\lambda_k \beta \|w^k - x^*\|^2 + \lambda_k^2 L^2 \|w^k - x^*\|^2 \\
&= (1 - 2\lambda_k\beta + \lambda_k^2 L^2)\|w^k - x^*\|^2. \qquad (3.4)
\end{aligned}$$

It is well known that $A$ is strongly monotone and $\Omega \neq \emptyset$, so the problem $\mathrm{VIF}(\Omega, A)$ has a unique solution $x^* \in \Omega$. By Lemma 2.4 and $x^* \in \mathrm{Fix}(S_k)$, we have

$$\begin{aligned}
\|\bar{S}_k w^k - x^*\|^2 &\le \|w^k - x^*\|^2 \\
&\quad - \alpha_k(1 - \xi_k - \alpha_k)\|S_k w^k - w^k\|^2 \\
&\le \|w^k - x^*\|^2. \qquad (3.5)
\end{aligned}$$

Combining the scheme (3.3) and the relation (3.4), we obtain

$$\begin{aligned}
\|z^k - x^*\| &= \left\|(1-\gamma_k)\bar{S}_k w^k + \gamma_k \left[w^k - \lambda_k A(w^k)\right] - x^*\right\| \\
&\le \gamma_k \left\|[w^k - \lambda_k A(w^k)] - x^*\right\| + (1-\gamma_k)\|\bar{S}_k w^k - x^*\| \\
&\le \gamma_k \left\|[w^k - \lambda_k A(w^k)] - [x^* - \lambda_k A(x^*)]\right\| \\
&\quad + \gamma_k \lambda_k \|A(x^*)\| + (1-\gamma_k)\|\bar{S}_k w^k - x^*\| \\
&\le \gamma_k \sqrt{1 - 2\lambda_k\beta + \lambda_k^2 L^2}\|w^k - x^*\| \\
&\quad + \gamma_k \lambda_k \|A(x^*)\| + (1-\gamma_k)\|w^k - x^*\| \\
&= [1 - \gamma_k(1-\delta_k)]\|w^k - x^*\| + \gamma_k\lambda_k\|A(x^*)\|, \qquad (3.6)
\end{aligned}$$

where $\delta_k := \sqrt{1 - 2\lambda_k\beta + \lambda_k^2 L^2} \in (0, 1-a)$.

By a similar way as in (3.5), we have

$$\begin{aligned}
\|\bar{S}_k z^k - x^*\|^2 &\le \|z^k - x^*\|^2 \\
&\quad - \alpha_k(1 - \xi_k - \alpha_k)\|S_k z^k - z^k\|^2 \\
&\le \|z^k - x^*\|^2.
\end{aligned}$$

Combining this, (3.6) and (3.1), we obtain

$$\begin{aligned}
\|x^{k+1} - x^*\| &= \|(1-\beta_k)\bar{S}_k w^k + \beta_k \bar{S}_k z^k - x^*\| \\
&\le (1-\beta_k)\|\bar{S}_k w^k - x^*\| + \beta_k \|\bar{S}_k z^k - x^*\| \\
&\le (1-\beta_k)\|w^k - x^*\| + \beta_k \|z^k - x^*\| \\
&\le [1 - \beta_k\gamma_k(1-\delta_k)]\|w^k - x^*\| + \beta_k\gamma_k\lambda_k\|A(x^*)\| \\
&\le [1 - \beta_k\gamma_k(1-\delta_k)]\left(\|x^k - x^*\| + \theta_k\|x^k - x^{k-1}\|\right) \\
&\quad + \beta_k\gamma_k \frac{2\beta\|A(x^*)\|}{L^2} \\
&\le [1 - \beta_k\gamma_k(1-\delta_k)]\|x^k - x^*\| \\
&\quad + \beta_k\gamma_k\left(\frac{\theta_k}{\beta_k\gamma_k}\|x^k - x^{k-1}\| + \frac{2\beta\|A(x^*)\|}{L^2}\right) \\
&\le [1 - \beta_k\gamma_k(1-\delta_k)]\|x^k - x^*\| \\
&\quad + \beta_k\gamma_k(1-\delta_k)\left(\frac{\theta_k}{a\beta_k\gamma_k}\|x^k - x^{k-1}\| + \frac{2\beta\|A(x^*)\|}{aL^2}\right).
\end{aligned}$$

By using Step 1 and the conditions (3.1), we deduce

$$0 \le \frac{\theta_k}{\beta_k\gamma_k}\|x^k - x^{k-1}\| \le \frac{\tau_k}{c_1\gamma_k} \to 0 \quad \text{as } k \to \infty.$$

This implies $M := \sup_k \left\{\frac{\theta_k}{a\beta_k\gamma_k}\|x^k - x^{k-1}\| + \frac{2\beta\|A(x^*)\|}{aL^2}\right\} < +\infty$. Then, we have

$$\begin{aligned}
\|x^{k+1} - x^*\| &\le [1 - \beta_k\gamma_k(1-\delta_k)]\|x^k - x^*\| + \beta_k\gamma_k(1-\delta_k)M \\
&\le \max\left\{\|x^k - x^*\|, M\right\}.
\end{aligned}$$

By mathematical induction, we deduce that

$$\|x^k - x^*\| \le \max\left\{\|x^1 - x^*\|, M\right\}, \quad \forall k \ge 1.$$

So, $\{x^k\}$ is bounded. It follows from (3.3) that

$$\|w^k - x^k\| = \theta_k\|x^k - x^{k-1}\| < +\infty.$$

By using (3.6), we also have that both $\{z^k\}$ and $\{w^k\}$ are bounded. By (3.4) and the relation

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in \mathcal{H},$$

we get

$$
\begin{aligned}
& 2\|z^k - x^*\|^2 \\
= & \left\|(1-\gamma_k)(\bar{S}_k w^k - x^*) + \gamma_k[w^k - \lambda_k A(w^k) - (x^* - \lambda_k A(x^*))] - \gamma_k \lambda_k A(x^*)\right\|^2 \\
\leq & \|(1-\gamma_k)(\bar{S}_k w^k - x^*) + \gamma_k[w^k - \lambda_k A(w^k) - (x^* - \lambda_k A(x^*))]\|^2 \\
& - 2\gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle \\
\leq & (1-\gamma_k)\|\bar{S}_k w^k - x^*\|^2 + \gamma_k \|w^k - \lambda_k A(w^k) - (x^* - \lambda_k A(x^*))\|^2 \\
& - 2\gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle \\
\leq & (1-\gamma_k)\|w^k - x^*\|^2 + \gamma_k \delta_k^2 \|w^k - x^*\|^2 - 2\gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle \\
\leq & \left[1 - \gamma_k(1-\delta_k^2)\right] \|w^k - x^*\|^2 - 2\gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle. \qquad (3.7)
\end{aligned}
$$

From $w^k = x^k + \theta_k(x^k - x^{k-1})$, it implies

$$
\begin{aligned}
\|w^k - x^*\|^2 &= \|x^k - x^*\|^2 + \theta_k^2 \|x^k - x^{k-1}\|^2 + 2\theta_k \langle x^k - x^*, x^k - x^{k-1} \rangle \\
&\leq \|x^k - x^*\|^2 + \theta_k^2 \|x^k - x^{k-1}\|^2 + 2\theta_k \|x^k - x^*\| \|x^k - x^{k-1}\|. \qquad (3.8)
\end{aligned}
$$

By Lemma 2.4 with $x^* \in \text{Fix}(S_k)$, (3.7), (3.8) and $x^{k+1} = (1-\beta_k)\bar{S}_k w^k + \beta_k \bar{S}_k z^k$, we obtain

$$
\begin{aligned}
\|x^{k+1} - x^*\|^2 = & \|(1-\beta_k)(\bar{S}_k w^k - x^*) + \beta_k(\bar{S}_k z^k - x^*)\|^2 \\
= & (1-\beta_k)\|\bar{S}_k w^k - x^*\|^2 + \beta_k \|\bar{S}_k z^k - x^*\|^2 \\
& - \beta_k(1-\beta_k)\|\bar{S}_k w^k - \bar{S}_k z^k\|^2 \\
\leq & (1-\beta_k)\|w^k - x^*\|^2 + \beta_k \|z^k - x^*\|^2 \\
& - \beta_k(1-\beta_k)\|\bar{S}_k w^k - \bar{S}_k z^k\|^2 \\
\leq & (1-\beta_k)\|w^k - x^*\|^2 + \beta_k[1 - \gamma_k(1-\delta_k^2)]\|w^k - x^*\|^2 \\
& - 2\beta_k \gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle - \beta_k(1-\beta_k)\|\bar{S}_k w^k - \bar{S}_k z^k\|^2 \\
= & [1 - \beta_k \gamma_k(1-\delta_k^2)]\|w^k - x^*\|^2 - 2\beta_k \gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle \\
& - \beta_k(1-\beta_k)\|\bar{S}_k w^k - \bar{S}_k z^k\|^2 \\
\leq & [1 - \beta_k \gamma_k(1-\delta_k^2)]\|x^k - x^*\|^2 + \theta_k^2 \|x^k - x^{k-1}\|^2 \\
& + 2\theta_k \|x^k - x^*\| \|x^k - x^{k-1}\| \\
& - 2\beta_k \gamma_k \lambda_k \langle A(x^*), z^k - x^* \rangle - \beta_k(1-\beta_k)\|\bar{S}_k w^k - \bar{S}_k z^k\|^2 \\
\leq & [1 - \beta_k \gamma_k(1-\delta_k^2)]\|x^k - x^*\|^2 - \beta_k(1-\beta_k)\|\bar{S}_k w^k - \bar{S}_k z^k\|^2 \\
& + \beta_k \gamma_k (1-\delta_k^2)\sigma_k,
\end{aligned}
$$

where

$$\begin{aligned}\sigma_k :=&\frac{1}{1-\delta_k^2}\Big\{\frac{\theta_k^2}{\beta_k\gamma_k}\|x^k-x^{k-1}\|^2+\frac{2\theta_k}{\beta_k\gamma_k}\|x^k-x^*\|\|x^k-x^{k-1}\|\\&-2\lambda_k\langle A(x^*),z^k-x^*\rangle\Big\}\\ \leq&\frac{1}{a(2-a)}\Big\{-2\lambda_k\langle A(x^*),z^k-x^*\rangle+\left(\frac{\theta_k}{c_1\gamma_k}\|x^k-x^{k-1}\|\right)\theta_k\|x^k-x^{k-1}\|\\&+2\|x^k-x^*\|\left(\frac{\theta_k}{c_1\gamma_k}\|x^k-x^{k-1}\|\right)\Big\}.\end{aligned}$$

Since $\{x^k\}$ is bounded, we have $\sup_k \sigma_k < +\infty$. It follows that

$$\begin{aligned}\|x^{k+1}-x^*\|^2 \leq& [1-\beta_k\gamma_k(1-\delta_k^2)]\|x^k-x^*\|^2\\&-\beta_k(1-\beta_k)\|\bar{S}_kw^k-\bar{S}_kz^k\|^2+\beta_k\gamma_k(1-\delta_k^2)\sigma_k.\end{aligned}\tag{3.9}$$

Now we apply Lemma 2.5 for $s_k := \|x^k-x^*\|^2, \alpha_k := \beta_k\gamma_k(1-\delta_k^2) \in (0,1)$ and $p_k := \sigma_k$. It follows from (3.9) that

$$s_{k+1} \leq (1-\alpha_k)s_k+\alpha_k p_k.$$

Assume that $\{s_{k_i}\}$ is any subsequence of $\{s_k\}$ such that

$$\liminf_{i\to\infty}(s_{k_i+1}-s_{k_i}) \geq 0.$$

Then, using the conditions (3.1) and (3.9), we obtain

$$\begin{aligned}0 &\leq c_1(1-c_2)\limsup_{i\to\infty}\left\|\bar{S}_{k_i}w^{k_i}-\bar{S}_{k_i}z^{k_i}\right\|^2\\&\leq \limsup_{i\to\infty}\beta_{k_i}(1-\beta_{k_i})\left\|\bar{S}_{k_i}w^{k_i}-\bar{S}_{k_i}z^{k_i}\right\|^2\\&\leq \limsup_{i\to\infty}\left[s_{k_i}-s_{k_i+1}+\beta_{k_i}\gamma_{k_i}(1-\delta_{k_i}^2)\sigma\right]\\&\leq \limsup_{i\to\infty}(s_{k_i}-s_{k_i+1})\\&= -\liminf_{i\to\infty}(s_{k_i+1}-s_{k_i})\\&\leq 0.\end{aligned}$$

Consequently,

$$\lim_{i\to\infty}\left\|\bar{S}_{k_i}w^{k_i}-\bar{S}_{k_i}z^{k_i}\right\| = 0.\tag{3.10}$$

It follows from the scheme (3.3) that

$$\|z^k-\bar{S}_kw^k\| = \gamma_k\|w^k-\lambda_kA(w^k)-\bar{S}_kw^k\|,$$

and hence

$$\left\|z^{k_i} - \bar{S}_{k_i} w^{k_i}\right\| = \gamma_{k_i} \left\|w^{k_i} - \lambda_{k_i} A(w^{k_i}) - \bar{S}_{k_i} w^{k_i}\right\|.$$

Since (3.8) and $\{x^k\}$ is bounded, we deduce that $\{w^k\}$ is also bounded. From $\lim_{k\to\infty} \gamma_k = 0$, we get

$$\lim_{i\to\infty} \left\|z^{k_i} - \bar{S}_{k_i} w^{k_i}\right\| = 0. \tag{3.11}$$

Since $\bar{S}_{k_i} z^{k_i} = (1-\alpha_{k_i}) z^{k_i} + \alpha_{k_i} S_{k_i} z^{k_i}$, (3.10) and (3.11), we obtain

$$\begin{aligned}\alpha_{k_i} \|z^{k_i} - S_{k_i} z^{k_i}\| &= \left\|z^{k_i} - \bar{S}_{k_i} z^{k_i}\right\| \\ &\le \left\|z^{k_i} - \bar{S}_{k_i} w^{k_i}\right\| + \left\|\bar{S}_{k_i} w^{k_i} - \bar{S}_{k_i} z^{k_i}\right\| \\ &\to 0, \quad \text{as } i \to \infty.\end{aligned} \tag{3.12}$$

By (3.12), the assumption $\inf_k \alpha_k > 0$ leads to

$$\left\|z^{k_i} - S_{k_i} z^{k_i}\right\| \to 0, \quad \text{as } i \to \infty. \tag{3.13}$$

We next show that $\limsup_{i\to\infty} p_{k_i} \le 0$. Since the conditions (3.1), we have

$$\begin{aligned}p_k =& \sigma_k \\ \le& \frac{1}{a(2-a)} \Bigg\{ -2\lambda_k \langle A(x^*), z^k - x^* \rangle + \left( \frac{\theta_k}{c_1 \gamma_k} \|x^k - x^{k-1}\| \right) \theta_k \|x^k - x^{k-1}\| \\ &+ 2\|x^k - x^*\| \left( \frac{\theta_k}{c_1\gamma_k} \|x^k - x^{k-1}\| \right) \Bigg\} \\ \le& \frac{1}{a(2-a)} \Bigg\{ -2\lambda_k \langle A(x^*), z^k - x^* \rangle + \frac{\tau_k}{\gamma_k} \left( \frac{\mu_k \|x^k - x^{k-1}\|}{c_1} + \frac{2\|x^k - x^*\|}{c_1} \right) \Bigg\}.\end{aligned}$$

Since $\lambda_k \in (\frac{\beta}{L^2}, \frac{2\beta}{L^2})$, the boundedness of $\{x^k\}$ and $\{\mu_k\}$, we deduce that if $\limsup_{i\to\infty} \langle A(x^*), x^* - z^{k_i} \rangle \le 0$ then $\limsup_{i\to\infty} p_{k_i} \le 0$. Since $\{z^k\}$ is bounded, without loss of generality, we can assume that there exists a subsequence $\{\bar{z}^{k_i}\}$ of $\{z^{k_i}\}$ such that $\bar{z}^{k_i} \rightharpoonup \bar{x}$ and

$$\limsup_{i\to\infty} \langle A(x^*), x^* - z^{k_i} \rangle = \lim_{i\to\infty} \langle A(x^*), x^* - \bar{z}^{k_i} \rangle.$$

Using (3.13), if follows from the condition $(Z)$ of the sequence $\{S_i\}$ that $\bar{x} \in \Omega$. Therefore,

$$\limsup_{i\to\infty} \langle A(x^*), x^* - z^{k_i} \rangle = \langle A(x^*), x^* - \bar{x} \rangle \le 0.$$

By Lemma 2.5, we can conclude that $x^k \to x^*$ as $k \to \infty$. This completes the proof. □

## 4. Numerical results

This section provides some several numerical experiments to illustrate strong convergence of the proposed algorithm and compare them with two algorithms: The parallel projection algorithm ($PPA$) of Anh et al. in [4, Scheme (3.1)] and the hybrid steepest descent scheme $HSDA$ (1.1) of Yamada in [36].

**Example 4.1.** Consider an academic example, where $\mathcal{H} := l_2$, the mappings $S_i, A : \mathcal{H} \to \mathcal{H}$ are given as follows, for each $x \in \mathcal{H}, i \in I := \{1, 2, ...\}$,

$$\begin{aligned} &l_2 := \left\{ x = (x_1, x_2, ...)^\top : \sum_{i=1}^{\infty} x_i^2 < +\infty \right\}, \\ &A(x) := (2x_1, x_2, 2x_3, ..., 2x_{2i-1}, x_{2i}, ...)^\top \in \mathcal{H}, \\ &S_1 x := x, \\ &S_i x := \{ y \in \mathcal{H} : y_{2j} = x_{2j}, x_{2j-1} = 0, \quad \forall j \geq 2 \}, \quad \forall i \geq 2. \end{aligned}$$

Then for each $k \in I, S_k$ is $0-$demicontractive, $A$ is $1-$strongly monotone and $2-$Lipschitz continuous. It is easy to see that the common fixed point set is defined in the form:

$$\begin{aligned} \Omega &= \bigcap_{k \in I} \mathrm{Fix}(S_k) \\ &= \left\{ x = (x_1, x_2, ..., x_{2i-1}, x_{2i}, ...)^\top \in \mathcal{H} : x_{2i-1} = 0, \quad \forall i \geq 1 \right\}. \end{aligned} \tag{4.1}$$

Choose $\mu_k = 1, \beta_k = \frac{1}{2}, \gamma_k = \tau_k = \frac{1}{k+1}, \lambda_k = \frac{3}{10}, \alpha_k = \frac{1}{2} \in (0, 1 - \xi_k]$ where $\xi_k = 0$, and hence $\sqrt{1 - 2\lambda_k \beta + \lambda_k^2 L^2} = \frac{\sqrt{19}}{5} \in (0, 1)$. Taking any sequence $\{x^k := (x_1^k, x_2^k, ...)^\top\}$ such that $\lim_{k\to\infty} \|S_k x^k - x^k\| = 0$, we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|S_k x^k - x^k\| \\ &= \lim_{k\to\infty} \|(x_1^k, 0, x_3^k, 0, ..., x_{2i-1}^k, 0, ...)^\top\| \\ &= \lim_{k\to\infty} \sqrt{(x_1^k)^2 + (x_3^k)^2 + ... + (x_{2i-1}^k)^2 + ...}\,. \end{aligned}$$

This implies that $\{x^k\}$ converges strongly to a point in $\Omega$ and hence the condition $(Z)$ is satisfied. Thus, the assumptions $(A_1) - (A_3)$ and the condition $(Z)$ hold. Take $x^0, x^1 \in \mathcal{H}$. By the algorithm 3.1, for each $k \geq 1$, we have

$$\begin{aligned}
\theta_k =& \min\left\{1, \frac{1}{(k+1)\|x^k - x^{k-1}\|}\right\} \quad \text{if} \quad \|x^k - x^{k-1}\| = 0, \quad \text{else} \quad \theta_k = 0,\\
w^k =& x^k + \theta_k(x^k - x^{k-1}),\\
z^1 =& (1-\gamma_1)\bar{S}_1 w^1 + \gamma_1[w^1 - \lambda_1 A(w^1)]\\
=& (1-\gamma_1)\bar{S}_1 w^1 + \gamma_1[w^1 - \lambda_1 A(w^1)]\\
=& \frac{1}{2}\bar{S}_1 w^1 + \frac{1}{2}\left[w^1 - \frac{1}{2}A(w^1)\right]\\
=& \left(\frac{7}{10}w_1^1, \frac{17}{20}w_2^1, ..., \frac{7}{10}w_{2i-1}^1, \frac{17}{20}w_{2i}^1, ...\right)^\top,\\
z^k =& (1-\gamma_k)\bar{S}_k w^k + \gamma_k[w^k - \lambda_k A(w^k)]\\
=& \frac{k}{k+1}\left(\frac{1}{2}w_1^k, w_2^k, \frac{1}{2}w_3^k, w_4^k, ...\right)^\top\\
&+ \frac{1}{k+1}\left[w^k - \frac{3}{10}\left(2w_1^k, w_2^k, 2w_3^k, w_4^k, ...\right)^\top\right]\\
=& \left(\frac{4-k}{10(k+1)}w_1^k, \frac{7}{10}w_2^k, ..., \frac{4-k}{10(k+1)}w_{2i-1}^k, \frac{7}{10}w_{2i}^k, ...\right)^\top, \quad \forall k \geq 2,
\end{aligned}$$

and

$$\begin{aligned}
x^2 =& (1-\beta_1)\bar{S}_1 w^1 + \beta_1 \bar{S}_1 z^1\\
=& \frac{1}{2}z^1 + \frac{1}{2}\bar{S}_1 z^1,\\
x^{k+1} =& (1-\beta_k)\bar{S}_k w^k + \beta_k \bar{S}_k z^k\\
=& \frac{1}{2}z^k + \frac{1}{2}\bar{S}_k z^k\\
=& \left(\frac{4-k}{20(k+1)}w_1^k, \frac{7}{10}w_2^k, ..., \frac{4-k}{20(k+1)}w_{2i-1}^k, \frac{7}{10}w_{2i}^k, ...\right)^\top, \quad \forall k \geq 2.
\end{aligned}$$

**Example 4.2.** Let us take $\mathcal{H} := \mathcal{R}^5$, the mappings $S_i : \mathcal{R}^5 \to \mathcal{R}^5 (i = 1, 2 \cdots)$ are defined by, for each $x = (x_1, x_2, ..., x_5)^\top \in \mathcal{R}^5$,

$$\begin{aligned}
S_1 x &= \left(x_1, \sin x_2, \frac{1}{3}x_3, x_4, \sin^3 x_5\right)^\top,\\
S_k x &= \left(x_1, \frac{1}{2}x_2, \sin x_3, \sin^2 x_4, \frac{1}{4}x_5\right)^\top, \quad \forall k \geq 2.
\end{aligned}$$

The cost mapping $A : \mathcal{R}^5 \to \mathcal{R}^5$ is given in the form $A(x) = sx + Qx + q$, where $P$ is a $5 \times 5$ matrix, $H$ is a $5 \times 5$ skew-symmatric matric, $K$ is a $5 \times 5$ diagonal matrix, $Q = PP^\top + H + K$ used in [2, 7] and $\|Q\| < s \in \mathcal{R}, q \in \mathcal{R}^n$.

Then, it is easy to see that $A$ is strongly monotone with constant $\beta := s - \|Q\|$ and Lipschitz continuous with constant $L := \|sE + Q\|$ where $E$ is the identity matrix, the mappings $S_k(k \geq 1)$ are $0-$demicontractive. The common fixed point set of $\{S_k\}$ is computed by

$$\Omega = \{(x_1, 0, 0, 0, 0)^\top : x_1 \in \mathcal{R}\}.$$

Suppose that the sequence $\{x^k\} \subset \mathcal{R}^5$ satisfies $\lim_{k\to\infty} \|S_k(x^k) - x^k\| = 0$. Then,

$$0 = \lim_{k\to\infty} \|S_k(x^k) - x^k\| = \lim_{k\to\infty} \left\| \left(0, \frac{1}{2}x_2^k, x_3^k - \sin x_3^k, x_4^k - \sin^2 x_4^k, \frac{3}{4}x_5^k\right)^\top \right\| = 0,$$

and hence $\lim_{k\to\infty} x_i^k = 0$ for all $i = 2, 3, 4, 5$. So, the condition $(Z)$ holds.

**Test 1.** The matrices $P, H, K$ and the vector $q$ are randomly chosen:

$$P = \begin{pmatrix} 2 & 3 & 0 & 4 & 1 \\ 3 & 2 & 1 & 0 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 4 & 1 & 3 & 1 & 0 \\ 1 & 0 & 1 & -1 & 3 \end{pmatrix}, H = \begin{pmatrix} 0 & 1 & 2 & 1 & 4 \\ 1 & 3 & 2 & 0 & 2 \\ 2 & -2 & 1 & 1 & -3 \\ 3 & 0 & -1 & 1 & 0 \\ 5 & -2 & 3 & 0 & 2 \end{pmatrix},$$

$$K = \begin{pmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix}, q = \begin{pmatrix} -2 \\ 5 \\ 7 \\ 10 \\ 2 \end{pmatrix}.$$

Taking $s = 80$. Then, we get that $\|Q\| \simeq 78.2072$, $A$ is $\beta-$strongly monotone and $L-$Lipschitz continuous, where $\beta = s - \|Q\| \simeq 80 - 78.2072 = 1.7928, L = \|sE + Q\| \simeq 158.1860$. For each $k \geq 1$, the other parameters are chosen as follows:

$$\mu_k = 10, \gamma_k = \frac{1}{k+3}, \tau_k = \frac{1}{k^2+1}, \alpha_k = 0.1 + \frac{1}{k+10},$$
$$\lambda_k = 0.0001 \in (6.8644e - 05, 1.3729e - 04) = \left(\frac{\beta}{L^2}, \frac{2\beta}{L^2}\right), \beta_k = 0.5 + \frac{1}{2k+9}.$$

We obtain that the conditions (3.1) hold and the numerical results of the Algorithm 3.1 in Figure 1 and Table 1. As usual, the tolerance error is $\epsilon-$solution, if $\|x^{k+1} - x^k\| \leq \epsilon$.

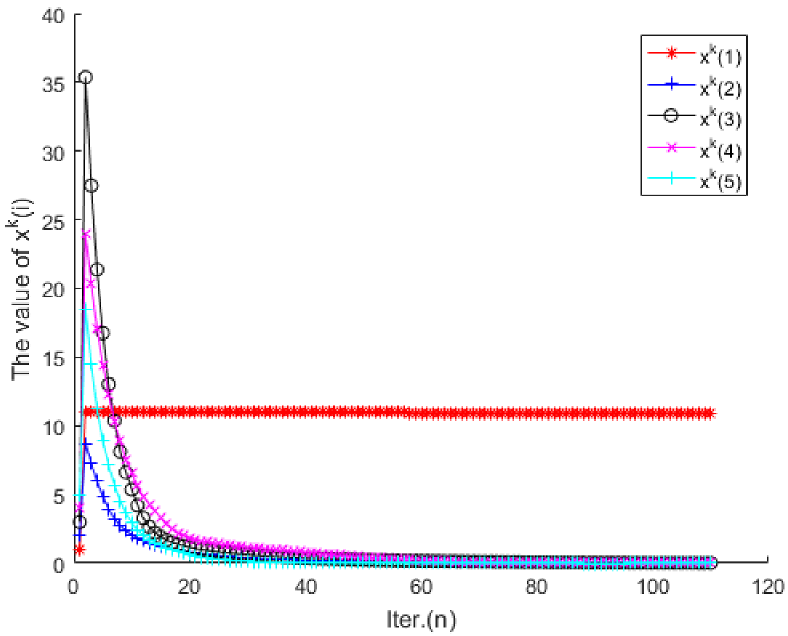


FIGURE 1. The Algorithm 3.1 with $x^0 = (1,2,3,4,5)^\top$, $x^1 = (0,1,-1,2,3)^\top$, the tolerance $\epsilon = 10^{-3}$.

Table 1 contains the numerical results of the Algorithm 3.1 for 10 choices of parameters.

| Case | $\mu_k$ | $\gamma_k$ | $\tau_k$ | $\alpha_k$ | $\lambda_k$ | $\beta_k$ | No. Iter. | CPU times |
|---|---|---|---|---|---|---|---|---|
| 1 | 10 | $\frac{1}{k+3}$ | $\frac{1}{k^2+1}$ | $0.10 + \frac{1}{k+10}$ | 0.00010 | $0.5 + \frac{1}{2k+9}$ | 109 | 0.0012 |
| 2 | 15 | $\frac{1}{2k+3}$ | $\frac{1}{k^2+10}$ | $0.15 + \frac{1}{k+10}$ | 0.00012 | $0.5 + \frac{1}{2k+7}$ | 112 | 0.0625 |
| 3 | 25 | $\frac{1}{3k+10}$ | $\frac{1}{k^2+10}$ | $0.15 + \frac{1}{k+10}$ | 0.00012 | $0.5 + \frac{1}{2k+7}$ | 120 | 0.0313 |
| 4 | 25 | $\frac{1}{3k+10}$ | $\frac{1}{2k^2+10}$ | $0.15 + \frac{1}{3k+10}$ | 0.00012 | $0.5 + \frac{1}{2k+7}$ | 121 | 0.0469 |
| 5 | 25 | $\frac{1}{3k+10}$ | $\frac{1}{2k^2+10}$ | $0.16 + \frac{1}{k+1}$ | 0.00010 | $0.7 + \frac{1}{5k+2}$ | 116 | 0.0156 |
| 6 | 2 | $\frac{1}{10k+1}$ | $\frac{1}{2k^2+10}$ | $0.16 + \frac{1}{k+1}$ | 0.00010 | $0.7 + \frac{1}{5k+2}$ | 92 | 0.0469 |
| 7 | 20 | $\frac{1}{10k+1}$ | $\frac{1}{2k^2+10}$ | $0.16 + \frac{1}{k+1}$ | 0.00019 | $0.7 + \frac{1}{5k+2}$ | 111 | 0.0156 |
| 8 | 50 | $\frac{1}{10k+1}$ | $\frac{1}{2k^2+10}$ | $0.10 + \frac{1}{k+100}$ | 0.00019 | $0.7 + \frac{1}{5k+2}$ | 156 | 0.0313 |
| 9 | 70 | $\frac{1}{k+1}$ | $\frac{1}{k^2+10}$ | $0.10 + \frac{1}{k+100}$ | 0.00019 | $0.7 + \frac{1}{5k+2}$ | 206 | 0.0469 |
| 10 | 100 | $\frac{1}{k+10}$ | $\frac{1}{2k^2+15}$ | $0.17 + \frac{1}{k+16}$ | 0.00014 | $0.3 + \frac{1}{5k+1}$ | 294 | 0.0156 |

TABLE 1. The Algorithm 3.1 with different parameters and $\epsilon = 10^{-3}$.

Table 2 presents the numerical results of the Algorithm 3.1 with different starting points.

| Case | Start. point $x^0$ | Start. point $x^1$ | No. Iter. | CPU times |
|---|---|---|---|---|
| 1 | $(1,2,3,4,5)^\top$ | $(0,1,-1,2,3)^\top$ | 17 | 0.6875 |
| 2 | $(-1,2,-3,4,-5)^\top$ | $(0,-1,1,-2,-3)^\top$ | 111 | 0.0156 |
| 3 | $(0,2,0,4,0)^\top$ | $(1,1,1,1,1)^\top$ | 102 | 0.0469 |
| 4 | $(2,4,6,8,10)^\top$ | $(3,5,7,9,11)^\top$ | 90 | 0.0469 |
| 5 | $(1,2,0.5,3,0)^\top$ | $(1,2,0.5,3,0)^\top$ | 67 | 0.0313 |
| 6 | $(1.2,2.2,3.3,4.4,5.5)^\top$ | $(-2.1,3.2,-4.3,5.4,-6.5)^\top$ | 125 | 0.0469 |
| 7 | $(1,2,0.5,3,0)^\top$ | $(-10,2,-3,4,-5)^\top$ | 324 | 0.0156 |
| 8 | $(10,20,30,40,50)^\top$ | $(-10,2,-3,4,-5)^\top$ | 604 | 0.0156 |
| 9 | $(10,20,30,40,50)^\top$ | $(0,0,0,0,0)^\top$ | 318 | 0.0469 |
| 10 | $(1.5,2.7,0.1,5.3,1.9)^\top$ | $(-1,-2,-5,-7,9)^\top$ | 118 | 0.0313 |

TABLE 2. The algorithm 3.1 with different starting points, where $\epsilon = 10^{-3}$.

**Test 2.** Compare the Algorithm 3.1 (Alg.1) with the algorithm $PPA$ and the algorthm $HSDA$. The stopping criterion of the algorithms is $\|x^{k+1} - x^k\| \le \epsilon$. Choosing randomly $x^0 = (1,2,3,4,5)^\top, x^1 = (0,0,0,0,0)^\top$. Let all entries $P, H$, and $K$ be randomly generated by using the commands $P = 2*5*rand(5,5) - 5; H = skewdec(5,1); K = diag(1:5)$. The comparative results are reported in Table 3 for $q = (3,7,9,10,-17)^\top$. Data of the algorithms are given as follows:

(1) **Alg.1**: $\mu_k = 15, \gamma_k = \frac{1}{2k+1}, \tau_k = \frac{1}{k^2+5}, \alpha_k = 0.5 + \frac{1}{k+10}, \lambda_k = 0.00012, \beta_k = 0.7 + \frac{1}{2k+100}$;

(2) **PPA**: $f(x,y) = \langle F(x), y - x\rangle\ \forall x, y \in \mathcal{R}^5, \alpha_{k,i} = 0.01 + \frac{1}{2k+19}\ (i = 1,2), \epsilon_k = 0, \gamma_k = \frac{1}{9k+15}$;

(3) **HSDA**: $\mu = 0.1, \lambda_k = \frac{1}{2k+1}$ for all $k \in \mathcal{N}$.

| Case | Iter. | | | CPU times | | |
|---|---|---|---|---|---|---|
| | $Alg.1$ | $PPA$ | $HSDA$ | $Alg.1$ | $PPA$ | $HSDA$ |
| 1 | 110 | 24 | 509 | 0.0469 | 0.0156 | 0.0781 |
| 2 | 111 | 937 | 541 | 0.0337 | 0.1250 | 0.0625 |
| 3 | 114 | 40 | 520 | 0.0483 | 0.0156 | 0.0625 |
| 4 | 108 | 30 | 516 | 0.0469 | 0.0313 | 0.0313 |
| 5 | 117 | 281 | 509 | 0.0905 | 0.0938 | 0.0156 |
| 6 | 96 | 44 | 500 | 0.0532 | 0.0441 | 0.0313 |
| 7 | 120 | 30 | 516 | 0.0716 | 0.0074 | 0.0156 |
| 8 | 114 | 159 | 507 | 0.0550 | 0.0663 | 0.0156 |
| 9 | 101 | 81 | 502 | 0.0712 | 0.0860 | 0.0469 |
| 10 | 119 | 28 | 522 | 0.0622 | 0.0052 | 0.0156 |

TABLE 3. The comparative results for Test 2 with $\epsilon = 10^{-3}$.

All the programs are written in in MATLAB R2016a running on a PC with Intel Core i7-7800X CPU @ 3.50 GHz 32 GB Ram. From the computational results of the hybrid inertial subgradient algorithm $Alg.1$, the the parallel projection algorithm $PPA$ and the hybrid steepest decent algorithm $HSDA$ reported in the tables, we observe that:

(i) The convergent speed of the Algorithm 3.1 is quite sensitive to the choice of the parameter sequences $\{\mu_k\}, \{\gamma_k\}, \{\tau_k\}, \{\lambda_k\}$ and $\{\beta_k\}$;
(ii) Test on $\mathcal{R}^5$, the CPU time and the iteration numbers of our Algorithm 3.1 are less than of the algorithms $PPA$ and $HSDA$.

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