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POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER THREE-POINT SINGULAR DIFFERENTIAL EQUATIONS

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Abstract. This paper investigates the existence of positive solutions for second-order singular three point boundary value problems. A necessary and sufficient condition for the existence of $C[0,1]$ as well as $C^1[0,1]$ positive solutions is given by constructing lower and upper solutions and with the maximal theorem. Also, the uniqueness of the $C[0,1]$ positive solutions is studied. Our nonlinearity may be singular at $t=0$ and/or $t=1$.

1. Introduction and the Main Results

The singular ordinary differential equations arises in the fields of gas dynamics, Newtonian fluid mechanics, the theory of boundary layer and so on. The theory of second order three-point boundary value problems has become an important area of investigation in recent years (see [1]-[3], [9], [10], [13], [14] and the references therein). In this paper, we will consider the positive solutions to the following nonlinear singular three-point boundary value problems of second-order ordinary differential equation

$$-u''(t) = f(t, u(t)),\ 0 < t < 1, \tag{1.1}$$

with

$$u(0) = au(\eta),\ u(1) = 0, \tag{1.2}$$

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where $0 < a < 1,\ 0 < \eta < 1$, and f satisfies the following hypothesis.

(H) $f(t,u) : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous, decreasing on u for each fixed $t \in (0,1)$, and there exists constant $b > 0$ such that $f(t,cu) \leq c^{-b} f(t,u),\ \forall c \in (0,1),\ (t,u) \in (0,1) \times (0,+\infty)$.

Clearly, from the condition (H), we have

$$f(s,u) = f(s, \frac{u}{v}v) \leq \left(\frac{v}{u}\right)^b f(s,v),\ \forall\ 0 < u \leq v. \tag{1.3}$$

By means of (1.3), we also have

$$f(s,u) \leq \left(\frac{u+v+|u-v|}{2u}\right)^b f(s,v),\ \forall\ u,\ v \in R^+. \tag{1.4}$$

By singularity we mean that the functions f in (1.1) are allowed to be unbounded at the points $t = 0$ and/or $t = 1$.

A function $u(t) \in C[0,1] \cap C^2(0,1)$ is called a $C[0,1]$ (positive) solution of (1.1) and (1.2) if it satisfies (1.1) and (1.2) ($u(t) > 0$, for $t \in (0,1)$). A $C[0,1]$ (positive) solution of (1.1) and (1.2) is called a $C^1[0,1]$ (positive) solution if both $u'(0+)$ and $u'(1-)$ exist ($u(t) > 0$, for $t \in (0,1)$).

When the function $f \in C([0,1] \times R, R)$ in (1.1), i.e. f is continuous, problem (1.1) and (1.2) is nonsingular, the existence of solutions to (1.1) and (1.2) has been studied by many authors using nonlinear alternative of Leray-Schauder, coincidence degree theory and fixed point theorem in cone (see [1]-[3], [9], [10] and references therein).

Very recently, the existence and multiplicity of positive solutions to the singular boundary value problem (1.1) and (1.2) have been widely studied by many authors, see, for example, [13]-[14]. But there are few papers concerned with the sufficient and necessary conditions of the multi-point boundary problems. The objective of the present paper is to fill this gap.

To seek necessary and sufficient conditions for the existence of solutions to the above problems is important and interesting, but difficult. Thus, researches in this respect are rare up to now. In this paper, we shall study the existence of positive solutions to the second-order singular three-point boundary value problem (1.1) and (1.2). A necessary and sufficient conditions for the existence of $C[0,1]$ as well as $C^1[0,1]$ positive solutions is given by constructing lower and upper solutions and with the maximal theorem. Also, the uniqueness of the $C[0,1]$ positive solutions is studied.

A function $\alpha(t)$ is called a *lower solution* to the problem (1.1), (1.2), if $\alpha(t) \in C[0,1] \bigcap C^2(0,1)$ and satisfies

$$\begin{cases} \alpha''(t) + f(t,\alpha(t)) \geq 0,\ t \in (0,1), \\ \alpha(0) - a\alpha(\eta) \leq 0,\ \alpha(1) \leq 0. \end{cases}$$

Upper solution is defined by reversing the above inequality signs. If there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ to problem (1.1), (1.2) such that $\alpha(t) \leq \beta(t)$, then $(\alpha(t), \beta(t))$ is called a couple of upper and lower solution to problem (1.1), (1.2).

In order to prove the main results, we need the following Lemma which can be found in [13].

Lemma 1.1. (maximum principle) *Suppose that* $0 < \eta < b_n$, *and*

$$F_n = \{x \in C[0, b_n] \cap C^2(0, b_n),\ x(0) - ax(\eta) \geq 0,\ x(b_n) \geq 0\}.$$

If $x \in F_n$ *such that* $-x''(t) \geq 0,\ t \in (0, b_n)$, *then* $x(t) \geq 0,\ t \in [0, b_n]$.

Now we state the main results of this paper as follows.

Theorem 1.1. *Suppose that* (H) *holds. Then a necessary and sufficient condition for problem (1.1) and (1.2) to have* $C[0,1]$ *positive solution is that*

$$0 < \int_0^1 e(s) f(s, 1) ds < +\infty, \tag{1.5}$$

where $e(s) = G(s,s) = s(1-s)$, $G(t,s)$ is the **Green**'s function of the problem $u'' = 0$ subject to the boundary value condition $u(0) = u(1) = 0$, which can be written as

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s < t \leq 1. \end{cases}$$

It is obvious that

$$e(t)e(s) \leq G(t,s) \leq e(t),\ \forall\ t, s \in I. \tag{1.6}$$

Theorem 1.2. *Suppose (H) holds, then a necessary and sufficient condition for problem (1.1) and (1.2) to have* $C^1[0,1]$ *positive solution is that*

$$0 < \int_0^1 f(s, 1-s) ds < +\infty. \tag{1.7}$$

Theorem 1.3. *Suppose (H) holds, then the* $C[0,1]$ *positive solution of (1.1) and (1.2) is unique.*

2. The proof of Theorem 1.1

2.1. Necessity of Theorem 1.1.

First we shall prove that the first inequality of (1.5) holds.

Suppose $w(t)$ is a positive solution of (1.1) (1.2), and $\overline{w} = \max_{t \in I} w(t)$. In view of (1.4), we have

$$0 \leq f(s, w(s)) \leq \left(\frac{w(s) + 1 + |w(s) - 1|}{2w(s)} \right)^b f(s, 1),\ \forall s \in J. \tag{2.1}$$

If $f(s,1) \equiv 0$ on J, then from (2.1), it is easy to see that $f(t, w(t)) \equiv 0$, that is, $w(t) \equiv 0$. This contradicts the assumption that $w(t)$ is a positive solution of (1.1) and (1.2). Hence $f(s,1) \not\equiv 0$. So the first inequality of (1.5) holds.

Next we shall prove that the second inequality of (1.5) holds.

From (1.2), there is $t_0 \in J$ such that $w'(t_0) = 0$. So

$$\begin{aligned}\int_{t_0}^{t} f(s, w(s))ds &= -\int_{t_0}^{t} w''(s)ds \\ &= -w'(t),\ t \in J.\end{aligned} \tag{2.2}$$

This implies

$$\begin{aligned}Sgn(t-t_0)\int_{t_0}^{t} f(s,\overline{w})ds &\le Sgn(t-t_0)\int_{t_0}^{t} f(s,w(s))ds \\ &= Sgn(t-t_0)(-w'(t)),\ t \in J,\end{aligned} \tag{2.3}$$

where $Sgn(t)$ denotes the symbolic function of t. It follows that

$$\begin{aligned}\int_0^{t_0} sf(s,\overline{w})ds &= \int_0^{t_0} ds \int_0^{s} f(s,\overline{w})dt \\ &= \int_0^{t_0} dt \int_t^{t_0} f(s,\overline{w})ds \\ &\le \int_0^{t_0} w'(t)dt \\ &= w(t_0) - w(0),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\int_{t_0}^{1} (1-s)f(s,\overline{w})ds &= \int_{t_0}^{1} ds \int_s^{1} f(s,\overline{w})dt \\ &= \int_{t_0}^{1} dt \int_{t_0}^{t} f(s,\overline{w})ds \\ &\le \int_{t_0}^{1} (-w'(t))dt \\ &= w(t_0) - w(1).\end{aligned} \tag{2.5}$$

Thus by (1.4) (2.4) and (2.5), we get

$$\begin{aligned}\int_0^{t_0} sf(s,1)ds &\le \left(\frac{1+\overline{w}+|1-\overline{w}|}{2}\right)^b \int_0^{t_0} sf(s,\overline{w})ds \\ &< \infty,\end{aligned} \tag{2.6}$$

and

$$\int_{t_0}^{1}(1-s)f(s,1)ds \le \left(\frac{1+\overline{w}+|1-\overline{w}|}{2}\right)^{b}\int_{t_0}^{1}(1-s)f(s,\overline{w})ds < \infty. \tag{2.7}$$

Therefore, (2.6) and (2.7) imply that the second inequality (1.5) holds.

2.2. **The existence of lower solutions and upper solutions.**

Suppose that (1.5) is satisfied. Choosing the real number m such that $mb > 1$. Let

$$\begin{aligned}
&h(t) = \int_0^1 G(t,s)f(s,1)ds + \frac{a(1-t)}{1-a+a\eta}\int_0^1 G(\eta,s)f(s,1)ds,\\
&g(t) = \int_0^1 G(t,s)f(s,k(s))ds + \frac{a(1-t)}{1-a+a\eta}\int_0^1 G(\eta,s)f(s,k(s))ds + k(t),\ t\in I,\\
&l = \min\Big\{1, \Big(\frac{1+a\eta}{1-a+a\eta}\int_0^1 e(s)f(s,1)ds\Big)^{-1}\Big\},\\
&L = \max\Big\{1, \Big(\frac{1+a\eta}{1-a+a\eta}\int_0^1 e(s)f(s,1)ds\Big)^{-1},\\
&\qquad (\overline{k})^b, (\overline{k})^b\Big(\frac{1+a\eta}{1-a+a\eta}\int_0^1 e(s)f(s,1)ds\Big)^{-1}\Big\},\\
&H(t) = lh(t),\ Q(t) = Lg(t),\ t\in I,
\end{aligned} \tag{2.8}$$

where $k(t) = (h(t))^{\frac{1}{mb}},\ t\in I,\ \overline{k} = \max_{t\in I} k(t)$.

It follows from (1.5) and (1.6) that

$$h(t) \le \frac{1+a\eta}{1-a+a\eta}\int_0^1 e(s)f(s,1)ds < \infty,$$

which implies that $h(t),\ k(t),\ l,\ L,\ \overline{k},\ H(t)$ are well defined on I.

In the following we shall show that $g(t)$ exists. In fact, we only need to prove that

$$\int_0^1 e(s)f(s,k(s))ds < \infty \tag{2.9}$$

By (2.8), it is easy to see that

$$\begin{aligned}
&\frac{1}{2}\int_0^t sf(s,1)ds \le (1-t)\int_0^t sf(s,1)ds \le h(t),\ t\in[0,\frac{1}{2}],\\
&\frac{1}{2}\int_t^1 (1-s)f(s,1)ds \le t\int_t^1 (1-s)f(s,1)ds \le h(t),\ t\in[\frac{1}{2},1].
\end{aligned} \tag{2.10}$$

In view of (2.10) and $m > 1$, we have

$$(h(t))^{-\frac{1}{m}} \le 2^{\frac{1}{m}} \left(\int_0^t sf(s,1)ds \right)^{-\frac{1}{m}},\ t \in [0, \frac{1}{2}],$$
$$(h(t))^{-\frac{1}{m}} \le 2^{\frac{1}{m}} \left(\int_0^t (1-s)f(s,1)ds \right)^{-\frac{1}{m}},\ t \in [\frac{1}{2}, 1].$$

Since the function $\left(\int_0^t sf(s,1)ds\right)^{1-\frac{1}{m}}$ is increasing on $[0, \frac{1}{2}]$ and the function $\left(\int_t^1 (1-s)f(s,1)ds\right)^{1-\frac{1}{m}}$ is decreasing on $[\frac{1}{2}, 1]$, their derivatives are integrable on the interval $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ respectively. So by the condition (H) and the above reasons, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}} e(t)f(s,k(t))ds \\
&\le d^{-b} \int_0^{\frac{1}{2}} t(1-t)(h(t))^{-\frac{1}{m}} f(t,1)dt \\
&\le d^{-b} 2^{\frac{1}{m}} \int_0^{\frac{1}{2}} t(1-t) \left(\int_0^t sf(s,1) \right)^{-\frac{1}{m}} f(t,1)dt \\
&\le d^{-b} 2^{\frac{1}{m}} \frac{m}{m-1} \int_0^{\frac{1}{2}} (1-t) \left[(\int_0^t sf(s,1)ds)^{1-\frac{1}{m}} \right]' dt,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^t e(t)f(s,k(t))ds \\
&\le d^{-b} \int_{\frac{1}{2}}^1 t(1-t)(h(t))^{-\frac{1}{m}} f(t,1)dt \\
&\le d^{-b} 2^{\frac{1}{m}} \int_{\frac{1}{2}}^1 t(1-t) \left(\int_t^1 (1-s)f(s,1) \right)^{-\frac{1}{m}} f(t,1)dt \\
&\le d^{-b} 2^{\frac{1}{m}} \frac{m}{m-1} \int_{\frac{1}{2}}^1 t \left[(\int_t^1 (1-s)f(s,1)ds)^{1-\frac{1}{m}} \right]' dt.
\end{aligned} \tag{2.12}$$

Consequently by (2.11) and (2.12), we have

$$\int_0^1 e(s)f(s,k(s))ds < \infty.$$

Hence, $g(t),\ Q(t)$ are well defined.

It is obvious that $H(t) > 0,\ t \in J$ and $H(t),\ Q(t) \in C(I) \bigcap C^2(J)$,

$$\begin{cases} H(0) = aH(\eta),\ H(1) = 0, \\ Q(0) - aQ(\eta) \geq 0,\ Q(1) = 0. \end{cases} \tag{2.13}$$

Clearly, (2.8) implies that

$$L \min\{1, (\overline{k})^{-b}\} \geq \min\left\{1, \left(\frac{1+a\eta}{1-a+a\eta}\int_0^1 e(s)f(s,1)ds\right)^{-1}\right\} = l,$$

and thus

$$\begin{aligned} Q(t) &\geq L\left[\int_0^1 G(t,s)f(s,k(s))ds + \frac{a(1-t)}{1-a+a\eta}\int_0^1 G(\eta,s)f(s,k(s))ds\right] \\ &\geq L\left[\int_0^1 G(t,s)f(s,\overline{k})ds + \frac{a(1-t)}{1-a+a\eta}\int_0^1 G(\eta,s)f(s,\overline{k})ds\right] \\ &\geq L\min\{1,\overline{k}^{-b}\}\left[\int_0^1 G(t,s)f(s,1)ds + \frac{a(1-t)}{1-a+a\eta}\int_0^1 G(\eta,s)f(s,1)ds\right] \\ &\geq H(t),\ t \in I. \end{aligned} \tag{2.14}$$

By the condition (H) and (2.8), we obtain

$$\begin{aligned} &H''(t) + f(t,H(t)) = f(t,lh(t)) - lf(t,1) \geq 0,\ t \in J, \\ &k''(t) = \frac{1}{mb}\left(\frac{1}{mb}-1\right)(h(t))^{\left(\frac{1}{mb}-2\right)}(h'(t))^2 + \frac{1}{mb}(h(t))^{\left(\frac{1}{mb}-1\right)}h''(t) \leq 0, \\ &Q''(t) + f(t,Q(t)) \leq f(t,Q(t)) - Lf(t,k(t)) \leq 0,\ t \in J. \end{aligned} \tag{2.15}$$

Thus (2.13) (2.14) and (2.15) imply that $H(t)$ and $Q(t)$ are lower solution and upper solution to (1.1) (1.2) respectivly.

2.3. **Sufficiency of Theorem 1.1.**

First of all, we define a partial ordering in $C(I) \bigcap C^2(J)$ by $x \leq y$ if and only if

$$x(t) \leq y(t),\ t \in I.$$

Then, we shall define an auxillary function. $\forall x(t) \in C(I) \bigcap C^2(J)$,

$$g(t,x) = \begin{cases} f(t,H(t)), & \text{if } H(t) \not\geq x(t), \\ f(t,x(t)), & \text{if } H(t) \leq x(t) \leq Q(t), \\ f(t,Q(t)), & \text{if } x(t) \not\leq Q(t). \end{cases} \tag{2.16}$$

By the condition (H), we have $g\colon J \times R \to [0,+\infty)$ is continuous.

Let $\{b_n\}$ be a sequence satisfying $0 < \eta < b_1 < \cdots < b_n < b_{n+1} < \cdots < 1$, and $b_n \to 1$ as $n \to \infty$, and let $\{r_n\}$, be a sequence satisfying

$$H(b_n) \le r_n \le Q(b_n), \quad n = 1, 2, \cdots .$$

For each n, consider the nonsingular problem

$$\begin{cases} -x''(t) = g(t, x),\ t \in [0, b_n], \\ x(0) - ax(\eta) = 0,\ x(b_n) = r_n. \end{cases} \tag{2.17}$$

obviously, the problem (2.17) is equivalent to the integral equation

$$\begin{aligned} x(t) = A_n x(t) = &\frac{((1-a)t + a\eta)}{b_n(1-a) + a\eta} r_n + \int_0^{b_n} G_n(t,s) g(s, x(s)) ds \\ &+ \frac{a(b_n - t)}{b_n(1-a) + a\eta} \int_0^{b_n} G_n(\eta, s) g(s, x(s)) ds,\ t \in [0, b_n], \end{aligned} \tag{2.18}$$

where

$$G_n(t,s) = \frac{1}{b_n} \begin{cases} (b_n - t)s,\ s < t, \\ (b_n - s)t,\ t \le s. \end{cases}$$

It is easy to verify that $A_n \colon X_n \to X_n = C[0, b_n]$ is completely continuous and $A_n(X_n)$ is a bounded set. Moreover, $x \in C[0, b_n]$ is a solution to (2.17) if and only if $A_n x = x$. Using the Schauder's fixed point theorem, we assert that A_n has at least one fixed point $x_n \in C^2[0, b_n]$.

We claim that

$$H(t) \le x_n(t) \le Q(t), \quad t \in [0, b_n], \tag{2.19}$$

and hence $x_n(t) \in C^2[0, b_n]$ and satisfies

$$-x''(t) = f(t, x(t)), \quad t \in [0, b_n]. \tag{2.20}$$

Indeed, suppose by contradiction that $x_n \not\le Q(t)$. By the definition of g, we have

$$g(t, x_n(t)) = f(t, Q(t)), \quad t \in [0, b_n],$$

and therefore

$$-x_n''(t) = f(t, Q(t)), \quad t \in [0, b_n]. \tag{2.21}$$

On the other hand, since $Q(t)$ is an upper solution of (1.1) and (1.2), we also have

$$-Q''(t) \ge f(t, Q(t)), \quad t \in J. \tag{2.22}$$

Then setting

$$z(t) = Q(t) - x_n(t), \quad t \in [0, b_n].$$

By (2.17) (2.21) and (2.22), we obtain

$$-z''(t) \ge 0,\ t \in (0, b_n),\ z \in C[0, b_n] \bigcap C^2(0, b_n),\ z(0) - az(\eta) \ge 0,\ z(b_n) \ge 0.$$

By Lemma 1.1, we can conclude that $z(t) \geq 0,\ t \in [0, b_n]$, a contrdiction with the assumption $x_n(t) \not\leq Q(t)$. Therefore $x_n(t) \not\leq Q(t)$ is impossible.

Similarly, we can show that $H(t) \leq x_n(t)$. So, we have shown that (2.19) holds.

Using the method of [11] and the Theorem 3.2 in [7], we can obtain that there is a $C(I)$ positive solution $x(t)$ to (1.1) (1.2) such that $H(t) \leq x(t) \leq Q(t)$, and a subsequence of $\{x_n(t)\}$ converges to $x(t)$ on any compact subintervals of J. This completes the proof of theorem 1.1.

3. The proof of Theorem 1.2

3.1. **Necessity of Theorem 1.2.**

Assume that $w(t)$ is a $C^1[0,1]$ positive solution to (1.1) and (1.2). Then $w''(t) \leq 0,\ \forall t \in J$. So $w(t)$ is a concave function on $[0,1]$. It is well known that $w(t)$ can be stated as

$$w(t) = \int_0^1 G(t,s)f(s,w(s))ds + \frac{a(1-t)}{1-a+a\eta}\int_0^1 G(\eta,s)f(s,w(s))ds, \tag{3.1}$$

where $G(t,s)$ is defined in Theorem 1.1.

It is easy to see that

$$w(0) > 0,\ w(t) \geq t(1-t)\|w\|,\ t \in I. \tag{3.2}$$

Here, $\|w\| = \max_{t\in[0,1]} |w(t)|$. For $t \in I$, from the concavity of w and (3.2) we have that

$$w(t) \geq \frac{1-t}{1-\eta}w(\eta) \geq \frac{1-t}{1-\eta}\eta(1-\eta)\|w\| = \eta\|w\|(1-t),\ t \in I. \tag{3.3}$$

Since $w(t)$ is a $C^1[0,1]$ positive solution to (1.1) and (1.2), from (1.2), we have

$$w(t) = \int_t^1 (-w'(s))ds \leq \max_{t\in[0,1]} |w'(t)|(1-t),\ t \in I. \tag{3.4}$$

Setting $I_1 = \eta\|x\|,\ I_2 = \|x'\|$, then from (3.3) and (3.4) we know that there are constants $0 < I_1 < I_2$, such that

$$I_1(1-t) \leq w(t) \leq I_2(1-t),\ t \in I. \tag{3.5}$$

Without loss of generality, we may assume that $0 < I_1 < 1 < I_2$. This together with the condition (H) implies that

$$\int_0^1 f(t, 1-t)dt \leq {I_2}^b \int_0^1 f(t,w(t))dt = {I_2}^b[w'(1-) - w'(0+)] < +\infty. \tag{3.6}$$

Hence, the second part of (1.7) holds.

Assume that $f(t, 1-t) \equiv 0,\ t \in J$. By (H) and (3.5) it is easy to see that

$$0 \le f(t, w(t)) \le f(t, I_1(1-t)) \le I_1{}^{-b} f(t, 1-t),\ t \in J. \tag{3.7}$$

Then $f(t, w(t)) \equiv 0$, that is, $w(t) \equiv 0$, which contradicts the assumption that $w(t)$ is a positive solution to (1.1) and (1.2). Thus $f(t, 1-t) \not\equiv 0$. Consequently the first part of (1.7) holds.

3.2. **Sufficiency of Theorem 1.2.**

In view of the condition (H) and (1.7) we have

$$\int_0^1 e(s) f(s,1) ds \le \int_0^1 e(s) f(s, 1-s) ds \le \int_0^1 f(s, 1-s) ds < +\infty.$$

Hence, from the proof of theorem 1.1, there exists a positive solution $w(t)$ to (1.1) and (1.2) such that

$$H(t) \le w(t) \le Q(t),\ \ \forall t \in I, \tag{3.8}$$

where $H(t),\ Q(t)$ are defined in (2.8).

It follows from (1.7) and (2.8) that

$$w(t) \ge \frac{la(1-t)}{1-a+a\eta} \int_0^1 G(\eta, s) f(s,1) ds = lk_1(1-t),\ \ t \in I,$$

where $k_1 = \frac{a}{1-a+a\eta} \int_0^1 G(\eta, s) f(s,1) ds$. Thus,

$$f(t, w(t)) \le f(t, lk_1(1-t)) \le \max\{1, (lk_1)^{-b}\} f(t, 1-t). \tag{3.9}$$

From (3.9) we know that $f(t, w(t))$ is integratable on J, that is, the derivable function $w''(t)$ of $w'(t)$ is integratable on J. Therefore, $w'(0+)$, $w'(1-0)$ exist, that is $w(t) \in C^1(I) \bigcap C^2(J)$. This completes the proof of theorem 1.2.

4. The proof of theorem 1.3

Suppose that $u_1(t),\ u_2(t)$ are $C(I)$ positive solution to (1.1) and 1.2. We may assume without loss of generality that there exists $t^* \in J$ such that $u_2(t^*) - u_1(t^*) = \max_{t \in I} u_2(t^*) - u_1(t^*) > 0$. Let

$$\alpha = \inf\{t_1 \mid 0 \le t_1 < t^*,\ u_2(t) \ge u_1(t),\ \text{for } t \in (t_1, t^*]\},$$

$$\beta = \sup\{t_2 \mid t^* < t_2 \le 1\ \ u_2(t) \ge u_1(t),\ \text{for } t \in (t^*, t_2]\}$$

$$z(t) = u_2(t) - u_1(t),\ \ t \in I.$$

Evidently, $t^* \in (\alpha, \beta),\ u_2(t) \ge u_1(t),\ f(t, u_2(t)) \le f(t, u_1(t)),\ t \in [\alpha, \beta]$, hence, $z''(t) = f(t, u_1(t)) - f(t, u_2(t)) \ge 0,\ t \in [\alpha, \beta]$.

By (1.2), it is easy to check that there exist the following two possible cases

$$(1)\ z(\alpha) = z(\beta) = 0, \quad (2)\ z(\alpha) > 0,\ z(\beta) = 0.$$

For case (1). From $z''(t) \geq 0$, and $z(\alpha) = z(\beta) = 0$ we derive that $z(t) \leq 0,\ t \in [\alpha, \beta]$, which is in contradiction with $u_2(t^*) > u_1(t^*)$.

For case (2). In this case $\alpha = 0,\ z'(t^*) = 0$. Since $z'(t)$ is increasing on $[\alpha, \beta]$, thus, $z'(t) \geq 0, t \in [t^*, \beta]$, that is, $z(t)$ is increasing on $[t^*, \beta]$. From $z(\beta) = 0$, we see $z(t^*) \leq 0$, which is in contradiction with $u_2(t^*) > u_1(t^*)$.

This completes the proof of theorem 1.3.

5. Concerned remarks and applications

Remark 1. This paper generalizes the results of the concrete functions

$$f(t, u) = p(t)u^{\lambda}$$

in [11] to a class of functions which possess the abstract property (H). Most of the proof in the known literature are not suitable for the class of functions in our paper, and the conclusions in this paper are not weakened, on the contrary, they are strengthened. Therefore this paper is essentially difficult.

Remark 2. Suppose $a_i(t)\ (i = 0, 1, 2, \cdots, m)$ are nonnegative continuous functions on (0,1), which may be unbounded at end points of (0,1). F is the set of functions f(t,u) which satisfy the condition (H). Then we have the following conclusions:

(1) $a_i(t) \in F,\ u^{-\lambda} \in F,$ where $0 < \lambda < \infty$;

(2) If $0 < b_i < +\infty\ (i = 1, 2, \cdots, m),\ then\ [a_0(t) + \sum_{i=1}^{m} a_i(t)u^{-b_i}] \in F$;

(3) if $\sum_{i=0}^{m} a_i(t) > 0,\ t \in J,\ then\ [a_0(t) + \sum_{i=1}^{m} a_i(t)u^{b_i}]^{-1} \in F$;

(4) If $f(t, u) \in F,\ then\ a_i(t)f(t, u) \in F$;

(5) If $f_i(t, u) \in F,\ (i = 1, 2, \cdots, m),\ then\ \max_{1 \leq i \leq m} \{f_i(t, u)\} \in F$;

(6) If $f_i(t, u) \in F,\ (i = 1, 2, \cdots, m),\ then\ \min_{1 \leq i \leq m} \{f_i(t, u)\} \in F$.

The above five facts can be verified directly. This indicates that functions which satisfy the condition (H) are rather wide.

Using the results obtained in above section, we study the problem

$$u''(t) + \sum_{i=1}^{n} a_i(t)u^{-a_i}(t) = 0,\ 0 < t < 1. \tag{5.1}$$

with the boundary condition (1.2), where $a_i > 0,\ a_i(t)\ (i = 1, 2 \cdots n)$ are nonnegative continuous functions (may be unbounded) on J and

$$\sum_{i=1}^{n} a_i(t) > 0,\ t \in J,$$

we have

(i) a necessary and sufficient conditions for (1.1) and (1.2)to have $C(I)$ positive solutions is that

$$\int_0^1 t(1-t)a_i(t)dt < +\infty,\ i = 1, 2, \cdots, n; \tag{5.2}$$

(ii) a necessary and sufficient condition for (1.1) and (1.2) to have $C^1(I)$ positive solution is that

$$\int_0^1 a_i(t)(1-t)^{-a_i}dt < +\infty,\ i = 1, 2, \cdots, n; \tag{5.3}$$

(iii) if $u^*(t)$ is a $C(I)$ positive solution to (1.1) and (1.2), then it is unique.

Remark 3. Consider (1.1) and the singular three-point boundary value problem

$$u(0) = 0, \quad u(1) = au(\eta). \tag{5.4}$$

By analogous methods, we have the following results.

Assume that $u(t)$ is a $C(I)$ positive solution to (1.1) and (5.4). Then $u(t)$ can be stated

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds + \frac{at}{1-a\eta}\int_0^1 G(\eta,s)f(s,u(s))ds.$$

Theorem A *Suppose (H) holds, then a necessary and sufficient condition for problem (1.1) and (5.4) to have C[0,1] positive solutions is that*

$$0 < \int_0^1 t(1-t)f(t,1)dt < +\infty.$$

Theorem B *Suppose (H) holds, then a necessary and sufficient condition for problem (1.1) and (5.4) to have $C^1[0,1]$ positive solutions is that*

$$0 < \int_0^1 f(t,t)dt < +\infty.$$

Theorem C *Suppose (H) holds, then the $C[0,1]$ positive solutions of (1.1) (5.4) is unique.*

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