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SOME PROPERTIES OF CERTAIN MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. The main object of the present paper is to investigate some interesting properties of certain meromorphically multivalent functions associated with a linear operator $L_p(a, c)$.

1. Introduction and preliminaries

Let $\sum_p$ denote the class of meromorphically multivalent functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the punctured unit disk

$$U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}.$$

For functions $f \in \sum_p$ given by (1.1) and $g \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in N), \tag{1.2}$$

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we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.3}$$

In terms of the Pochhammer symbol (or the shifted factorial) $(\lambda)_n$ given by

$$(\lambda)_0 = 1 \quad \text{and } (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad (n \in N), \tag{1.4}$$

we now define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \tag{1.5}$$

$$(z \in U^*; a \in R; c \in R \setminus Z_0^-; Z_0^- = \{0, -1, -2, \cdots\}).$$

Corresponding to the function $\phi_p(a, c; z)$, we introduce here a linear operator $L_p(a, c)$ which is defined by means of the following Hadamard product (or convolution):

$$L_p(a, c)f(z) = \phi_p(a, c; z) * f(z) \quad (f \in \Sigma_p). \tag{1.6}$$

It is easily verified from the definitions (1.5) and (1.6) that

$$z(L_p(a, c)f(z))' = aL_p(a+1, c)f(z) - (a+p)L_p(a, c)f(z). \tag{1.7}$$

The definition (1.6) of the linear operator $L_p(a, c)$ was first introduced and investigated by Liu and Srivastava [3]. A linear operator $L_p(a, c)$, analogous to $L_p(a, c)$ defined here, was considered earlier by Saitoh [7] on the space of analytic and p-valent functions in U. We remark in passing that a much more general convolution operator than the operator $L_p(a, c)$ considered by Saitoh [7], involving the generalized hypergeometric function in the defining Hadamard product (or convolution), was introduced and studied recently by Dziok and Srivastava [1,2].

Given two functions $f(z)$ and $g(z)$, which are analytic in U, we say that the function $g(z)$ is subordinate to $f(z)$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $g(z) = f(w(z))$ $(z \in U)$. In particular, if $f(z)$ is univalent in U, we have the following equivalence

$$g(z) \prec f(z) \quad (z \in U) \Longleftrightarrow g(0) = f(0) \quad \text{and} \quad g(U) \subset f(U).$$

Further, we define a function $H(z)$ by

$$H(z) = (1 - \lambda(a+p+1))L_p(a, c)f(z) + \lambda a L_p(a+1, c)f(z) \tag{1.8}$$

for $f \in \sum_p$, $\lambda > 0$, $a \in R$ and $c \in R \setminus Z_0^-$.

We shall need the following lemmas.

Lemma 1.1. ([4]) *Let $h(z)$ be convex univalent in U, $h(0) = 1$, and let $g(z) = 1 + b_1 z + \cdots$ be analytic in U. If*

$$g(z) + \frac{1}{c} z g'(z) \prec h(z),$$

then for $c \neq 0$ and $Rec \geq 0$

$$g(z) \prec \frac{c}{z^c} \int_0^z t^{c-1} h(t) dt.$$

Lemma 1.2. ([5,6]) *Let a function $p(z) = 1 + c_1 z + \cdots$ be analytic in U and $p(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$ such that*

$$|argp(z)| < \pi\gamma/2 \quad (|z| < |z_0|) \qquad and \qquad |argp(z_0)| = \pi\gamma/2 \quad (0 < \gamma \leq 1),$$

then we have $z_0 p'(z_0)/p(z_0) = ik\gamma$, where

$$\begin{aligned} k &\geq \frac{1}{2}(a + \frac{1}{a}) \quad (\text{ where } argp(z_0) = \pi\gamma/2), \\ k &\leq -\frac{1}{2}(a + \frac{1}{a}) \quad (\text{ where } argp(z_0) = -\pi\gamma/2), \end{aligned}$$

and $(p(z_0))^{1/\gamma} = \pm ia$ $(a > 0)$.

In this paper, we shall derive several interesting properties of $H(z)$ defined by (1.8).

2. Main results

Theorem 2.1. *Let $f \in \sum_p$ and let $H(z)$ be defined by (1.8). If*

$$\frac{H^{(j)}(z)}{(-1)^j z^{-p-j}} \prec (1 - \lambda - \lambda p)(p)_j \frac{1 + Az}{1 + Bz}, \tag{2.1}$$

then

$$\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^j z^{-p-j}} \prec \frac{(1 - \lambda - \lambda p)(p)_j}{\lambda} \int_0^1 u^{(1-\lambda-\lambda p)/\lambda - 1} \left(\frac{1 + Auz}{1 + Buz} \right) du, \tag{2.2}$$

where $j \geq 0$, $\lambda > 0$, $|B| \leq 1$ and $A \neq B$.

Proof. From (1.7) and (1.8), we have

$$\begin{aligned} H^{(j)}(z) &= (1 - \lambda(a + p + 1))(L_p(a,c)f(z))^{(j)} + \lambda a (L_p(a+1,c)f(z))^{(j)} \\ &= (1 - \lambda + \lambda j)(L_p(a,c)f(z))^{(j)} + \lambda z (L_p(a,c)f(z))^{(j+1)}. \end{aligned} \tag{2.3}$$

Putting

$$g(z) = \frac{1}{(p)_j} \cdot \frac{(L_p(a,c)f(z))^{(j)}}{(-1)^j z^{-p-j}} \tag{2.4}$$

for $f \in \sum_p$, we see that $g(z) = 1 + b_1 z + \cdots$ is analytic in U. Note that

$$\frac{H^{(j)}(z)}{(-1)^j z^{-p-j}} = (1-\lambda-\lambda p)(p)_j \left(g(z) + \frac{\lambda}{1-\lambda-\lambda p)} zg'(z) \right). \tag{2.5}$$

Then by (2.1), we obtain

$$g(z) + \frac{\lambda}{1-\lambda-\lambda p} zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Since $h(z) = (1+Az)/(1+Bz)$ is convex univalent in U, an application of Lemma 1 yields

$$g(z) \prec \frac{1-\lambda-\lambda p}{\lambda} z^{-(1-\lambda-\lambda p)/\lambda} \int_0^z t^{(1-\lambda-\lambda p)/\lambda - 1} \left(\frac{1+At}{1+Bt} \right) dt.$$

This proves (2.2). □

Theorem 2.2. *Let $f \in \sum_p$ and let $H(z)$ be defined by (1.8). If*

$$\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^j z^{-p-j}} \prec (p)_j \frac{1+(1-2\alpha)z}{1-z} \quad (z \in U), \tag{2.6}$$

then

$$\frac{H^{(j)}(z)}{(-1)^j z^{-p-j}} \prec (1-\lambda-\lambda p)(p)_j \frac{1+(1-2\alpha)z}{1-z} \quad (|z| < \rho), \tag{2.7}$$

where $j \geq 0, 0 \leq \alpha < 1$, $0 < \lambda < 1/(p+1)$ and

$$\rho = \left[1 + \left(\frac{\lambda}{1-\lambda-\lambda p} \right)^2 \right]^{1/2} - \frac{\lambda}{1-\lambda-\lambda p}. \tag{2.8}$$

The bound $\rho \in (0,1)$ is best possible.

Proof. Put

$$\varphi(z) = (1-\beta)\frac{z}{1-z} + \beta \frac{z}{(1-z)^2} \quad (z \in U),$$

where $\beta = \lambda/(1-\lambda-\lambda p) > 0$ for $0 < \lambda < 1/(p+1)$. We now show that

$$Re \left\{ \frac{\varphi(\rho z)}{\rho z} \right\} > \frac{1}{2} \quad (z \in U), \tag{2.9}$$

where $\rho = (1+\beta^2)^{1/2} - \beta$ and $0 < \rho < 1$.

Let $1/(1-z) = Re^{i\theta}$ and $|z| = r < 1$. In view of

$$cos\theta = \frac{1+R^2(1-r)}{2R}, \quad R \geq \frac{1}{1+r},$$

we have

$$\begin{aligned} 2Re\left\{\frac{\varphi(z)}{z}-\frac{1}{2}\right\} &= 2(1-\beta)Rcos\theta+2\beta R^2cos2\theta-1 \\ &= R^4\beta(1-r^2)^2+R^2((1-\beta)(1-r^2)-2\beta r^2) \\ &\geq R^2(\beta(1-r)^2+(1-\beta)(1-r^2)-2\beta r^2) \\ &= R^2(1-2\beta r-r^2)>0 \end{aligned}$$

for $|z|=r<\rho$, which gives (2.9). Thus the function φ has the integral representation

$$\frac{\varphi(\rho z)}{\rho z}=\int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z\in U), \tag{2.10}$$

where $\mu(x)$ is a probability measure on $|x|=1$.

Now putting

$$g(z)=\frac{1}{(p)_j}\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^jz^{-p-j}},$$

we see that $g(z)=1+b_1z+\cdots$ is analytic in U and it follows from (2.6) that

$$Reg(z)>\alpha \quad (0\leq\alpha<1; z\in U). \tag{2.11}$$

Since we can write

$$g(z)+\beta zg'(z)=\left(\frac{\varphi(z)}{z}\right)*g(z),$$

it follows from (2.10) and (2.11) that

$$\begin{aligned} Re\{g(\rho z)+\beta\rho zg'(\rho z)\} &= Re\left\{\left(\frac{\varphi(\rho z)}{\rho z}\right)*g(z)\right\} \\ = Re\left\{\int_{|x|=1}g(xz)d\mu(x)\right\} &> \alpha \quad (z\in U). \end{aligned} \tag{2.12}$$

Thus, from (2.5) in the proof of Theorem 1 and (2.12), we conclude that (2.7) holds.

To show that the bound ρ is sharp we take $f\in\sum_p$ defined by

$$\frac{1}{(p)_j}\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^jz^{-p-j}}=\alpha+(1-\alpha)\frac{1+z}{1-z}.$$

Noting that

$$\begin{aligned} \frac{1}{(p)_j(1-\lambda-\lambda p)}\frac{H^{(j)}(z)}{(-1)^jz^{-p-j}} &= \alpha+(1-\alpha)\frac{1+z}{1-z}+\beta(1-\alpha)z\left(\frac{1+z}{1-z}\right)' \\ &= \alpha+(1-\alpha)\frac{1+2\beta z-z^2}{(1-z)^2} \\ &= \alpha \end{aligned}$$

for $z = \rho e^{i\pi}$, the proof is completed. □

Theorem 2.3. *Let $f \in \sum_p$ and let $H(z)$ be defined by (1.8). If*

$$\left| arg\left(\frac{H^{(j)}(z)}{(-1)^j z^{-p-j}}\right)\right| < \frac{\pi}{2}\gamma \quad (z \in U), \tag{2.13}$$

then

$$\left| arg\left(\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^j z^{-p-j}}\right)\right| < \frac{\pi}{2}\gamma \quad (z \in U), \tag{2.14}$$

where $0 < \gamma \leq 1$,$j \geq 0$ and $0 < \lambda < 1/(p+1)$.

Proof. Let

$$g(z) = \frac{1}{(p)_j}\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^j z^{-p-j}}$$

for $f \in \sum_p$. Then $g(z) = 1 + b_1 z + \cdots$ is analytic in U. Suppose that there exists a point $z_0 \in U$ such that

$$|argg(z)| < \frac{\pi}{2}\gamma \quad (|z| < |z_0|) \quad \text{and } |argg(z_0)| = \frac{\pi}{2}\gamma.$$

Then, By Lemma 2, we can write that $z_0 g'(z_0)/g(z_0) = ik\gamma$ and $(g(z_0))^{1/\gamma} = \pm ia$ $(a > 0)$.
Therefore, if $argg(z_0) = \pi\gamma/2$, then by (2.5)

$$\begin{aligned}\frac{H^{(j)}(z_0)}{(-1)^j z_0^{-p-j}} &= (1-\lambda-\lambda p)(p)_j g(z_0)\left(1 + \frac{\lambda}{1-\lambda-\lambda p}\frac{z_0 g'(z_0)}{g(z_0)}\right)\\ &= (1-\lambda-\lambda p)(p)_j a^\gamma e^{i\pi\gamma/2}\left(1 + \frac{\lambda}{1-\lambda-\lambda p}\cdot ik\gamma\right).\end{aligned}$$

This implies that

$$\begin{aligned}arg\left(\frac{H^{(j)}(z_0)}{(-1)^j z_0^{-p-j}}\right) &= \frac{\pi}{2}\gamma + arg\left(1 + \frac{k\lambda\gamma i}{1-\lambda-\lambda p}\right)\\ &= \frac{\pi}{2}\gamma + tan^{-1}\left(\frac{k\lambda\gamma}{1-\lambda-\lambda p}\right)\\ &\geq \frac{\pi}{2}\gamma \quad (\text{ where } k \geq \frac{1}{2}(a + \frac{1}{a}) \geq 1),\end{aligned}$$

which contradicts the condition (2.13).

Similarly, if $argg(z_0) = -\pi\gamma/2$, then we obtain that

$$arg\left(\frac{H^{(j)}(z_0)}{(-1)^j z_0^{-p-j}}\right) \leq -\frac{\pi}{2}\gamma,$$

which also contradicts the condition (2.13).
Thus, the function $g(z)$ has to satisfy $|argg(z)| < \pi\gamma/2$ $(z \in U)$.
This show that

$$\left| arg\left(\frac{(L_p(a,c)f(z))^{(j)}}{(-1)^j z^{-p-j}}\right)\right| < \frac{\pi}{2}\gamma \quad (z \in U).$$

The proof is now complete. □

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References

[1] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103** (1999), 1–13.

[2] J. Dziok and H. M. Srivastava, *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct. **14** (2003), 7–18.

[3] J. -L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. **259** (2001), 566–581.

[4] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.

[5] M. Nunokawa, *On properties of non-caratheodory functions*, Proc. Japan Acad. Ser.A, Math. Sci. **68** (1992), 152–153.

[6] M. Nunokawa et al., *Some results for strongly starlike functions*, J. Math. Anal. Appl. **212** (1997), 98–106.

[7] H. Saitoh, *A linear operator and its applications of first order differential subordinations*, Math. Japon. **44** (1996), 31–38.

[8] D. G. Yang, *Some properties of certain analytic functions*, Soochow J. Math. **23** (1997), 63–70.