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ON THE CONVERGENCE OF NOOR ITERATION PROCESS FOR ZAMFIRESCU MAPPING IN ARBITRARY BANACH SPACES

Samir Dashputre[1] and S. D. Diwan[2]

[1]Department of Applied Mathematics,
Shri Shankaracharya College of Engg. and Tech,
Junwani, Bhilai, 490020, India
e-mail: samir_nmrg@yahoomail.com

[2]Department of Applied Mathematics,
Shri Shankaracharya College of Engg. and Tech,
Junwani, Bhilai, 490020, India
e-mail: sdd11@rediffmail.com

Abstract. In this paper, convergence theorem of Rafiq [16], regarding to approximation of fixed point of Zamfirescu mappings in arbitrary Banach spaces using Mann iteration process with errors in the sense of Liu [10] is extended to Noor iteration process with errors in the sense of [15], [14]. Our result generalizes and improves corresponding results of Berinde [2], [3], Rhoades ([18], [19]), Rafiq [16] in arbitrary Banach spaces.

1. Introduction

Let C be a nonempty subset of a metric space (X, d) and T be a mapping from C into itself. Then T is said to be

(i) *contraction* [1] if there exists a number $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq k\ d(x, y)\ for\ all\ x, y \in C. \tag{1.1}$$

(ii) *Kannan type mapping* [8] if there exists a number $k \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k\ \{d(x, Tx) + d(y, Ty)\}\ for\ all\ x, y \in C. \tag{1.2}$$

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(iii) *Chatterjee type mapping* [4] if there exists a number $k \in (0, \frac{1}{2})$

$$d(Tx, Ty) \leq k\{d(x, Ty) + d(y, Tx)\} \; for \; all \; x, y \in C. \tag{1.3}$$

Banach [1] introduced the first contractive definition in a complete metric space in the year 1922, which is known as Banach contraction principle and it states that the contractive mapping has a unique fixed point, which can be reached from any starting value x_0 in the space. In 1968, Kannan [8] proved a fixed point theorem for a discontinuous mapping. Following Kannan's theorem a lot of papers are devoted to obtain various class of contractive mappings (see e.g., [2], [3], [4], [5], [6], [8], [12], [17], [21], [22]). Although the mappings in these papers more general than either Banach's [1] or Kannan's [8] contractive mappings.

It is known, (see Rhoades [20]) that (1.1) and (1.2), (1.1) and (1.3) respectively are independent contractive conditions.

In this sequel Zamfirescu [23] combined contractive conditions of Banach [1], Kannan [8] and Chatterjee [4] and proved the following theorem:

Theorem 1.1. (Zamfirescu [23]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b, c satisfying $0 < a < 1,\ 0 < b, c < \frac{1}{2}$ such that for all pair of $x, y \in X$ at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b\{d(x, Tx) + d(y, Ty)\}$,

(z_3) $d(Tx, Ty) \leq c\{d(x, Ty) + d(y, Tx)\}$.

Then T has a unique fixed point p and Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \; for \; all \; n \in N, \tag{1.4}$$

converges to p for any arbitrary but fixed $x_0 \in X$.

Remark 1.2. A mapping T which satisfies the contractive condition in Theorem 1.1 will be called a Zamfirescu mapping (Z-mapping, Z-operator). The class of Zamfirescu mapping is one of the most studied class of quasi-contractive type mappings. In this class all important fixed point iteration processes i.e. the Picard, Mann [13], Ishikawa [7] are known to converge to unique fixed point of T.

On the other hand, the following iteration processes have been extensively studied by many authors for approximating either fixed points of nonlinear mappings (when these mappings are already known to have fixed point) or solutions of nonlinear operators.

(**a**) **The Mann iteration process** (Mann [13]) which is defined as follows: For C a convex subset of a Banach space X and T a mapping of C into itself, the sequence $\{x_n\}$ in C is defined by:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \end{cases} \tag{1.5}$$

where

(i) $0 \leq \alpha_n < 1, n \geq 0,$

(ii) $\lim\limits_{n\to\infty} \alpha_n = 0,$

(iii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty.$

(**b**) **The Ishikawa iteration process** (Ishikawa [7]) which is defined as follows: With C and T as in (a), the sequence $\{x_n\}$ is defined by:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(iv) $0 \leq \alpha_n \leq \beta_n < 1, n \geq 0,$

(v) $\lim\limits_{n\to\infty} \beta_n = 0,$

(vi) $\sum\limits_{n=1}^{\infty} \alpha_n\beta_n = \infty.$

(**c**) **The Noor iteration process with errors**(Noor [15],[14]) which is defined as follows: With C and T as in (a), the sequence $\{x_n\}$ is defined by:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T z_n + c'_n v_n, \\ z_n = a''_n x_n + b''_n T x_n + c''_n w_n \quad n \geq 0, \end{cases} \tag{1.7}$$

where $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are arbitrary bounded sequences in C and $\{a_n\}, \{b_n\}$, $\{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}$ $\{a''_n\}, \{b''_n\}$ and $\{c''_n\}$ are six real sequences in $[0,1]$ such that

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$, for all $n \geq 0$.

It is clear that, Noor, Ishikawa, Mann iteration sequences are all special case of Noor, Ishikawa and Mann iteration sequences with errors, respectively (see Noor [15], [14]).

Rhoades ([18], [19]) proved the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu mappings in uniformly convex Banach spaces. Berinde ([2], [3]) and Rafiq [16] improve and generalize the results of Rhoades ([18], [19]) by extending it from uniformly convex Banach spaces to arbitrary Banach spaces via the Mann, the Ishikawa and the Mann iteration process with errors in the sense of Liu [10] respectively.

In this view, we have the following Question:

Question 1.3. *Can the Ishikawa iteration process be replace by that of Noor iteration with error for Zamfirescu mapping in an arbitrary Banach space?*

Our purpose in this paper is to study the convergence of Noor iteration process with errors for Zamfirescu mappings in arbitrary Banach spaces. Our result extends Theorem 4 of Rhoades [18], Theorem 8 of Rhoades [19], Theorems 1 and 2 of Berinde [3] and Theorem 3 of Rafiq [16].

2. Preliminaries

For our main result we need the following lemmas:

Lemma 2.1. [9] *Let C be a nonempty subset of Banach space X and $T : C \to C$ a Zamfirescu mapping. Then*

$$\|Tx - Ty\| \quad \leq \quad \delta\|x - y\| + 2\delta\|x - Tx\|, \tag{2.1}$$

holds for all $x, y \in C$, where $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$.

Lemma 2.2. (Liu [11]) *Suppose that $\{\rho_n\}_{n\geq 0}, \{w_n\}_{n\geq 0}$, and $\{t_n\}_{n\geq 0}$ are non-negative sequences such that*

$$\rho_{n+1} \leq (1 - w_n)\rho_n + t_n w_n \ for\ all\ n \geq 0$$

where $\{w_n\}_{n\geq 0} \subset [0, 1], \sum_{n=0}^{\infty} w_n = \infty$ and $\lim_{n\to\infty} t_n = 0$. Then $\lim_{n\to\infty} \rho_n = 0$.

3. Main result

Theorem 3.1. *Let C be a nonempty closed convex subset of an arbitrary Banach space X. Let $T : C \to C$ be a Z mapping. Suppose that $\{u_n\}, \{v_n\}$ and $\{w_n\}$ are arbitrary bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{c_n^{'}\}, \{a_n^{"}\}, \{b_n^{"}\}$ and $\{c_n^{"}\}$ are six sequences in $[0, 1]$ satisfying:*

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{"} + b_n^{"} + c_n^{"} = 1,$

(ii) $c_n(1 - r_n) = r_n b_n, \forall n \geq 0,$

(iii) $\lim_{n\to\infty} c_n^{'} = \lim_{n\to\infty} c_n^{"} = \lim_{n\to\infty} r_n = 0$,

(iv) $\sum_{n=0}^{\infty} \alpha_n = \infty$, *where* $\alpha_n = b_n + c_n$,

(v) $\beta_n = b_n^{'} + c_n^{'}$ *and* $\gamma_n = b_n^{"} + c_n^{"}$.

The sequence $\{x_n\}$ *is defined by (1.7) converges strongly to a unique fixed point of* T.

Proof. By Theorem 1.1, we know that T has unique fixed point in C, say $p \in C$. Using Lemma 2.1 and (1.7), we have

$$\begin{aligned}
\|z_n - p\| &= \|a_n^{"}x_n + b_n^{"}Tx_n + c_n^{"}w_n - p\| \\
&= \|a_n^{"}(x_n - p) + b_n^{"}(Tx_n - p) + c_n^{"}(w_n - p)\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| + c_n^{"}\|w_n - Tx_n\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\delta\|x_n - p\| + c_n^{"}M_1 \\
&\leq (1-\gamma_n(1-\delta))\|x_n - p\| + c_n^{"}M_1, \qquad (3.1)
\end{aligned}$$

for some $M_1 \geq 0$. Using (1.7), (3.1) and Lemma 2.1, we have

$$\begin{aligned}
\|y_n - p\| &= \|a_n^{'}x_n + b_n^{'}Tz_n + c_n^{'}v_n - p\| \\
&= \|a_n^{'}(x_n - p) + b_n^{'}(Tz_n - p) + c_n^{'}(v_n - p)\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|Tz_n - p\| + c_n^{'}\|w_n - Tz_n\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\delta\|z_n - p\| + c_n^{'}M_2 \\
&\leq (1-\beta_n)\|x_n - p\| + c_n^{'}M_2 \\
&\quad +\beta_n\delta((1-\gamma_n(1-\delta))\|x_n - p\| + c_n^{"}M_1) \\
&\leq (1-\beta_n + \beta_n\delta_n(1-\gamma_n(1-\delta))\|x_n - p\| + \beta_n\delta c_n^{"}M_1 + c_n^{'}M_2 \\
&\leq (1-\beta_n + \beta_n\delta - \beta_n\gamma_n\delta(1-\delta))\|x_n - p\| \\
&\quad +\beta_n\delta c_n^{"}M_1 + c_n^{'}M_2 \\
&\leq [1-\beta_n(1-\delta)(1+\gamma_n\delta)]\|x_n - p\| + c_n^{'}M_2 + \beta_n\delta c_n^{"}M_1 \qquad (3.2)
\end{aligned}$$

for some $M_2 \geq 0$. Using (1.7), (3.2) and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n x_n + b_n T y_n + c_n u_n - p\| \\
&= \|a_n(x_n - p) + b_n(Ty_n - p) + c_n(u_n - p)\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|Ty_n - p\| + c_n\|u_n - Ty_n\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\delta\|y_n - p\| + c_n M_3 \\
&\leq (1-\alpha_n)\|x_n - p\| + c_n M_3 \\
&\quad + \alpha_n\delta[1 - \beta_n(1-\delta)(1+\gamma_n\delta)]\|x_n - p\| \\
&\quad + \alpha_n\delta(\beta_n\delta c_n^{"} M_1 + c_n^{'} M_2) \\
&\leq \left[(1-\alpha_n) + \alpha_n\delta[1 - \beta_n(1-\delta)(1+\gamma_n\delta)]\right]\|x_n - p\| \\
&\quad + c_n M_3 + \alpha_n\delta(\beta_n\delta c_n^{"} M_1 + c_n^{'} M_2) \\
&\leq [1 - \alpha_n(1-\delta)(1+\beta_n\delta(1+\gamma_n\delta))]\|x_n - p\| \\
&\quad + c_n M_3 + \alpha_n\delta(\beta_n\delta c_n^{"} M_1 + c_n^{'} M_2) \\
&\leq (1 - \alpha_n(1-\delta))\|x_n - p\| + c_n M_3 \\
&\quad + \alpha_n\delta(\beta_n\delta c_n^{"} M_1 + c_n^{'} M_2). \qquad (3.3)
\end{aligned}$$

Put $\rho_n := \|x_n - p\|$, $w_n := k\alpha_n$, where $k = (1-\delta)$ and $t_n := (\beta_n\delta c_n^{"} M_1 + c_n^{'} M_2 + r_n M_3)k^{-1}$ for all $n \geq 0$. Using Lemma 2.2, $(ii), (iii), (iv)$ and (3.3), we conclude immediately that $\rho \to 0$ as $n \to \infty$. Therefore the sequence $\{x_n\}$ converges strongly to the unique fixed point of T. $\square$

Now we furnish with an example to illustrate Theorem 3.1.

Example 3.2. Let $H = (-\infty, \infty)$ with the usual norm and $C = [0, 1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{2}, & for \quad x \in [0,1) \\ \frac{1}{4} & for \quad x = 1. \end{cases}$$

Clearly $F(T) = \{\frac{1}{2}\}$. Observer that for $x \in [0, 1)$ and $y = 1$, we have

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{1}{2} - \frac{1}{4}\| \\
&= \frac{1}{4} \\
&\leq c\left[\|x - \frac{1}{2}\| + \frac{3}{4}\right]. \qquad (3.4)
\end{aligned}$$

Hence for $x = 0$ and $y = 1$, we get $c = \frac{1}{5}$, therefore, $\delta = \frac{1}{4}$. Hence mapping T satisfies Zamfirescu mapping.

A prototype for $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$, $\{c_n^{'}\}$, $\{a_n^{"}\}, \{b_n^{"}\}$ and $\{c_n^{"}\}$ in our theorem is

$$\begin{aligned}
a_n &= 1 - (n+1)^{\frac{-1}{4}}, \\
b_n &= (n+1)^{\frac{-1}{4}} - (n+1)^{\frac{-1}{2}}, \\
c_n &= (n+1)^{\frac{-1}{2}}, \\
r_n &= (n+1)^{\frac{-1}{4}}, \\
a_n^{'} &= a_n^{"} = \frac{1}{n+2}, \\
b_n^{'} &= b_n^{"} = \frac{n}{n+2}, \\
c_n^{'} &= c_n^{"} = \frac{1}{n+2}.
\end{aligned}$$

It is easy to verify that all the condition of Theorem 3.1 are fulfilled, that mean sequence generated from (1.7) is converges strongly to unique fixed point of T.

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