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THE NONCOMMUTATIVE $\ell_1 - \ell_2$ INEQUALITY FOR HILBERT C*-MODULES AND THE EXACT CONSTANT

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Abstract. Let $\mathcal{A}$ be a unital C*-algebra. Then it follows that

$$\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}} \leq \sqrt{n}\left(\sum_{i=1}^{n}a_ia_i^*\right)^{\frac{1}{2}}, \quad \forall n \in \mathbb{N}, \ \ \forall a_1, \dots, a_n \in \mathcal{A}.$$

By modifications of arguments of Botelho-Andrade, Casazza, Cheng, and Tran given in 2019, for certain n-tuple $x = (a_1, \dots, a_n) \in \mathcal{A}^n$, we give a method to compute a positive element c_x in the C*-algebra $\mathcal{A}$ such that the equality

$$\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}} = c_x\sqrt{n}\left(\sum_{i=1}^{n}a_ia_i^*\right)^{\frac{1}{2}}$$

holds. We give an application for the integral of Kasparov. We also derive a formula for the exact constant for the continuous $\ell_1 - \ell_2$ inequality.

1. Introduction

Let $\mathbb{K}$ be the field of real or complex scalars and $x \in \mathbb{K}^n$. Universally known $\ell_1 - \ell_2$ inequality for Hilbert spaces states that $\|x\|_1 \leq \sqrt{n}\|x\|_2$. In 2019,

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Botelho-Andrade et al. [1] gave a characterization which allows to compute a constant c_x, for a given x such that $\|x\|_1 = c_x\sqrt{n}\|x\|_2$. First we recall this result.

Definition 1.1. ([1]) A vector $x = \frac{1}{\sqrt{n}}(c_1, \cdots, c_n) \in \mathbb{K}^n$ is said to be a constant modulus vector if $|c_i| = 1$, for all $i = 1, \cdots, n$.

Theorem 1.2. ([1]) *Let $x = (a_1, \cdots, a_n) \in \mathbb{K}^n$. The following are equivalent.*

(i)
$$\|x\|_1 = \left(1 - \frac{c_x}{2}\right)\sqrt{n}\|x\|_2.$$

(ii)
$$\sum_{i=1}^{n} \left| \frac{|a_i|}{\|x\|_2} - \frac{1}{\sqrt{n}} \right|^2 = c_x.$$

(iii) *The infimum of the distance from $\frac{x}{\|x\|_2}$ to the constant modulus vector is $\sqrt{c_x}$.*

In particular,

$$\|x\|_1 \leq \sqrt{s}\|x\|_2 \iff \left(1 - \frac{c_x}{2}\right)\sqrt{n} \leq \sqrt{s} \iff 1 - \frac{c_x}{2} \leq \sqrt{\frac{s}{n}}.$$

Theorem 1.2 says that as long as we have equality connecting one-norm and two-norm, the constant can be determined using two-norm and the dimension of space. Further, it also helps to find the distance between $\frac{x}{\|x\|_2}$ to certain types of vectors (constant modulus vectors). This result is useful in nonlinear diffusion and diffusion state distances [2, 6]. A variation of Theorem 1.2 which concerns subspaces is the following.

Theorem 1.3. ([1]) *Let W be a subspace of $\mathbb{K}^n$ and let $P : \mathbb{K}^n \to W$ be onto orthogonal projection. Then the followings are equivalent.*

(i) *For every unit vector $x \in W$, $\|x\|_1 \leq \left(1 - \frac{c_x}{2}\right)\sqrt{n}$.*

(ii) *The distance of any unit vector in W to any constant modulus vector $x \in W$ is greater than or equal to $\sqrt{c_x}$.*

(iii) *For every constant modulus vector $x \in W$, $\|Px\|_2 \leq 1 - \frac{c_x}{2}$.*

We organized this paper as follows. In Section 2, we obtain a result (Theorem 2.1), which is similar to first two implications of Theorem 1.2, in the context of Hilbert C*-modules. A partial result is obtained (Proposition 2.4) which corresponds to (iii) in Theorem 1.2. In Section 3, we derive results which are similar to Theorems 1.2 and 1.3, namely Theorems 3.2 and 3.4, respectively, for the function space $\mathcal{L}^2(X)$ whenever $\mu(X) < \infty$.

2. The noncommutative $\ell_1 - \ell_2$ inequality for Hilbert C*-modules and the exact constant

Let $\mathcal{A}$ be a unital C*-algebra. Then the space $\mathcal{A}^n$ becomes (left) Hilbert C*-module over the C*-algebra $\mathcal{A}$ with respect to the inner product

$$\langle x, y\rangle \coloneqq \sum_{i=1}^{n} a_i b_i^*, \quad \forall x = (a_1, \cdots, a_n), y = (b_1, \cdots, b_n) \in \mathcal{A}^n$$

and the norm

$$\|x\| \coloneqq \|\langle x, x\rangle\|^{\frac{1}{2}} = \left\|\sum_{i=1}^{n} a_i a_i^*\right\|^{\frac{1}{2}}, \quad \forall x = (a_1, \cdots, a_n) \in \mathcal{A}^n$$

(see [4, 8] for Hilbert C*-modules). Let $a_1, \cdots, a_n \in \mathcal{A}$ and let

$$x = ((a_1 a_1^*)^{\frac{1}{2}}, \cdots, (a_n a_n^*)^{\frac{1}{2}}), \quad y = (1, \cdots, 1) \in \mathcal{A}^n.$$

By applying the Cauchy-Schwarz inequality in Hilbert C*-modules (Proposition 1.1 in [4]) for this pair we get

$$\left(\sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}}\right)^2 \leq n \sum_{i=1}^{n} a_i a_i^*, \quad \forall\, n \in \mathbb{N}, \ \forall a_1, \cdots, a_n \in \mathcal{A}.$$

By taking C*-algebraic square root (see Theorem 1.4.11 in [5])

$$\sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}} \leq \sqrt{n} \left(\sum_{i=1}^{n} a_i a_i^*\right)^{\frac{1}{2}}, \quad \forall\, n \in \mathbb{N}, \ \forall a_1, \cdots, a_n \in \mathcal{A}. \tag{2.1}$$

We call the Inequality (2.1) as the noncommutative $\ell_1 - \ell_2$ inequality for Hilbert C*-modules. A standard result in C*-algebra is that an element $a \in \mathcal{A}$ is positive if and only if $a = bb^*$ for some $b \in \mathcal{A}$. Thus Inequality (2.1) can also be written as

$$\sum_{i=1}^{n} a_i a_i^* \leq \sqrt{n} \left(\sum_{i=1}^{n} (a_i a_i^*)^2\right)^{\frac{1}{2}}, \quad \forall\, n \in \mathbb{N}, \ \forall a_1, \cdots, a_n \in \mathcal{A}.$$

Note that Inequality (2.1) is the $\ell_1 - \ell_2$ inequality for Hilbert spaces whenever the C*-algebra is the field of scalars.

Theorem 2.1. *Let $x = (a_1, \cdots, a_n) \in \mathcal{A}^n$ be such that $\langle x, x\rangle$ is invertible. Then the followings are equivalent.*

(i)

$$\left(\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\right)\langle x,x\rangle^{\frac{1}{2}}+\langle x,x\rangle^{\frac{1}{2}}\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}=\sqrt{n}\langle x,x\rangle^{\frac{1}{2}}(2-c_x)\langle x,x\rangle^{\frac{1}{2}}.$$

(ii)

$$\sum_{i=1}^{n}\left(\langle x,x\rangle^{\frac{-1}{2}}(a_ia_i^*)^{\frac{1}{2}}-\frac{1}{\sqrt{n}}\right)\left(\langle x,x\rangle^{\frac{-1}{2}}(a_ia_i^*)^{\frac{1}{2}}-\frac{1}{\sqrt{n}}\right)^*=c_x.$$

Proof. We make expansion and see

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\langle x,x\rangle^{\frac{-1}{2}}(a_ia_i^*)^{\frac{1}{2}}-\frac{1}{\sqrt{n}}\right)\left(\langle x,x\rangle^{\frac{-1}{2}}(a_ia_i^*)^{\frac{1}{2}}-\frac{1}{\sqrt{n}}\right)^*\\
&\quad=\langle x,x\rangle^{\frac{-1}{2}}\left(\sum_{i=1}^{n}a_ia_i^*\right)\langle x,x\rangle^{\frac{-1}{2}}+1-\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\\
&\qquad-\left(\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\right)\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}\\
&\quad=2-\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}-\left(\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\right)\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}=c_x
\end{aligned}$$

if and only if

$$\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}+\left(\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\right)\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}=2-c_x$$

if and only if

$$\left(\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\right)\langle x,x\rangle^{\frac{1}{2}}+\langle x,x\rangle^{\frac{1}{2}}\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}=\sqrt{n}\langle x,x\rangle^{\frac{1}{2}}(2-c_x)\langle x,x\rangle^{\frac{1}{2}}.$$

This completes the proof. □

A particular case of Theorem 2.1 which is very similar to Theorem 1.2 is the following.

Corollary 2.2. *Let $x=(a_1,\cdots,a_n)\in\mathcal{A}^n$ be such that $\langle x,x\rangle$ is invertible and commutes with $\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}$. Then*

$$\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}=\left(1-\frac{c_x}{2}\right)\sqrt{n}\langle x,x\rangle^{\frac{1}{2}}=\sqrt{n}\langle x,x\rangle^{\frac{1}{2}}\left(1-\frac{c_x}{2}\right)$$

if and only if

$$\sum_{i=1}^{n}\left(\langle x,x\rangle^{\frac{-1}{2}}(a_ia_i^*)^{\frac{1}{2}}-\frac{1}{\sqrt{n}}\right)\left(\langle x,x\rangle^{\frac{-1}{2}}(a_ia_i^*)^{\frac{1}{2}}-\frac{1}{\sqrt{n}}\right)^*=c_x.$$

In particular,

$$\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\leq\sqrt{s}\langle x,x\rangle^{\frac{1}{2}}\iff\left(1-\frac{c_x}{2}\right)\sqrt{n}\leq\sqrt{s}\iff 1-\frac{c_x}{2}\leq\sqrt{\frac{s}{n}}.$$

We next derive a result which gives one sided implication in Theorem 1.2. For this, we need to generalize Definition 1.1.

Definition 2.3. A vector $x=\frac{1}{\sqrt{n}}(c_1,\cdots,c_n)\in\mathcal{A}^n$ is said to be a constant modulus vector if $c_ic_i^*=1$, for all $i=1,\cdots,n$.

Recall that an element a in a unital C*-algebra is said to be an isometry if $a^*a=1$. Thus a vector is a constant modulus vector if adjoint of each of its coordinates is an isometry upto scalar.

Proposition 2.4. *Let $x=(a_1,\cdots,a_n)\in\mathcal{A}^n$ be such that $a_ia_i^*$ is invertible for each i. Define*

$$c_x:=2-\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}-\left(\sum_{i=1}^{n}(a_ia_i^*)^{\frac{1}{2}}\right)\frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}}.$$

Then the infimum of the distance from $\langle x,x\rangle^{\frac{-1}{2}}x$ to the constant modulus vector is less than or equal to $\sqrt{\|c_x\|}$.

Proof. Note that the condition $a_ia_i^*$ is invertible for each i implies that $\langle x,x\rangle$ is invertible. Now consider the vector $\frac{1}{\sqrt{n}}((a_1a_1^*)^{\frac{-1}{2}}a_1,\cdots,(a_na_n^*)^{\frac{-1}{2}}a_n)$, which is of unit modulus. Using the definition of infimum and by an expansion we get

$$\inf\left\{\|\langle x,x\rangle^{\frac{-1}{2}}x-y\|:y=\frac{1}{\sqrt{n}}(c_1,\cdots,c_n)\in\mathcal{A}^n\text{ is a constant modulus vector}\right\}$$

$$=\inf\left\{\left\|\sum_{i=1}^{n}\left(\langle x,x\rangle^{\frac{-1}{2}}a_i-\frac{c_i}{\sqrt{n}}\right)\left(\langle x,x\rangle^{\frac{-1}{2}}a_i-\frac{c_i}{\sqrt{n}}\right)^*\right\|^{\frac{1}{2}}\right.$$
$$\left.:c_i\in\mathcal{A},c_ic_i^*=1,i=1,\cdots,n\right\}$$

$$\leq\left\|\sum_{i=1}^{n}\left(\langle x,x\rangle^{\frac{-1}{2}}a_i-\frac{(a_ia_i^*)^{\frac{-1}{2}}a_i}{\sqrt{n}}\right)\left(\langle x,x\rangle^{\frac{-1}{2}}a_i-\frac{(a_ia_i^*)^{\frac{-1}{2}}a_i}{\sqrt{n}}\right)^*\right\|^{\frac{1}{2}}$$

$$\leq \left\| \sum_{i=1}^{n} \langle x,x\rangle^{\frac{-1}{2}} a_i a_i^* \langle x,x\rangle^{\frac{-1}{2}} - \frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}} \sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}} - \left(\sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}}\right) \frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}} + 1 \right\|^{\frac{1}{2}}$$

$$= \left\| 2 - \frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}} \sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}} - \left(\sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}}\right) \frac{\langle x,x\rangle^{\frac{-1}{2}}}{\sqrt{n}} \right\|^{\frac{1}{2}} = \|c_x\|^{\frac{1}{2}}.$$

□

Proposition 2.4 and Theorem 1.2 lead to the following question: Does converse of Proposition 2.4 hold? We see that when $n = 1$, $c_x = 0$ and hence converse holds. It is not known that for $n \geq 2$. Next we derive a result which concerns the $\ell_1 - \ell_2$ inequality for submodules of Hilbert C*-modules.

Proposition 2.5. *Let $\mathcal{N}$ be a submodule of $\mathcal{A}^n$ and $x \in \mathcal{N}$ be a vector such that $\langle x, x\rangle = 1$. If the distance of x to the constant modulus vector is greater than or equal to c_x, then*

$$c_x \leq \left\| 2 - \frac{2}{\sqrt{n}} \sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}} \right\|^{\frac{1}{2}}.$$

Proof. By doing a similar calculation as in the proof of Proposition 2.4 we get that

$$c_x \leq \inf\left\{ \|x - y\| : y = \frac{1}{\sqrt{n}}(c_1, \cdots, c_n) \in \mathcal{A}^n \text{ is a constant modulus vector} \right\}$$

$$= \inf\left\{ \left\| \sum_{i=1}^{n} \left(a_i - \frac{c_i}{\sqrt{n}}\right)\left(a_i - \frac{c_i}{\sqrt{n}}\right)^* \right\|^{\frac{1}{2}} : c_i \in \mathcal{A}, c_i c_i^* = 1, i = 1, \cdots, n \right\}$$

$$\leq \left\| 2 - \frac{2}{\sqrt{n}} \sum_{i=1}^{n} (a_i a_i^*)^{\frac{1}{2}} \right\|^{\frac{1}{2}}.$$

□

Again we look at Proposition 2.5 and Theorem 2.4 in [1] which lead to the following question: Does converse of Proposition 2.5 hold?

In the spirit of Theorem 3.1 in [1], we next give an application of the previous result. For this we need some concepts.

Let G be a compact Lie group and μ be the left Haar measure on G such that $\mu(G) = 1$ (see [9]). If $f, g : G \to \mathcal{A}$ are continuous functions, then we define

$$\langle f, g\rangle := \int_G f(x)g(x)^*\, d\mu(x),$$

where the integral is in the sense of Kasparov (see [3, 7]). Now we can state the result.

Theorem 2.6. *Let G be a compact Lie group, $\mu(G) = 1$, $f : G \to \mathcal{A}$ be continuous, $f(x) \geq 0$, for all $x \in G$ and $\langle f, f\rangle = 1$. Then the followings are equivalent.*

(i) $\int_G f(x)\, d\mu(x) = 1 - \frac{c}{2}$.

(ii) $\langle f - 1, f - 1\rangle = c$.

Proof. Consider $4 = \langle f - 1, f - 1\rangle + \langle f + 1, f + 1\rangle = \langle f - 1, f - 1\rangle + 1 + 1 + 2\int_G f(x)\, d\mu(x) = \langle f - 1, f - 1\rangle + 2 + 2\int_G f(x)\, d\mu(x)$ which implies $\langle f - 1, f - 1\rangle = 2 - 2\int_G f(x)\, d\mu(x)$. Conclusion follows by taking $c = \langle f - 1, f - 1\rangle = 2 - 2\int_G f(x)\, d\mu(x)$. □

3. Exact constant for the continuous $\ell_1 - \ell_2$ inequality

Let X be a measure space with finite measure. Continuous Cauchy-Schwarz inequality tells that $\|f\|_1 \leq \sqrt{\mu(X)}\|f\|_2$. Given $f \in \mathcal{L}^2(X)$, we now derive a method for the exact constant in the equality

$$\|f\|_1 = c_f\sqrt{\mu(X)}\|f\|_2.$$

For this, we reform the Definition 1.1.

Definition 3.1. A function $f \in \mathcal{L}^2(X)$ is said to be a constant modulus function if $|f(x)| = \frac{1}{\sqrt{\mu(X)}}$ for all $x \in X$.

Definition 3.1 says that a function is a constant modulus function if its image lies in the circle of radius $\frac{1}{\sqrt{\mu(X)}}$, centered at origin.

Theorem 3.2. *For $f \in \mathcal{L}^2(X)$, the followings are equivalent.*

(i) $\|f\|_1 = \left(1 - \frac{c_f}{2}\right)\sqrt{\mu(X)}\|f\|_2$.

(ii) $\int_X \left|\frac{|f(x)|}{\|f\|_2} - \frac{1}{\sqrt{\mu(X)}}\right|^2 d\mu(x) = c_f$.

(iii) *The infimum of the distance from $\frac{f}{\|f\|_2}$ to the constant modulus function is $\sqrt{c_f}$.*

In particular,

$$\|f\|_1 \leq \sqrt{s}\|f\|_2 \iff \left(1 - \frac{c_f}{2}\right)\sqrt{\mu(X)} \leq \sqrt{s} \iff 1 - \frac{c_f}{2} \leq \sqrt{\frac{s}{\mu(X)}}.$$

Proof. (i) $\iff$ (ii) Starting from the integral in (ii) we see that

$$\begin{aligned}\int_X \left|\frac{|f(x)|}{\|f\|_2} - \frac{1}{\sqrt{\mu(X)}}\right|^2 d\mu(x) &= \frac{1}{\|f\|_2^2}\int_X |f(x)|^2\, d\mu(x) + \frac{1}{\mu(X)}\int_X d\mu(x) \\ &\quad - 2\frac{1}{\|f\|_2\sqrt{\mu(X)}}\int_X |f(x)|\, d\mu(x) \\ &= 2\left(1 - \frac{1}{\|f\|_2\sqrt{\mu(X)}}\int_X |f(x)|\, d\mu(x)\right) = c_f\end{aligned}$$

if and only if

$$\frac{1}{\|f\|_2\sqrt{\mu(X)}}\int_X |f(x)|\, d\mu(x) = 1 - \frac{c_f}{2}$$

if and only if

$$\int_X |f(x)|\, d\mu(x) = \left(1 - \frac{c_f}{2}\right)\|f\|_2\sqrt{\mu(X)}.$$

(i) $\iff$ (iii) This follows from the calculation.

$$\begin{aligned}&\inf\left\{\left\|\frac{f}{\|f\|_2} - g\right\|_2 : g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\} \\ &= \inf\left\{\left(\int_X \left|\frac{f(x)}{\|f\|_2} - g(x)\right|^2 d\mu(x)\right)^{\frac{1}{2}} : \right. \\ &\qquad\qquad \left. g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\} \\ &= \inf\left\{\left(\int_X \left|\frac{f(x)}{\|f\|_2}\right|^2 d\mu(x) + \int_X |g(x)|^2\, d\mu(x) \right.\right. \\ &\qquad\qquad \left.\left. - \frac{2}{\|f\|_2}\mathrm{Re}\left(\int_X f(x)\overline{g(x)}\, d\mu(x)\right)\right)^{\frac{1}{2}} : \right. \\ &\qquad\qquad \left. g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\}\end{aligned}$$

$$= \inf\left\{ \left(1 + 1 - \frac{2}{\|f\|_2}\mathrm{Re}\left(\int_X f(x)\overline{g(x)}\, d\mu(x)\right)\right)^{\frac{1}{2}} : \right.$$
$$\left. g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\}$$
$$= \left(2 - \frac{2}{\sqrt{\mu(X)}\|f\|_2}\int_X |f(x)|\, d\mu(x)\right)^{\frac{1}{2}}.$$

□

To obtain further results we need a result whose proof will follow from the routine argument using Hilbert projection theorem.

Theorem 3.3. *Let $\mathcal{K}$ be a closed subspace of a Hilbert space $\mathcal{H}$ and let $P : \mathcal{H} \to \mathcal{K}$ be a surjective orthogonal projection. Then for each $h \in \mathcal{H}$ with $Ph \neq 0$, $\frac{Ph}{\|Ph\|}$ is the closest unit vector in $\mathcal{K}$ to h.*

We now use Theorem 3.3 to obtain relations between closed subspaces of $\mathcal{L}^2(X)$ and continuous $\ell_1 - \ell_2$ inequality.

Theorem 3.4. *Let W be a closed subspace of $\mathcal{L}^2(X)$ and let $P : \mathcal{L}^2(X) \to W$ be a surjective orthogonal projection. Then the followings are equivalent.*

(i) *For every unit vector $f \in W$, $\|f\|_1 \leq \left(1 - \frac{c_f}{2}\right)\sqrt{\mu(X)}$.*
(ii) *The distance of any unit vector in W to any constant modulus function $f \in W$ is greater than or equal to $\sqrt{c_f}$.*
(iii) *For every constant modulus function $f \in W$, $\|Pf\|_2 \leq 1 - \frac{c_f}{2}$.*

Proof. (i) $\iff$ (ii) We do a similar calculation as in the proof of Theorem 3.2 and get

$$\inf\left\{\|f - g\|_2 : g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\}$$
$$= \inf\left\{ \left(\int_X |f(x) - g(x)|^2\, d\mu(x)\right)^{\frac{1}{2}} : \right.$$
$$\left. g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\}$$
$$= \inf\left\{ \left(\int_X |f(x)|^2\, d\mu(x) + \int_X |g(x)|^2\, d\mu(x) - 2\mathrm{Re}\left(\int_X f(x)\overline{g(x)}\, d\mu(x)\right)\right)^{\frac{1}{2}} : \right.$$
$$\left. g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\}$$

$$= \inf\left\{ \left(1+1-2\mathrm{Re}\left(\int_X f(x)\overline{g(x)}\,d\mu(x)\right)\right)^{\frac{1}{2}} : g \in \mathcal{L}^2(X) \text{ is a constant modulus function}\right\}$$

$$= \left(2 - \frac{2}{\sqrt{\mu(X)}}\int_X |f(x)|\,d\mu(x)\right)^{\frac{1}{2}}.$$

Therefore

$$\sqrt{c_f} \leq \left(2 - \frac{2}{\sqrt{\mu(X)}}\int_X |f(x)|\,d\mu(x)\right)^{\frac{1}{2}}$$

if and only if

$$\|f\|_1 = \int_X |f(x)|\,d\mu(x) \leq \left(1 - \frac{c_f}{2}\right)\sqrt{\mu(X)}.$$

(ii) $\iff$ (iii) Let $f \in W$ be a constant modulus function. In view of Theorem 3.3 we calculate

$$\begin{aligned}\left\|\frac{Pf}{\|Pf\|} - f\right\|^2 &= 1 + 1 - \left\langle \frac{Pf}{\|Pf\|}, f\right\rangle - \left\langle f, \frac{Pf}{\|Pf\|}\right\rangle \\ &= 2 - \left\langle \frac{P^2 f}{\|Pf\|}, f\right\rangle - \left\langle f, \frac{P^2 f}{\|Pf\|}\right\rangle \\ &= 2 - 2\|Pf\|.\end{aligned}$$

Therefore

$$c_f \leq \left\|\frac{Pf}{\|Pf\|} - f\right\| \quad \text{if and only if} \quad \|Pf\| \leq 1 - \frac{c_f}{2}.$$

This completes the proof. □

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