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BEST PROXIMITY POINT OF CYCLIC GENERALIZED φ-WEAK CONTRACTION MAPPING IN METRIC SPACES

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Abstract. The purpose of this paper is to introduce a new generalization class of cyclic mappings, called cyclic generalized φ-weak contraction and obtain a corresponding best proximity point theorem for this cyclic mapping under certain conditions.

1. Introduction

Let (X, d) be a metric space and A be nonempty subsets of X. For a given self-mapping T is defined on A, if there exists x such that $Tx = x$, we say that x is a fixed point of T. A fundamental result in fixed point theory is the Banach contraction principle and it has wide applications in many branches of applied sciences. Banach fixed point theorem states that when (X, d) be a complete metric space and $T : X \to X$ is a contraction, then T has a unique fixed point in X. Also, there are several extensions and generalizations of this principle. One of the interesting extensions was given by Kirk et al. [19] as follows.

Theorem 1.1. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$, $T(A) \subset B$ and $T(B) \subset A$ such that $d(Tx, Ty) \leq kd(x, y)$ for some $k \in (0, 1)$ and for all $x \in A$ and $y \in B$. Then T has a unique fixed point in $A \cap B$.*

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First, we give the definition of cyclic mapping.

Definition 1.2. Let A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is called a cyclic mapping provided that $T(A) \subset B$ and $T(B) \subset A$.

Let T be cyclic mapping. We say that

(1) T is said to be a cyclic contraction ([9]) if

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)d(A, B)$$

for some $\alpha \in (0, 1)$ and for all $x \in A$ and $y \in B$, where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\};$$

(2) $x \in A \cup B$ is a best proximity point of T if $d(x, Tx) = d(A, B)$.

Eldred and Veeramani ([9]) extended Theorem 1.1, to include the case $A \cap B = \emptyset$, by the following existence result of best proximity point.

Theorem 1.3. *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space X and $T : A \cup B \to A \cup B$ be a cyclic contraction mapping. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $\|x - Tx\| = d(A, B)$.*

Recently, several authors presented many results for cyclic mappings satisfying various (nonlinear) contractive conditions based on altering distance function φ which were appeared in the literature [2, 7, 13, 14, 15, 16, 17, 18].

In 2009, Al-Thagafi and Shahzad ([2]) introduced a class of mappings, called cyclic φ-contractions, which contains the cyclic contraction mappings as a subclass (see [3]). For such mappings, they obtained convergence and existence result of best proximity points.

Definition 1.4. ([2]) Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic contraction mapping. The mapping T is said to be a cyclic φ-contraction if $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing mapping and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) \tag{1.1}$$

for all $x \in A$ and $y \in B$.

Remark 1.5. If $F(T)$ is the set of fixed points of a cyclic φ-contraction mapping $T : A \cup B \to A \cup B$, then $F(T) \subset A \cap B$.

Theorem 1.6. ([2]) *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction mapping. For $x_0 \in A \cup B$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then $d(x_n, x_{n+1}) \to d(A, B)$ as $n \to \infty$.*

With the fixed point problem, another obvious problem arises. If T is a nonself-mapping from A to B, where A and B are nonempty subsets of X, solution of equation $Tx = x$ may not exist, particularly when $A \cap B = \emptyset$, then we want to find a solution x^* such that

$$d(Tx^*, x^*) = \min d(Tx, x),$$

where $x \in A$. This is the problem to be solved by best proximity problem. Therefore, the best proximity point problem becomes a hot topic recently.

Motivated by the above mentioned results and the on-going research, the purpose of this paper is to introduce a new generalization class of cyclic mappings, called cyclic generalized φ-weak contraction and obtain a corresponding best proximity point theorem for this cyclic mapping under certain conditions.

2. Preliminaries

To establish our results, we introduce the following new class of mappings.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X, d). A cyclic mapping $T : A \cup B \to A \cup B$ is called a cyclic generalized φ-weak contraction if there exists a strictly increasing mapping $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)) + \varphi(d(A, B)), \quad \forall\, x \in A, y \in B \tag{2.1}$$

holds.

Example 2.2. Let $X = \mathbb{R}$ with Euclidean metric and $A = B = [0, 1]$. Define $T : A \cup B \to A \cup B$ and $\varphi : [0, \infty) \to [0, \infty)$ by

$$Tx = \frac{2}{5}(x - x^2), \quad \varphi(t) = \frac{4}{3}t.$$

For the verification of cyclic generalized φ-weak contraction condition (2.1) the following cases arise: Since $A = B = [0, 1]$, we have $\varphi(d(A, B)) = 0$. So, we get

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= \frac{2}{5}|(x - y) - (x - y)(x + y)| \\ &= \frac{2}{5}|x - y| \cdot |1 - x - y| \\ &\leq |x - y| - |x - y| \cdot |1 - x - y| + \frac{2}{5}|(x - y)| \cdot |1 - x - y| \\ &= |x - y| - \frac{3}{5}|x - y| \cdot |1 - x - y| \end{aligned}$$

$$
\begin{aligned}
&< |x-y| - \frac{8}{15}|x-y|\cdot|1-x-y| \\
&= |x-y| - \frac{4}{3}\cdot\frac{2}{5}|x-y|\cdot|1-x-y| \\
&= d(x,y) - \varphi(d(Tx,Ty)) + \varphi(d(A,B)) \quad \forall\, x,y \in [0,1].
\end{aligned}
$$

But,

$$
\begin{aligned}
d(Tx,Ty) &= |Tx - Ty| \\
&= \frac{2}{5}|(x-y) - (x-y)(x+y)| \\
&= \frac{2}{5}|x-y|\cdot|1-x-y| \\
&\geq -\frac{1}{3}|x-y| \\
&= |x-y| - \frac{4}{3}|x-y| \\
&= d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B)) \quad \forall\, x,y \in [0,1].
\end{aligned}
$$

Therefore, T is not a cyclic φ-contraction mapping.

3. Main results

To establish our results, we needed the following lemma.

Lemma 3.1. *Let A and B be nonempty subsets of a metric space (X,d) and let $T : A\cup B \to A\cup B$ be a cyclic generalized φ-weak contraction mapping. For $x_0 \in A\cup B$, define $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}$. Then*

(1) $-\varphi(d(Tx,Ty)) + \varphi(d(A,B)) \leq 0$ *for all $x \in A$ and $y \in B$,*
(2) $d(Tx,Ty) \leq d(x,y)$ *for all $x \in A$ and $y \in B$,*
(3) $d(x_{n+2},x_{n+1}) = d(Tx_{n+1},Tx_n) \leq d(x_{n+1},x_n)$ *for all $n \geq 0$.*

Proof. (1) Since T is cyclic mapping, that is, $T(A) \subset B$ and $T(B) \subset A$, we have

$$
\begin{aligned}
d(A,B) &= \inf\{d(x,y) : x \in A, y \in B\} \\
&\leq d(Tx,Ty), \quad \forall\, x \in A, y \in B.
\end{aligned}
$$

Since φ is a strictly increasing mapping, we get

$$\varphi(d(A,B)) \leq \varphi(d(Tx,Ty))$$

for all $x \in A$ and $y \in B$. Thus, we have the conclusion (1).

(2) Since T is a cyclic generalized φ-weak contraction mapping, from (2.1) and above property (1), we get

$$d(Tx,Ty) \leq d(x,y), \quad \forall\, x \in A, y \in B.$$

(3) From (2), we have the conclusion (3) immediately. □

Theorem 3.2. *Let A and B be nonempty subsets of a metric space (X,d). Suppose that $T : A\cup B \to A\cup B$ is a cyclic generalized φ-weak contraction and there exists $x_0 \in A$. Define $x_{n+1} = Tx_n$ for any $n \in \mathbb{N}$. Then $d(x_{n+1}, x_n) \to d(A,B)$ as $n \to \infty$.*

Proof. Let $d_n = d(x_{n+1}, x_n)$. First we show that the sequence $\{d_n\}$ is nonincreasing. By the assumption and Lemma 3.1-(1), we have

$$\begin{aligned} d_{n+1} &= d(x_{n+2}, x_{n+1}) \\ &= d(Tx_{n+1}, Tx_n) \\ &\le d(x_{n+1}, x_n) - \varphi(d(x_{n+2}, x_{n+1})) + \varphi(d(A,B)) \qquad (3.1) \\ &\le d(x_{n+1}, x_n) = d_n. \end{aligned}$$

Thus the sequence $\{d_n\}$ is non-increasing and bounded below, it is obvious that $\lim_{n\to\infty} d_n$ exists.

Case I. If $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, obviously, $d_n \to 0$ and $d(A,B) = 0$. That is,

$$d_n \to d(A,B).$$

Case II. If $d_n \neq 0$ for all $n \in \mathbb{N}$. Put $d_n \to \gamma$. Then $\gamma \ge d(A,B)$. Since φ is a strictly increasing mapping, we have

$$\varphi(\gamma) \ge \varphi(d(A,B)). \qquad (3.2)$$

From (3.1), we get

$$\varphi(d(x_{n+2}, x_{n+1})) \le d(x_{n+1}, x_n) - d(x_{n+2}, x_{n+1}) + \varphi(d(A,B)).$$

It follows that

$$\begin{aligned} \varphi(\gamma) &\le \lim_{n\to\infty} \varphi(d(x_{n+2}, x_{n+1})) \\ &\le \varphi(d(A,B)). \qquad (3.3) \end{aligned}$$

Therefore, by (3.2) and (3.3), $\gamma = d(A,B)$, that is,

$$d_n \to d(A,B).$$

This completes the proof. □

Remark 3.3. Theorem 3 of [2] is a special case of Theorem 3.2.

Theorem 3.4. *Let (X,d) be a complete metric space. Suppose that $T : X \to X$ is a cyclic generalized φ-weak contraction and there exists $x_0 \in X$. Define $x_{n+1} = Tx_n$ for any $n \in \mathbb{N}$. Then there exists a unique point $x \in X$ such that $Tx = x$.*

Proof. By the assumption and Theorem 3.2, we have $d(x_{n+1}, x_n) \to 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Suppose $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{n_k}\}$ and $\{x_{n_l}\}$ of $\{x_n\}$ such that n_l is the smallest index for which

$$n_l > n_k > n, \quad d(x_{n_k}, x_{n_l}) \geq \varepsilon, \tag{3.4}$$

from which it follows that

$$d(x_{n_k}, x_{n_l-1}) < \varepsilon. \tag{3.5}$$

From (3.4), (3.5) and the triangular inequality, we get that

$$\begin{aligned} \epsilon &\leq d(x_{n_k}, x_{n_l}) \\ &\leq d(x_{n_k}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_l}) \\ &\leq \varepsilon + d(x_{n_l-1}, x_{n_l}). \end{aligned}$$

Letting $n \to \infty$ in the above inequalities and using $d(x_{n+1}, x_n) \to 0$, we get that

$$\lim_{n\to\infty} d(x_{n_k}, x_{n_l}) = \varepsilon. \tag{3.6}$$

Again, regarding (3.4) and the triangular inequality, we have

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{n_l}) \\ &\leq d(x_{n_l}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_k}) \\ &\leq d(x_{n_l}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq d(x_{n_l}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &= d(x_{n_l}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_k}) + 2d(x_{n_k+1}, x_{n_k}) \\ &\leq d(x_{n_l}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_l}) + d(x_{n_l}, x_{n_k}) + 2d(x_{n_k+1}, x_{n_k}) \\ &= 2d(x_{n_l}, x_{n_l-1}) + d(x_{n_l}, x_{n_k}) + 2d(x_{n_k+1}, x_{n_k}). \end{aligned}$$

Letting $n \to \infty$ in the above inequalities, using $d(x_{n+1}, x_n) \to 0$ and (3.6), we get that

$$\begin{aligned} \lim_{n\to\infty} d(x_{n_k}, x_{n_l}) &= \lim_{n\to\infty} d(x_{n_k}, x_{n_l-1}) \\ &= \lim_{n\to\infty} d(x_{n_k+1}, x_{n_l-1}) = \varepsilon. \end{aligned}$$

Since

$$\begin{aligned} d(x_{n_k+1}, x_{n_l}) &= d(Tx_{n_k}, Tx_{n_l-1}) \\ &\leq d(x_{n_k}, x_{n_l-1}) - \varphi(d(x_{n_k+1}, x_{n_l})), \end{aligned}$$

letting $n \to \infty$ and considering the continuity of φ, we have

$$\varepsilon \leq \varepsilon - \varphi(\varepsilon).$$

Hence $\varphi(\varepsilon) = 0$, so $\varepsilon = 0$. This is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in (X, d). Since X is complete, there exists $x \in X$ such that $x_n \to x$. We have

$$\begin{aligned} d(x, Tx) &\le d(x, x_{n+1}) + d(x_{n+1}, Tx) \\ &= d(x, x_{n+1}) + d(Tx_n, Tx) \\ &\le d(x, x_{n+1}) + d(x_n, x) - \varphi(d(x_{n+1}, Tx)). \end{aligned}$$

Taking $n \to \infty$, we get that

$$d(x, Tx) \le -\varphi(d(x, Tx)).$$

From the condition of φ,

$$d(x, Tx) = 0,$$

that is

$$Tx = x.$$

This shows that x is a fixed point of T. For the uniqueness of fixed point of T, we can suppose that there exists $x^* \in X$ such that $Tx^* = x^*$ but $x^* \neq x$. Since $T : X \to X$ is a cyclic generalized φ-weak contraction, we get

$$\begin{aligned} d(x, x^*) &= d(Tx, Tx^*) \\ &\le d(x, x^*) - \varphi(d(Tx, Tx^*)) \\ &= d(x, x^*) - \varphi(d(x, x^*)), \end{aligned}$$

that is, $\varphi(d(x, x^*)) \le 0$.

By the condition of φ, we have

$$d(x, x^*) = 0.$$

Therefore,

$$x = x^*.$$

This completes the proof. □

Theorem 3.5. *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic generalized φ-weak contraction mapping. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \ge 0$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.*

Proof. Let $\{x_{2n_k}\}$ be a subsequence of sequence $\{x_{2n}\}$ with $x_{2n_k} \to x \in A$. By Lemma 3.1-(2),

$$\begin{aligned} d(A, B) &\le d(x_{2n_k}, Tx) \\ &\le d(x_{2n_k}, Tx_{2n_k}) + d(Tx_{2n_k}, Tx) \\ &\le d(x_{2n_k}, Tx_{2n_k}) + d(x_{2n_k}, x) \end{aligned} \tag{3.7}$$

for each $k \geq 1$. From Theorem 3.2,

$$d(x_{2n_k}, Tx_{2n_k}) = d(x_{2n_k}, x_{2n_{k+1}}) \to d(A, B) \tag{3.8}$$

as $k \to \infty$. Since $x_{2n_k} \to x$, combining (3.7) and (3.8), we obtain

$$d(x, Tx) = \lim_{k\to\infty} d(x_{2n_k}, Tx) = d(A, B).$$

This completes the proof. □

Remark 3.6. Theorem 4 of [2] is a special case of Theorem 3.5.

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