Nonlinear Functional Analysis and Applications
Vol. 27, No. 2 (2022), pp. 271-287
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2022.27.02.05
http://nfaa.kyungnam.ac.kr/journal-nfaa

COMMON FIXED POINT THEOREMS FOR TWO SELF MAPS SATISFYING ξ-WEAKLY EXPANSIVE MAPPINGS IN DISLOCATED METRIC SPACE

Jong Kyu Kim[1], Manoj Kumar[2], Preeti[2], Poonam[3] and Won Hee Lim[4]

[1]Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam, 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

[2]Department of Mathematics, Baba Mastnath University
Asthal Bohar, Rohtak, Haryana, 124021, India
e-mail: manojantil18@gmail.com, preetichhikara2585@gmail.com

[3]Department of Mathematics, Govt. College for Women
Jassaur Kheri, Jhajjar
e-mail: poonam1291988@gmail.com

[4]Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam, 51767, Korea
e-mail: worry36@kyungnam.ac.kr

Abstract. In this article, we shall prove a common fixed point theorem for two weakly compatible self-maps $\mathcal{P}$ and $\mathfrak{Q}$ on a dislocated metric space (M, d^*) satisfying the following ξ-weakly expansive condition:

$$d^*(\mathcal{P}c, \mathcal{P}d) \geq d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)), \quad \forall\, c, d \in M,$$

where

$$\wedge(\mathfrak{Q}c, \mathfrak{Q}d) = \max\Big\{d^*(\mathfrak{Q}c, \mathfrak{Q}d), d^*(\mathfrak{Q}c, \mathcal{P}c), d^*(\mathfrak{Q}d, \mathcal{P}d), \frac{d^*(\mathfrak{Q}c, \mathcal{P}c)\cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathfrak{Q}c, \mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c, \mathcal{P}c)\cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathcal{P}c, \mathcal{P}d)}\Big\}.$$

Also, we have proved common fixed point theorems for the above mentioned weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

[0]Received September 21, 2021. Revised December 5, 2021. Accepted December 21, 2021.
[0]2020 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Fixed point, weakly compatible maps, E.A. property, (CLR) property, ξ-weakly expansive mapping.
[0]Corresponding author: M. Kumar(manojantil18@gmail.com).

1. Introduction

In 1984, Wang et al. [10] introduced the concept of expansive mapping as follows:

Definition 1.1. ([10]) Let $\mathcal{P}$ be a self-mapping of a metric space (M, d^*). Then $\mathcal{P}$ is said to be expansive if there exists a real number $h > 1$ such that $d^*(\mathcal{P}c, \mathcal{P}d) \geq hd^*(c, d)$ for all $c, d \in M$.

In 2014, Kang et al. [6] introduced ϕ-weakly expansive mappings as follows:

Definition 1.2. ([6]) Let $\mathcal{P}$ be a self-mapping of a metric space (M, d^*). Then $\mathcal{P}$ is said to be ϕ-weakly expansive if there exists a continuous mapping $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(\alpha) > \alpha$ for all $\alpha > 0$ such that

$$d^*(\mathcal{P}c, \mathcal{P}d) \geq d^*(c, d) + \phi(d^*(c, d))$$

for all $c, d \in M$.

Definition 1.3. Let $\mathcal{P}$ and $\mathfrak{Q}$ be two self- mappings of a metric space (M, d^*). Then $\mathcal{P}$ is said to be ϕ-weakly expansive with respect to $\mathfrak{Q} : M \to M$ if there exists a continuous mapping $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(\alpha) > \alpha$ for all $\alpha > 0$ such that

$$d^*(\mathcal{P}c, \mathcal{P}d) \geq d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \phi(d^*(\mathfrak{Q}c, \mathfrak{Q}d))$$

for all $c, d \in M$.

In 2000, Hitzler and Seda [4] introduced the concept of dislocated metric space (d^*-metric space) as follows:

Definition 1.4. ([2, 4]) Let M be a nonempty set and let $d^* : M \times M \to [0, \infty)$ be a function and for all $p, q, r \in M$, the following conditions are satisfied:

(1) $d^*(p, q) = d^*(q, p)$;
(2) $d^*(p, q) = 0$, then $p = q$;
(3) $d^*(p, q) \leq d^*(p, r) + d^*(r, q)$.

Then d^* is called dislocated metric (or simply d^*-metric) on M and the pair (M, d^*) is called dislocated metric space.

In 1996, Jungck [5] introduced the concept of weakly compatible maps as follows:

Definition 1.5. ([5]) Two self maps $\mathcal{P}$ and $\mathfrak{Q}$ defined on a metric space M are said to be weakly compatible if they commute at their coincidence points.

In 2002, Aamri et al. [1] introduced the notion of E.A. property as follows:

Definition 1.6. ([1, 7, 8]) Two self-mappings $\mathcal{P}$ and $\mathfrak{Q}$ of a metric space (M, d^*) are said to satisfy E.A. property if there exists a sequence $\{c_n\}$ in M such that

$$\lim_{n\to\infty} \mathcal{P}c_n = \lim_{n\to\infty} \mathfrak{Q}c_n = t$$

for some t in M.

In 2011, Sintunavarat et al. [9] introduced the notion of $(CRL_{\mathcal{P}})$ property as follows:

Definition 1.7. ([9]) Two self-mappings $\mathcal{P}$ and $\mathfrak{Q}$ of a metric space (M, d^*) are said to satisfy $(CLR_{\mathcal{P}})$ property if there exists a sequence $\{c_n\}$ in M such that

$$\lim_{n\to\infty} \mathcal{P}c_n = \lim_{n\to\infty} \mathfrak{Q}c_n = \mathcal{P}c$$

for some c in M.

2. Main results

Theorem 2.1. *Let $\mathcal{P}$ and $\mathfrak{Q}$ be two self-maps of a dislocate metric space (M, d^*) satisfying the followings:*

$$\mathfrak{Q}M \subseteq \mathcal{P}M. \tag{2.1}$$

There exists a continuous mapping $\xi : [0,\infty) \to [0,\infty)$ with $\xi(0) = 0$ and $\xi(\alpha) > \alpha$ for all $\alpha > 0$ such that:

$$d^*(\mathcal{P}c, \mathcal{P}d) \geq d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)), \quad \forall c, d \in M, \tag{2.2}$$

where

$$\wedge(\mathfrak{Q}c, \mathfrak{Q}d) = \max\Big\{d^*(\mathfrak{Q}c, \mathfrak{Q}d), d^*(\mathfrak{Q}c, \mathcal{P}c), d^*(\mathfrak{Q}d, \mathcal{P}d), \frac{d^*(\mathfrak{Q}c, \mathcal{P}c)\cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathfrak{Q}c, \mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c, \mathcal{P}c)\cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathcal{P}c, \mathcal{P}d)}\Big\}.$$

If $\mathcal{P}$ and $\mathfrak{Q}$ are weakly compatible and $\mathcal{P}M$ or $\mathfrak{Q}M$ is complete, then $\mathcal{P}$ and $\mathfrak{Q}$ have a unique common fixed point.

Proof. Let c_0 be an arbitrary point in M. From (2.1), we can define a sequence $\{c_n\}$ such that

$$\mathfrak{Q}c_n = \mathcal{P}c_{n+1},$$

since $\mathfrak{Q}M \subseteq \mathcal{P}M$. Define a sequence $\{d_n\}$ in M by

$$d_n = \mathfrak{Q}c_n = \mathcal{P}c_{n+1}. \tag{2.3}$$

If $d_n = d_{n+1}$ for some n in $\mathbb{N}$, then there is nothing to prove. Now we assume that $d_n \neq d_{n+1}$ for all n in $\mathbb{N}$. We prove that

$$\lim_{n\to\infty} d^*(d_n, d_{n+1}) = 0. \tag{2.4}$$

Substituting, $c = c_n$, $d = c_{n+1}$ in (2.2) and using (2.3), we get

$$\begin{aligned} d^*(\mathcal{P}c_n, \mathcal{P}c_{n+1}) &\geq d^*(\mathfrak{Q}c_n, \mathfrak{Q}c_{n+1}) + \xi(\wedge(\mathfrak{Q}c_n, \mathfrak{Q}c_{n+1})), \\ d^*(d_{n-1}, d_n) &\geq d^*(d_n, d_{n+1}) + \xi(\wedge(d_n, d_{n+1})), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \wedge(d_n, d_{n+1}) &= \wedge(\mathfrak{Q}c_n, \mathfrak{Q}c_{n+1}) \\ &= \max\Big\{ d^*(\mathfrak{Q}c_n, \mathfrak{Q}c_{n+1}), d^*(\mathfrak{Q}c_n, \mathcal{P}c_n), \\ &\quad d^*(\mathfrak{Q}c_{n+1}, \mathcal{P}c_{n+1}), \frac{d^*(\mathfrak{Q}c_n, \mathcal{P}c_n) d^*(\mathfrak{Q}c_{n+1}, \mathcal{P}c_{n+1})}{1 + d^*(\mathfrak{Q}c_n, \mathfrak{Q}c_{n+1})}, \\ &\quad \frac{d^*(\mathfrak{Q}c_n, \mathcal{P}c_n) d^*(\mathfrak{Q}c_{n+1}, \mathcal{P}c_{n+1})}{1 + d^*(\mathcal{P}c_n, \mathcal{P}c_{n+1})} \Big\} \\ &= \max\Big\{ d^*(d_n, d_{n+1}), d^*(d_n, d_{n-1}), d^*(d_{n+1}, d_n), \\ &\quad \frac{d^*(d_n, d_{n-1}) d^*(d_{n+1}, d_n)}{1 + d^*(d_n, d_{n+1})}, \frac{d^*(d_n, d_{n-1}) \cdot d^*(d_{n+1}, d_n)}{1 + d^*(d_{n-1}, d_n)} \Big\} \\ &= \max\{d^*(d_n, d_{n+1}), d^*(d_{n-1}, d_n)\}. \end{aligned}$$

If $d^*(d_{n+1}, d_n) < d^*(d_n, d_{n-1})$, then from (2.5), we have

$$d^*(d_{n-1}, d_n) > d^*(d_n, d_{n+1}) + d^*(d_n, d_{n-1}).$$

That is

$$d^*(d_n, d_{n+1}) < 0,$$

which is a contradiction. If $d^*(d_n, d_{n-1}) < d^*(d_{n+1}, d_n)$, then from (2.5), we have

$$d^*(d_{n-1}, d_n) > d^*(d_n, d_{n+1}) + \xi(d^*(d_n, d_{n+1})). \tag{2.6}$$

This implies that

$$d^*(d_{n-1}, d_n) > d^*(d_n, d_{n+1}).$$

Hence the sequence $\{d^*(d_{n+1}, d_n)\}$ is strictly decreasing and bounded below. Thus, there exists $r \geq 0$, such that

$$\lim_{n\to\infty} d^*(d_n, d_{n+1}) = r,$$

letting $n \to \infty$ in (2.6), we get

$$r \geq r + \xi(r),$$

which is a contradiction, hence we have $r = 0$. Therefore,

$$\lim_{n\to\infty} d^*(d_n, d_{n+1}) = 0. \tag{2.7}$$

Next, we prove that $\{d_n\}$ is a d^*-Cauchy sequence. Suppose that $\{d_n\}$ is not a d^*-Cauchy sequence. Then there exists $\epsilon > 0$, such that for $k \in \mathbb{N}$, there are $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) > k$ satisfying:

(1) $m(k)$ and $n(k)$ are positive integers.
(2) $d^*(d_{n(k)}, d_{m(k)}) > \epsilon$.
(3) $m(k)$ is the smallest even number such that the condition (ii) holds, that is, $d^*(d_{n(k)}, d_{m(k)-1}) \leq \epsilon$.

Therefore,

$$\begin{aligned} \epsilon &< d^*(d_{n(k)}, d_{m(k)}) \\ &\leq d^*(d_{n(k)}, d_{m(k)-1}) + d^*(d_{(m(k)-1)}, d_{m(k)}) \\ &\leq \epsilon + d^*(d_{m(k)-1}, d_{m(k)}). \end{aligned} \tag{2.8}$$

Letting $k \to \infty$, we obtain

$$\lim_{n\to\infty} d^*(d_{n(k)}, d_{m(k)}) = \epsilon. \tag{2.9}$$

Now, we have

$$\begin{aligned} \epsilon &\leq d^*(d_{n(k)-1}, d_{m(k)-1}) \\ &\leq d^*(d_{n(k)-1}, d_{m(k)-2}) + d^*(d_{(m(k)-2)}, d_{m(k)-1}) \\ &\leq \epsilon + d^*(d_{m(k)-2}, d_{m(k)-1}). \end{aligned}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d^*(d_{n(k)-1}, d_{m(k)-1}) = \epsilon.$$

Substituting $c = c_{n(k)}, d = c_{m(k)}$ in (2.2), we get

$$\begin{aligned} &d^*(\mathcal{P}c_{n(k)}, \mathcal{P}c_{m(k)}) \geq d^*(\mathfrak{Q}c_{n(k)}, \mathfrak{Q}c_{m(k)}) + \xi(\wedge(\mathfrak{Q}c_{n(k)}, Qc_{m(k)})), \\ &d^*(d_{n(k)-1}, d_{m(k)-1}) \geq d^*(d_{n(k)}, d_{m(k)}) + \xi(\wedge(d_{n(k)}, d_{m(k)})), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \wedge(d_{n(k)}, d_{m(k)}) &= \wedge(\mathfrak{Q}c_{n(k)}, \mathfrak{Q}c_{m(k)}) \\ &= \max\{d^*(\mathfrak{Q}c_{n(k)}, \mathfrak{Q}c_{m(k)}), d^*(\mathfrak{Q}c_{n(k)}, \mathcal{P}c_{n(k)}), \\ &\quad d^*(\mathfrak{Q}c_{m(k)}, \mathcal{P}c_{m(k)}), \\ &\quad \frac{d^*(\mathfrak{Q}c_{n(k)}, \mathcal{P}c_{n(k)}) \cdot d^*(\mathfrak{Q}c_{m(k)}, \mathcal{P}c_{m(k)})}{1 + d^*(\mathfrak{Q}c_{n(k)}, \mathfrak{Q}c_{m(k)})}, \\ &\quad \frac{d^*(\mathfrak{Q}c_{n(k)}, \mathcal{P}c_{n(k)}) \cdot d^*(\mathfrak{Q}c_{m(k)}, \mathcal{P}c_{m(k)})}{1 + d^*(\mathcal{P}c_{n(k)}, \mathcal{P}c_{m(k)})}\} \\ &= \max\Big\{d^*(d_{n(k)}, d_{m(k)}), d^*(d_{n(k)}, d_{n(k)-1}), d^*(d_{m(k)}, d_{m(k)-1}), \\ &\quad \frac{d^*(d_{n(k)}, d_{n(k)-1}) \cdot d^*(d_{m(k)}, d_{m(k)-1})}{1 + d^*(d_{n(k)}, d_{m(k)})}, \\ &\quad \frac{d^*(d_{n(k)}, d_{n(k)-1}) \cdot d^*(d_{m(k)}, d_{m(k)-1})}{1 + d^*(d_{n(k)-1}, d_{m(k)-1})}\Big\}. \end{aligned}$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} \wedge(d_{n(k)}, d_{m(k)}) = \max\{\epsilon, \epsilon, \epsilon, \frac{\epsilon \cdot \epsilon}{1+\epsilon}, \frac{\epsilon \cdot \epsilon}{1+\epsilon}\} = \epsilon.$$

Now, from (2.10), we get

$$\epsilon \geq \epsilon + \xi(\epsilon),$$

which is a contradiction, since $\xi(\epsilon) \geq 0$. Which implies that $\{d_n\}$ is a d^*-Cauchy sequence in M. Now, since $\mathcal{P}M$ is complete, there exists a point p in $\mathcal{P}M$ such that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \mathcal{P}c_{n+1} = p = \lim_{n\to\infty} \mathfrak{Q}c_n. \tag{2.11}$$

Since $p \in \mathcal{P}M$, we can find q in M such that $\mathcal{P}q = p$.

Now, we claim that $\mathcal{P}q = \mathfrak{Q}q$, let if possible $\mathcal{P}q \neq \mathfrak{Q}q$.

Put $c = c_{n+1}$, $d = q$ in (2.2), we have

$$\begin{aligned} d^*(\mathfrak{Q}c_n, \mathcal{P}q) &= d^*(\mathcal{P}c_{n+1}, \mathcal{P}q) \\ &\geq d^*(\mathfrak{Q}c_{n+1}\mathfrak{Q}q) + \xi(\wedge(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q)) \\ &= d^*(\mathcal{P}q, \mathfrak{Q}q) + \xi(\wedge(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q)), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q) = \max\Big\{ & d^*(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q), d^*(\mathfrak{Q}c_{n+1}, \mathcal{P}c_{n+1}), d^*(\mathfrak{Q}q, \mathcal{P}q), \\ & \frac{d^*(\mathfrak{Q}c_{n+1}, \mathcal{P}c_{n+1}) \cdot d^*(\mathfrak{Q}q, \mathcal{P}q)}{1 + d^*(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q)}, \\ & \frac{d^*(\mathfrak{Q}c_{n+1}, \mathcal{P}c_{n+1}) \cdot d^*(\mathfrak{Q}q, \mathcal{P}q)}{1 + d^*(\mathcal{P}c_{n+1}, \mathcal{P}q)} \Big\}. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} \wedge(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q) &= \max\Big\{ d^*(\mathcal{P}q, \mathfrak{Q}q), d^*(\mathcal{P}q, \mathcal{P}q), d^*(\mathfrak{Q}q, \mathcal{P}q), \\ & \qquad \frac{d^*(\mathcal{P}q, \mathcal{P}q) \cdot d^*(\mathfrak{Q}q, \mathcal{P}q)}{1 + d^*(\mathcal{P}q, \mathfrak{Q}q)}, \frac{d^*(\mathcal{P}q, \mathcal{P}q) \cdot d^*(\mathfrak{Q}q, \mathcal{P}q)}{1 + d^*(\mathcal{P}q, \mathcal{P}q)} \Big\} \\ &= \max\{d^*(\mathfrak{Q}q, \mathcal{P}q), d^*(\mathcal{P}q, \mathcal{P}q)\}. \end{aligned}$$

Now, there are two cases arise.

Case I: Let $\wedge(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q) = d^*(\mathcal{P}q, \mathfrak{Q}q)$.

From (2.12), we have

$$\begin{aligned} d^*(\mathcal{P}q, \mathcal{P}q) &\geq d^*(\mathcal{P}q, \mathfrak{Q}q) + \xi(d^*(\mathcal{P}q, \mathfrak{Q}q)), \\ d^*(\mathcal{P}q, \mathcal{P}q) &> d^*(\mathcal{P}q, \mathfrak{Q}q) + d^*(\mathcal{P}q, \mathfrak{Q}q) > 2d^*(\mathcal{P}q, \mathfrak{Q}q). \end{aligned}$$

But by triangular inequality, we have

$$\begin{aligned} d^*(\mathcal{P}q, \mathcal{P}q) &\leq d^*(\mathcal{P}q, \mathfrak{Q}q) + d^*(\mathfrak{Q}q, \mathcal{P}q), \\ &\leq 2d^*(\mathcal{P}q, \mathfrak{Q}q), \end{aligned}$$

which is a contradiction.

Case II: Let $\wedge(\mathfrak{Q}c_{n+1}, \mathfrak{Q}q) = d^*(\mathcal{P}q, \mathcal{P}q)$.

From (2.12), we have

$$\begin{aligned} d^*(\mathcal{P}q, \mathcal{P}q) &\geq d^*(\mathcal{P}q, \mathfrak{Q}q) + \xi(d^*(\mathcal{P}q, \mathcal{P}q)), \\ d^*(\mathcal{P}q, \mathcal{P}q) &> d^*(\mathcal{P}q, \mathfrak{Q}q) + d^*(\mathcal{P}q, \mathcal{P}q). \end{aligned}$$

This means that

$$d^*(\mathcal{P}q, \mathfrak{Q}q) < 0,$$

which is a contradiction. Hence, $d^*(\mathfrak{Q}q, \mathcal{P}q) = 0$. Which implies that

$$\mathcal{P}q = \mathfrak{Q}q = p. \tag{2.13}$$

Therefore, q is a coincidence point of $\mathcal{P}$ and $\mathfrak{Q}$.

Now, we show that there exists a common fixed point of $\mathcal{P}$ and $\mathfrak{Q}$. Since $\mathcal{P}$ and $\mathfrak{Q}$ are weakly compatible, by (2.13), we have

$$\mathfrak{Q}\mathcal{P}q = \mathcal{P}\mathfrak{Q}q \text{ and } \mathfrak{Q}p = \mathfrak{Q}\mathcal{P}q = \mathcal{P}\mathfrak{Q}q = \mathcal{P}p.$$

Now, consider

$$\begin{aligned} d^*(Pq, \mathcal{P}p) &\geq d^*(\mathfrak{Q}q, \mathfrak{Q}p) + \xi(\wedge(\mathfrak{Q}q, \mathfrak{Q}p)), \\ d^*(p, \mathfrak{Q}p) &\geq d^*(p, \mathfrak{Q}p) + \xi(\wedge(\mathfrak{Q}q, \mathfrak{Q}p)), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}q, \mathfrak{Q}p) &= \max\Big\{d^*(\mathfrak{Q}q, \mathfrak{Q}p), d^*(\mathfrak{Q}q, \mathcal{P}q), d^*(\mathfrak{Q}p, \mathcal{P}p), \\ &\qquad \frac{d^*(\mathfrak{Q}q, \mathcal{P}q)\cdot d^*(\mathfrak{Q}p, \mathcal{P}p)}{1 + d^*(\mathfrak{Q}q, \mathfrak{Q}p)}, \frac{d^*(\mathfrak{Q}q, \mathcal{P}q)\cdot d^*(\mathfrak{Q}p, \mathcal{P}p)}{1 + d^*(\mathcal{P}q, \mathcal{P}p)}\Big\} \\ &= \max\{d^*(p, \mathfrak{Q}p), 0, d^*(\mathfrak{Q}p, \mathfrak{Q}p), 0, 0\} \\ &= \max\{d^*(p, \mathfrak{Q}p), d^*(\mathfrak{Q}p, \mathfrak{Q}p)\}. \end{aligned}$$

Now, also we have two cases:

Case I: Let $\wedge(\mathfrak{Q}q, \mathfrak{Q}p) = d^*(p, \mathfrak{Q}p)$.

From (2.14), we have

$$\begin{aligned} d^*(p, \mathfrak{Q}p) &\geq d^*(p, \mathfrak{Q}p) + \xi(d^*(p, \mathfrak{Q}p)), \\ d^*(p, \mathfrak{Q}p) &> d^*(p, \mathfrak{Q}p) + d^*(p, \mathfrak{Q}p), \\ d^*(p, \mathfrak{Q}p) &> 2d^*(p, \mathfrak{Q}p), \end{aligned}$$

which is a contradiction.

Case II: Let $\wedge(\mathfrak{Q}q, \mathfrak{Q}p) = d^*(\mathfrak{Q}p, \mathfrak{Q}p)$.

From (2.14), we have

$$\begin{aligned} d^*(p,\mathfrak{Q}p) &\geq d^*(p,\mathfrak{Q}p)+\xi(d^*(\mathfrak{Q}p,\mathfrak{Q}p)), \\ d^*(p,\mathfrak{Q}p) &> d^*(p,\mathfrak{Q}p)+d^*(\mathfrak{Q}p,\mathfrak{Q}p), \\ d^*(p,\mathfrak{Q}p) &> d^*(p,\mathfrak{Q}p), \end{aligned}$$

which is again a contradiction. Hence $\mathcal{P}p=\mathfrak{Q}p=p$. This implies p is common fixed point of $\mathcal{P}$ and $\mathfrak{Q}$.

For the uniqueness, let r and s be two common fixed points of $\mathcal{P}$ and $\mathfrak{Q}$, such that $r\neq s$. Then

$$\begin{aligned} d^*(r,s) &= d^*(\mathcal{P}r,\mathcal{P}s) \\ &\geq d^*(\mathfrak{Q}r,\mathfrak{Q}s)+\xi(d^*(\mathfrak{Q}r,\mathfrak{Q}s)) \\ &= d^*(r,s)+\xi(d^*(r,s)) \\ &> d^*(r,s)+d^*(r,s), \end{aligned}$$

which is a contradiction, hence $r=s$. This proves the uniqueness of the common fixed point. Hence completes the proof of the theorem. □

Corollary 2.2. *Let T be self-map on a dislocated metric space (M,d^*) satisfying the followings: There exists a continuous mapping $\xi:[0,\infty)\to[0,\infty)$ with $\xi(0)=0$ and $\xi(\alpha)>\alpha$ for all $\alpha>0$ such that*

$$d^*(Tc,Td)\geq d^*(c,d)+\xi(\wedge(c,d)),\quad \forall c,d\in M,$$

where

$$\wedge(c,d)=\max\Big\{d^*(c,d),d^*(c,Tc),d^*(d,Td), \frac{d^*(c,Tc)\cdot d^*(d,Td)}{1+d^*(c,d)},\frac{d^*(c,Tc)\cdot d^*(d,Td)}{1+d^*(Tc,Td)}\Big\}.$$

If TM is complete, then T has a unique fixed point.

Theorem 2.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be self mappings of a dislocated metric space (M,d^*) satisfying (2.2) and the followings:*

$$\mathcal{P} \text{ and } \mathfrak{Q} \text{ are weakly compatible,} \tag{2.15}$$

$$\mathcal{P} \text{ and } \mathfrak{Q} \text{ satisfy the E.A. property.} \tag{2.16}$$

If either $\mathcal{P}M$ or $\mathfrak{Q}M$ is a complete subspace of M, then $\mathcal{P}$ and $\mathfrak{Q}$ have a unique common fixed point in M.

Proof. Since $\mathcal{P}$ and $\mathfrak{Q}$ satisfy the E.A. property, there exists a sequence $\{c_n\}$ in M such that

$$\lim_{n\to\infty}\mathcal{P}c_n=\lim_{n\to\infty}\mathfrak{Q}c_n=c \quad \text{for some} \quad c\in M. \tag{2.17}$$

Now, suppose that $\mathcal{P}M$ is complete subspace of M. Then, there exists z in M such that $c = \mathcal{P}z$. Subsequently, we have

$$\lim_{n\to\infty} \mathcal{P}c_n = \lim_{n\to\infty} \mathfrak{Q}c_n = c = \mathcal{P}z.$$

Now, we show that $\mathcal{P}z = \mathfrak{Q}z$.

From (2.2), we have

$$d^*(\mathcal{P}c_n, \mathcal{P}z) \geq d^*(\mathfrak{Q}c_n, \mathfrak{Q}z) + \xi(\wedge(\mathfrak{Q}c_n, \mathfrak{Q}z)).$$

Letting limit $n \to \infty$, we have

$$d^*(\mathcal{P}z, \mathcal{P}z) \geq d^*(\mathcal{P}z, \mathfrak{Q}z) + lim_{n\to\infty}\xi(\wedge(\mathfrak{Q}c_n, \mathfrak{Q}z)), \tag{2.18}$$

where

$$\begin{aligned}\lim_{n\to\infty} \wedge(\mathfrak{Q}c_n, \mathfrak{Q}z) &= \lim_{n\to\infty} \max\Big\{(d^*(\mathfrak{Q}c_n, \mathfrak{Q}z), d^*(\mathfrak{Q}c_n, \mathcal{P}c_n), d^*(\mathfrak{Q}z, \mathcal{P}z),\\ &\qquad \frac{d^*(\mathfrak{Q}c_n, \mathcal{P}c_n)\cdot d^*(\mathfrak{Q}z, \mathcal{P}z)}{1 + d^*(\mathfrak{Q}c_n, \mathfrak{Q}z)}, \frac{d^*(\mathfrak{Q}c_n, \mathcal{P}c_n)\cdot d^*(\mathfrak{Q}z, \mathcal{P}z)}{1 + d^*(\mathcal{P}c_n, \mathcal{P}z)}\Big\}\\ &= \max\Big\{d^*(\mathcal{P}z, \mathfrak{Q}z), d^*(\mathcal{P}z, \mathcal{P}z), d^*(\mathfrak{Q}z, \mathcal{P}z),\\ &\qquad \frac{d^*(\mathcal{P}z, \mathcal{P}z)\cdot d^*(\mathfrak{Q}z, \mathcal{P}z)}{1 + d^*(\mathcal{P}z, \mathfrak{Q}z)}, \frac{d^*(\mathcal{P}z, \mathcal{P}z)\cdot d^*(\mathfrak{Q}z, \mathcal{P}z)}{1 + d^*(\mathcal{P}z, \mathcal{P}z)}\Big\}\\ &= \max\{d^*(\mathcal{P}z, \mathfrak{Q}z), d^*(\mathcal{P}z, \mathcal{P}z)\}.\end{aligned}$$

Now two cases arise:

Case I: Let $\lim_{n\to\infty} \wedge(\mathfrak{Q}c_n, \mathfrak{Q}z) = d^*(\mathcal{P}z, \mathfrak{Q}z)$.

Then from (2.18), we have

$$\begin{aligned} d^*(\mathcal{P}z, \mathcal{P}z) &\geq d^*(\mathcal{P}z, \mathfrak{Q}z) + \xi(d^*(\mathcal{P}z, \mathfrak{Q}z)),\\ d^*(\mathcal{P}z, \mathcal{P}z) &> d^*(\mathcal{P}z, \mathfrak{Q}z) + d^*(\mathcal{P}z, \mathfrak{Q}z),\\ d^*(\mathcal{P}z, \mathcal{P}z) &> 2d^*(\mathcal{P}z, \mathfrak{Q}z).\end{aligned}$$

But by triangular inequality, we have

$$\begin{aligned} d^*(\mathcal{P}z, \mathcal{P}z) &\leq d^*(\mathcal{P}z, \mathfrak{Q}z) + d^*(\mathfrak{Q}z, \mathcal{P}z),\\ d^*(\mathcal{P}z, \mathcal{P}z) &\leq 2d^*(\mathcal{P}z, \mathfrak{Q}z),\end{aligned}$$

which is a contradiction.

Case II: Let $\wedge(\mathcal{P}z, \mathfrak{Q}z) = d^*(\mathcal{P}z, \mathcal{P}z)$.

Then from (2.18), we have

$$\begin{aligned} d^*(\mathcal{P}z, \mathcal{P}z) &\geq d^*(\mathcal{P}z, \mathfrak{Q}z) + \xi(d^*(\mathcal{P}z, \mathcal{P}z)),\\ d^*(\mathcal{P}z, \mathcal{P}z) &> d^*(\mathcal{P}z, \mathfrak{Q}z) + d^*(\mathcal{P}z, \mathcal{P}z),\end{aligned}$$

which implies that

$$d^*(\mathcal{P}z, \mathfrak{Q}z) < 0,$$

which is a contradiction. This implies

$$d^*(\mathcal{P}z, \mathfrak{Q}z) = 0 \, or \, \mathcal{P}z = \mathfrak{Q}z.$$

Since $\mathcal{P}$ and $\mathfrak{Q}$ are weakly compatible, $\mathfrak{Q}\mathcal{P}z = \mathcal{P}\mathfrak{Q}z$ implies that,

$$\mathcal{P}\mathcal{P}z = \mathcal{P}\mathfrak{Q}z = \mathfrak{Q}\mathcal{P}Z = \mathfrak{Q}\mathfrak{Q}z.$$

Now, we claim that $\mathfrak{Q}z$ is the common fixed point of $\mathcal{P}$ and $\mathfrak{Q}$.
From (2.2), we have

$$\begin{aligned} d^*(\mathcal{P}z, \mathcal{P}\mathcal{P}z) &\geq d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) + \xi(\wedge(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z)), \\ d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) &\geq d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) + \xi(\wedge(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z)), \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) &= \max\{(d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z), d^*(\mathfrak{Q}z, \mathcal{P}z), d^*(\mathfrak{Q}\mathfrak{Q}z, \mathcal{P}\mathfrak{Q}z), \\ &\qquad \frac{d^*(\mathfrak{Q}z, \mathcal{P}z) \cdot d^*(\mathfrak{Q}\mathfrak{Q}z, \mathcal{P}\mathfrak{Q}z)}{1 + d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z)}, \frac{d^*(\mathfrak{Q}z, \mathcal{P}z) \cdot d^*(\mathfrak{Q}\mathfrak{Q}z, \mathcal{P}\mathfrak{Q}z)}{1 + d^*(\mathcal{P}z, \mathcal{P}\mathfrak{Q}z)}\} \\ &= \max\{d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z), 0, 0, 0, 0\}. \end{aligned}$$

This implies that

$$\wedge(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) = d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z).$$

Now, from (2.19), we have

$$\begin{aligned} d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) &\geq d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) + \xi(\wedge(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z)), \\ d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) &\geq d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) + \xi(d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z)), \\ d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) &> d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) + d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z), \\ d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z) &> 2d^*(\mathfrak{Q}z, \mathfrak{Q}\mathfrak{Q}z), \end{aligned}$$

it implies that

$$\mathfrak{Q}z = \mathfrak{Q}\mathfrak{Q}z = \mathcal{P}\mathfrak{Q}z.$$

Hence $\mathfrak{Q}z$ is common fixed point of $\mathcal{P}$ and $\mathfrak{Q}$.

For the uniqueness, let r and s be two common fixed points of $\mathcal{P}$ and $\mathfrak{Q}$. Then, from (2.2), we get

$$d^*(\mathcal{P}r, \mathcal{P}s) \geq d^*(\mathfrak{Q}r, \mathfrak{Q}s) + \xi(\wedge(\mathfrak{Q}r, \mathfrak{Q}s)), \tag{2.20}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}r, \mathfrak{Q}s) &= \max\Big\{d^*(\mathfrak{Q}r, \mathfrak{Q}s), d^*(\mathfrak{Q}r, \mathcal{P}r), d^*(\mathfrak{Q}s, \mathcal{P}s), \\ &\qquad \tfrac{d^*(\mathfrak{Q}r,\mathcal{P}r) \cdot d^*(\mathfrak{Q}s,\mathcal{P}s)}{1+d^*(\mathfrak{Q}r,\mathfrak{Q}s)}, \tfrac{d^*(\mathfrak{Q}r,\mathcal{P}r) \cdot d^*(\mathfrak{Q}s,\mathcal{P}s)}{1+d^*(\mathcal{P}r,\mathcal{P}s)}\Big\} \\ &= d^*(\mathfrak{Q}r, \mathfrak{Q}s), \end{aligned}$$

$$d^*(\mathcal{P}r,\mathcal{P}s) \geq d^*(\mathfrak{Q}r,\mathfrak{Q}s) + \xi(\wedge(\mathfrak{Q}r,\mathfrak{Q}s)),$$
$$d^*(\mathcal{P}r,\mathcal{P}s) \geq d^*(\mathfrak{Q}r,\mathfrak{Q}s) + d^*(\mathfrak{Q}r,\mathfrak{Q}s)),$$
$$d^*(\mathcal{P}r,\mathcal{P}s) > 2d^*(\mathfrak{Q}r,\mathfrak{Q}s),$$
$$d^*(r,s) > 2d^*(r,s),$$

which is a contradiction. This implies that $r = s$. This proves the uniqueness of common fixed point. This completes the proof. □

Theorem 2.4. *Let (M, d^*) be a dislocated metric space, let $\mathcal{P}$ and $\mathfrak{Q}$ be self maps on M satisfying (2.2), (2.15). If $\mathcal{P}$ and $\mathfrak{Q}$ satisfy $(CLR_{\mathcal{P}})$ property, then $\mathcal{P}$ and $\mathfrak{Q}$ have a unique common fixed point in M.*

Proof. Since $\mathcal{P}$ and $\mathfrak{Q}$ satisfy the $(CLR_{\mathcal{P}})$ property, there exists a sequence $\{c_n\}$ in M such that

$$\lim_{n\to\infty} \mathcal{P}c_n = \lim_{n\to\infty} \mathfrak{Q}c_n = \mathcal{P}c,$$

for some $c \in M$. First we prove that $\mathcal{P}c = \mathfrak{Q}c$. Let $\mathcal{P}c \neq \mathfrak{Q}c$. Then from (2.2), we have

$$d^*(\mathcal{P}c_n,\mathcal{P}c) \geq d^*(\mathfrak{Q}c_n,\mathfrak{Q}c) + \xi(\wedge(Qc_n,\mathfrak{Q}c)), \tag{2.21}$$

where

$$\wedge(\mathfrak{Q}c_n,\mathfrak{Q}c) = \max\Big\{(d^*(\mathfrak{Q}c_n,\mathfrak{Q}c), d^*(\mathfrak{Q}c_n,\mathcal{P}c_n), d^*(\mathfrak{Q}c,\mathcal{P}c), \frac{d^*(\mathfrak{Q}c_n,\mathcal{P}c_n)\cdot d^*(\mathfrak{Q}c,\mathcal{P}c)}{1+d^*(\mathfrak{Q}c_n,\mathfrak{Q}c)}, \frac{d^*(\mathfrak{Q}c_n,\mathcal{P}c_n)\cdot d^*(\mathfrak{Q}c,\mathcal{P}c)}{1+d^*(\mathcal{P}c_n,\mathcal{P}c)}\Big\}.$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} \wedge(\mathfrak{Q}c_n,\mathfrak{Q}c) &= \max\Big\{(d^*(\mathcal{P}c,\mathfrak{Q}c), d^*(\mathcal{P}c,\mathcal{P}c), d^*(\mathfrak{Q}c,\mathcal{P}c), \frac{d^*(\mathcal{P}c,\mathcal{P}c)\cdot d^*(\mathfrak{Q}c,\mathcal{P}c)}{1+d^*(\mathcal{P}c,\mathfrak{Q}c)}, \frac{d^*(\mathcal{P}c,\mathcal{P}c)\cdot d^*(\mathfrak{Q}c,\mathcal{P}c)}{1+d^*(\mathcal{P}c,\mathcal{P}c)}\Big\} \\ &= \max\{d^*(\mathcal{P}c,\mathfrak{Q}c), d^*(\mathcal{P}c,\mathcal{P}c)\}.\end{aligned}$$

Now, two cases arise:

Case I: Let $\lim_{n\to\infty} \wedge(\mathfrak{Q}c_n,\mathfrak{Q}c) = d^*(\mathcal{P}c,\mathfrak{Q}c)$.

From (2.21), we get

$$\begin{aligned} d^*(\mathcal{P}c,\mathcal{P}c) &\geq d^*(\mathcal{P}c,\mathfrak{Q}c) + \xi(\wedge(\mathcal{P}c,\mathfrak{Q}c)), \\ d^*(\mathcal{P}c,\mathcal{P}c) &\geq d^*(\mathcal{P}c,\mathfrak{Q}c) + \xi(d^*(\mathcal{P}c,\mathfrak{Q}c)), \\ d^*(\mathcal{P}c,\mathcal{P}c) &> d^*(\mathcal{P}c,\mathfrak{Q}c) + d^*(\mathcal{P}c,\mathfrak{Q}c), \\ d^*(\mathcal{P}c,\mathcal{P}c) &> 2d^*(\mathcal{P}c,\mathfrak{Q}c),\end{aligned}$$

but by triangular inequality, we have

$$d^*(\mathcal{P}c, \mathcal{P}c) \leq 2d^*(\mathcal{P}c, \mathfrak{Q}c),$$

which is a contradiction.

Case II: Let $\lim_{n\to\infty} \wedge(\mathfrak{Q}c_n, \mathfrak{Q}c) = d^*(\mathcal{P}c, \mathcal{P}c)$.

From (2.21), we get

$$\begin{aligned} d^*(\mathcal{P}c, \mathcal{P}c) &\geq d^*(\mathcal{P}c, \mathfrak{Q}c) + \xi(\wedge(\mathcal{P}c, \mathcal{P}c)), \\ d^*(\mathcal{P}c, \mathcal{P}c) &\geq d^*(\mathcal{P}c, \mathfrak{Q}c) + \xi(d^*(\mathcal{P}c, \mathcal{P}c)), \\ d^*(\mathcal{P}c, \mathcal{P}c) &> d^*(\mathcal{P}c, \mathfrak{Q}c) + d^*(\mathcal{P}c, \mathcal{P}c), \\ d^*(\mathcal{P}c, \mathcal{P}c) &< 0, \end{aligned}$$

this is possible only when $d^*(\mathcal{P}c, \mathfrak{Q}c) = 0$. Hence $\mathcal{P}c = \mathfrak{Q}c$.

Now, let $d = \mathcal{P}c = \mathfrak{Q}c$. Since $\mathcal{P}\mathfrak{Q}c = \mathfrak{Q}\mathcal{P}c$, implies that,

$$\mathcal{P}d = \mathcal{P}\mathfrak{Q}c = \mathfrak{Q}\mathcal{P}c = \mathfrak{Q}d.$$

Now, we claim that $\mathfrak{Q}d = d$.

From (2.2), we have

$$d^*(\mathfrak{Q}d, d) = d^*(\mathcal{P}d, \mathcal{P}c) \geq d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)), \tag{2.22}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}c, \mathfrak{Q}d) = \max \Big\{ & d^*(\mathfrak{Q}c, \mathfrak{Q}d), d^*(\mathfrak{Q}c, \mathcal{P}c), d^*(\mathfrak{Q}d, \mathcal{P}d), \\ & \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathfrak{Q}c, \mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathcal{P}c, \mathcal{P}d)} \Big\} \\ = {} & \max\{d^*(d, \mathfrak{Q}d), 0, 0, 0, 0\}, \\ = {} & d^*(\mathfrak{Q}d, d). \end{aligned}$$

From (2.22), we have

$$\begin{aligned} d^*(\mathfrak{Q}d, d) &\geq d^*(d, \mathfrak{Q}d) + \xi(d^*(d, \mathfrak{Q}d)), \\ d^*(\mathfrak{Q}d, d) &> d^*(d, \mathfrak{Q}d) + d^*(d, \mathfrak{Q}d), \\ d^*(\mathfrak{Q}d, d) &> 2d^*(d, \mathfrak{Q}d), \end{aligned}$$

this is possible only when $\mathfrak{Q}d = d$. Hence $\mathcal{P}d = \mathfrak{Q}d = d$. So, d is the common fixed point of $\mathcal{P}$ and $\mathfrak{Q}$.

For the uniqueness, let r, s be two common fixed points of $\mathcal{P}$ and $\mathfrak{Q}$. From (2.2), we get

$$d^*(\mathcal{P}r, \mathcal{P}s) \geq d^*(\mathfrak{Q}r, \mathfrak{Q}s) + \xi(\wedge(\mathfrak{Q}r, \mathfrak{Q}s)), \tag{2.23}$$

where

$$\begin{aligned}\wedge(\mathfrak{Q}r,\mathfrak{Q}s) &= \max\{d^*(\mathfrak{Q}r,\mathfrak{Q}s), d^*(\mathfrak{Q}r,\mathcal{P}r), d^*(\mathfrak{Q}s,\mathcal{P}s),\\ &\qquad \frac{d^*(\mathfrak{Q}r,\mathcal{P}r)\cdot d^*(\mathfrak{Q}s,\mathcal{P}s)}{1+d^*(\mathfrak{Q}r,\mathfrak{Q}s)}, \frac{d^*(\mathfrak{Q}r,\mathcal{P}r)\cdot d^*(\mathfrak{Q}s,\mathcal{P}s)}{1+d^*(\mathcal{P}r,\mathcal{P}s)}\}\\ &= d^*(r,s).\end{aligned}$$

From (2.23), we have

$$\begin{aligned}d^*(\mathcal{P}r,\mathcal{P}s) &\geq d^*(\mathfrak{Q}r,\mathfrak{Q}s)+\xi(\wedge(\mathfrak{Q}r,\mathfrak{Q}s)),\\ d^*(r,s) &> 2d^*(r,s),\end{aligned}$$

which is a contradiction, this implies that $r = s$. This proves the uniqueness of common fixed point. This completes the proof. □

Example 2.5. Let $M = [0, 2]$ be equipped with the dislocated metric space and $d^*(c, d) = \max\{|c|, |d|\}$ for all $c, d \in M$. Define $\mathcal{P}, \mathfrak{Q} : M \to M$ by

$$\mathcal{P}c = \begin{cases} 0, & \text{if } c = 0\\ \frac{2c}{3}, & \text{otherwise}\end{cases}$$

and

$$\mathfrak{Q}c = \begin{cases} 0, & \text{if } c = 0\\ \frac{c}{3}, & \text{otherwise.}\end{cases}$$

Then we have $\mathfrak{Q}M = [0, \frac{2}{3}] \subset [0, \frac{4}{3}] = \mathcal{P}M$.

Let $\{c_n\}$ be a sequence in M such that $\{c_n\} = \frac{1}{n}$ for each n. Also, let $\xi : [0,\infty) \to [0,\infty)$ be defined by:

$$\xi(t) = \begin{cases} \frac{t}{8}, & \text{if } t > 0\\ 0, & \text{otherwise.}\end{cases}$$

Clearly $\mathcal{P}(0) = \mathfrak{Q}(0) = 0$ and $\mathcal{P}\mathfrak{Q}(0) = \mathfrak{Q}\mathcal{P}(0) = 0$, this shows that $\mathcal{P}$ and $\mathfrak{Q}$ are weakly compatible. And let $c, d \in M$.

Now, we have to check the inequality of Theorem 2.1 for the following cases:

Case (I): Let $c = 0$ and $d = 0$.

$$d^*(\mathcal{P}c,\mathcal{P}d) = 0,$$

$$d^*(\mathfrak{Q}c,\mathfrak{Q}d) = 0\, and \wedge(\mathfrak{Q}c,\mathfrak{Q}d) = 0.$$

Also,

$$\xi(\wedge(\mathfrak{Q}c,\mathfrak{Q}d)) = 0,$$

hence

$$d^*(\mathcal{P}c,\mathcal{P}d) = d^*(Qc,Qd)+\xi(\wedge(\mathfrak{Q}c,\mathfrak{Q}d)).$$

Case (II): Let $c \neq 0$ and $d = 0$.

$$\begin{aligned} d^*(\mathcal{P}c, \mathcal{P}d) &= d^*(\frac{2c}{3}, 0) = \max\{\frac{2c}{3}, 0\} = \frac{2c}{3}, \\ d^*(\mathfrak{Q}c, \mathfrak{Q}d) &= d^*(\frac{c}{3}, 0) = \max\{\frac{c}{3}, 0\} = \frac{c}{3}, \end{aligned}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}c, \mathfrak{Q}d) &= \max\Big\{(d^*(\mathfrak{Q}c, \mathfrak{Q}d), d^*(\mathfrak{Q}c, \mathcal{P}c), d^*(\mathfrak{Q}d, \mathcal{P}d), \\ &\qquad \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathfrak{Q}c, \mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathcal{P}c, \mathcal{P}d)}\Big\} \\ &= \max\Big\{d^*(\frac{c}{3}, 0), d^*(\frac{c}{3}, \frac{2c}{3}), d^*(0, 0), \\ &\qquad \frac{d^*(\frac{c}{3}, \frac{2c}{3}) \cdot d^*(0, 0)}{1 + d^*(\frac{c}{3}, 0)}, \frac{d^*(\frac{c}{3}, \frac{2c}{3}) \cdot d^*(0, 0)}{1 + d^*(\frac{2c}{3}, 0)}\Big\} \\ &= \max\{\frac{c}{3}, \frac{2c}{3}, 0, 0, 0\}, \\ &= \frac{2c}{3}. \end{aligned}$$

Also, $\xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)) = \frac{1}{8}(\frac{2c}{3}) = \frac{c}{12}$, clearly

$$d^*(\mathcal{P}c, \mathcal{P}d) > d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)).$$

Case (III): Let $c = 0$ and $d \neq 0$.

$$\begin{aligned} d^*(\mathcal{P}c, \mathcal{P}d) = d^*(0, \tfrac{2d}{3}) = \max\{0, \tfrac{2d}{3}\} = \tfrac{2d}{3}, \\ d^*(\mathfrak{Q}c, \mathfrak{Q}d) = d^*(0, \tfrac{d}{3}) = \max\{0, \tfrac{d}{3}\} = \tfrac{d}{3}, \end{aligned}$$

where

$$\begin{aligned} \wedge(\mathfrak{Q}c, \mathfrak{Q}d) &= \max\Big\{(d^*(\mathfrak{Q}c, \mathfrak{Q}d), d^*(\mathfrak{Q}c, \mathcal{P}c), d^*(\mathfrak{Q}d, \mathcal{P}d), \\ &\qquad \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathfrak{Q}c, \mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathcal{P}c, \mathcal{P}d)}\Big\} \\ &= \max\Big\{(d^*(0, \frac{d}{3}), d^*(0, 0), d^*(\frac{d}{3}, \frac{2d}{3}), \\ &\qquad \frac{d^*(0, 0) \cdot d^*(\frac{d}{3}, \frac{2d}{3})}{1 + d^*(0, \frac{d}{3})}, \frac{d^*(0, 0) \cdot d^*(\frac{d}{3}, \frac{2d}{3})}{1 + d^*(0, \frac{2d}{3})}\Big\} \\ &= \max\{\frac{d}{3}, 0, \frac{2d}{3}, 0, 0\} \\ &= \frac{2d}{3}. \end{aligned}$$

Also, $\xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)) = \frac{1}{8}(\frac{2d}{3}) = \frac{d}{12}$, clearly

$$d^*(\mathcal{P}c, \mathcal{P}d) > d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)).$$

Case (IV): Let $c \neq 0$ and $d \neq 0$.
Now, we discuss three subcases:
Case (i): If $c > d$:

$$d^*(\mathcal{P}c, \mathcal{P}d) = d^*(\tfrac{2c}{3}, \tfrac{2d}{3}) = \max\{\tfrac{2c}{3}, \tfrac{2d}{3}\} = \tfrac{2c}{3},$$
$$d^*(\mathfrak{Q}c, \mathfrak{Q}d) = d^*(\tfrac{c}{3}, \tfrac{d}{3}) = \max\{\tfrac{c}{3}, \tfrac{d}{3}\} = \tfrac{c}{3},$$

where

$$\begin{aligned}\wedge(\mathfrak{Q}c, \mathfrak{Q}d) &= \max\Big\{(d^*(\mathfrak{Q}c, \mathfrak{Q}d), d^*(\mathfrak{Q}c, \mathcal{P}c), d^*(\mathfrak{Q}d, \mathcal{P}d) \\ &\qquad \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathfrak{Q}c, \mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c, \mathcal{P}c) \cdot d^*(\mathfrak{Q}d, \mathcal{P}d)}{1 + d^*(\mathcal{P}c, \mathcal{P}d)}\Big\} \\ &= \max\Big\{(d^*(\frac{c}{3}, \frac{d}{3}), d^*(\frac{c}{3}, \frac{2c}{3}), d^*(\frac{d}{3}, \frac{2d}{3}), \\ &\qquad \frac{d^*(\frac{c}{3}, \frac{2c}{3}) \cdot d^*(\frac{d}{3}, \frac{2d}{3})}{1 + d^*(\frac{c}{3}, \frac{d}{3})}, \frac{d^*(\frac{c}{3}, \frac{2c}{3}) \cdot d^*(\frac{d}{3}, \frac{2d}{3})}{1 + d^*(\frac{2c}{3}, \frac{2d}{3})}\Big\} \\ &= \max\Big\{\frac{c}{3}, \frac{2c}{3}, \frac{2d}{3}, \frac{\frac{2c}{3} \cdot \frac{2d}{3}}{1 + \frac{d}{3}}, \frac{\frac{2c}{3} \cdot \frac{2d}{3}}{1 + \frac{2c}{3}}\Big\}, \\ &= \frac{2c}{3}.\end{aligned}$$

Also, $\xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)) = \frac{1}{8}(\frac{2c}{3}) = \frac{c}{12}$, clearly

$$d^*(\mathcal{P}c, \mathcal{P}d) > d^*(\mathfrak{Q}c, \mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c, \mathfrak{Q}d)).$$

Case (ii): If $c < d$:

$$d^*(\mathcal{P}c, \mathcal{P}d) = d^*(\tfrac{2c}{3}, \tfrac{2d}{3}) = \max\{\tfrac{2c}{3}, \tfrac{2d}{3}\} = \tfrac{2d}{3},$$
$$d^*(\mathfrak{Q}c, \mathfrak{Q}d) = d^*(\tfrac{c}{3}, \tfrac{d}{3}) = \max\{\tfrac{c}{3}, \tfrac{d}{3}\} = \tfrac{d}{3},$$

where

$$\begin{aligned}\wedge(\mathfrak{Q}c,\mathfrak{Q}d) &= \max\Big\{(d^*(\mathfrak{Q}c,\mathfrak{Q}d), d^*(\mathfrak{Q}c,\mathcal{P}c), d^*(\mathfrak{Q}d,\mathcal{P}d),\\ &\qquad \frac{d^*(\mathfrak{Q}c,\mathcal{P}c)\cdot d^*(\mathfrak{Q}d,\mathcal{P}d)}{1+d^*(\mathfrak{Q}c,\mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c,\mathcal{P}c)\cdot d^*(\mathfrak{Q}d,\mathcal{P}d)}{1+d^*(\mathcal{P}c,\mathcal{P}d)}\Big\}\\ &= \max\Big\{(d^*(\frac{c}{3},\frac{d}{3}), d^*(\frac{c}{3},\frac{2c}{3}), d^*(\frac{d}{3},\frac{2d}{3}),\\ &\qquad \frac{d^*(\frac{c}{3},\frac{2c}{3})\cdot d^*(\frac{d}{3},\frac{2d}{3})}{1+d^*(\frac{c}{3},\frac{d}{3})}, \frac{d^*(\frac{c}{3},\frac{2c}{3})\cdot d^*(\frac{d}{3},\frac{2d}{3})}{1+d^*(\frac{2c}{3},\frac{2d}{3})}\Big\}\\ &= \max\Big\{\frac{d}{3},\frac{2c}{3},\frac{2d}{3},\frac{\frac{2c}{3}\cdot\frac{2d}{3}}{1+\frac{d}{3}},\frac{\frac{2c}{3}\cdot\frac{2d}{3}}{1+\frac{2d}{3}}\Big\},\\ &= \frac{2d}{3}.\end{aligned}$$

Also, $\xi(\wedge(\mathfrak{Q}c,\mathfrak{Q}d)) = \frac{1}{8}(\frac{2d}{3}) = \frac{d}{12}$, clearly

$$d^*(\mathcal{P}c,\mathcal{P}d) > d^*(\mathfrak{Q}c,\mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c,\mathfrak{Q}d)).$$

Case (iii): If $c = d \neq 0$:

$$d^*(\mathcal{P}c,\mathcal{P}d) = d^*(\tfrac{2c}{3},\tfrac{2d}{3}) = max\{\tfrac{2c}{3},\tfrac{2c}{3}\} = \tfrac{2c}{3},$$
$$d^*(\mathfrak{Q}c,\mathfrak{Q}d) = d^*(\tfrac{c}{3},\tfrac{d}{3}) = max\{\tfrac{c}{3},\tfrac{c}{3}\} = \tfrac{c}{3},$$

where

$$\begin{aligned}\wedge(\mathfrak{Q}c,\mathfrak{Q}d) &= \max\Big\{(d^*(\mathfrak{Q}c,\mathfrak{Q}d), d^*(\mathfrak{Q}c,\mathcal{P}c), d^*(\mathfrak{Q}d,\mathcal{P}d),\\ &\qquad \frac{d^*(\mathfrak{Q}c,\mathcal{P}c)\cdot d^*(\mathfrak{Q}d,\mathcal{P}d)}{1+d^*(\mathfrak{Q}c,\mathfrak{Q}d)}, \frac{d^*(\mathfrak{Q}c,\mathcal{P}c)\cdot d^*(\mathfrak{Q}d,\mathcal{P}d)}{1+d^*(\mathcal{P}c,\mathcal{P}d)}\Big\}\\ &= \max\Big\{(d^*(\frac{c}{3},\frac{c}{3}), d^*(\frac{c}{3},\frac{2c}{3}), d^*(\frac{c}{3},\frac{2c}{3}),\\ &\qquad \frac{d^*(\frac{c}{3},\frac{2c}{3})\cdot d^*(\frac{c}{3},\frac{2c}{3})}{1+d^*(\frac{c}{3},\frac{c}{3})}, \frac{d^*(\frac{c}{3},\frac{2c}{3})\cdot d^*(\frac{c}{3},\frac{2c}{3})}{1+d^*(\frac{2c}{3},\frac{2c}{3})}\Big\}\\ &= \max\Big\{\frac{c}{3},\frac{2c}{3},\frac{2c}{3},\frac{\frac{2c}{3}\cdot\frac{2c}{3}}{1+\frac{c}{3}},\frac{\frac{2c}{3}\cdot\frac{2c}{3}}{1+\frac{2c}{3}}|\Big\},\\ &= \frac{2c}{3}.\end{aligned}$$

Also, $\xi(\wedge(\mathfrak{Q}c,\mathfrak{Q}d)) = \frac{1}{8}(\frac{2c}{3}) = \frac{c}{12}$, clearly

$$d^*(\mathcal{P}c,\mathcal{P}d) > d^*(\mathfrak{Q}c,\mathfrak{Q}d) + \xi(\wedge(\mathfrak{Q}c,\mathfrak{Q}d)).$$

Hence the inequality of Theorem 2.1 holds for all the cases.

Now,

$$\lim_{n\to\infty} \mathcal{P}c_n = \lim_{n\to\infty} \frac{2}{3n} = \lim_{n\to\infty} \mathfrak{Q}c_n = \lim_{n\to\infty} \frac{1}{3n} = 0,$$

where $0 \in M$. This implies $\mathcal{P}$ and $\mathfrak{Q}$ satisfies E.A. property. Also, we have

$$\lim_{n\to\infty} \mathcal{P}c_n = \lim_{n\to\infty} \frac{2}{3n} = \lim_{n\to\infty} \mathfrak{Q}c_n = \lim_{n\to\infty} \frac{1}{3n} = 0 = \mathcal{P}(0),$$

where $0 \in M$. This implies $\mathcal{P}$ and $\mathfrak{Q}$ satisfies (CLR) property. Hence all the properties of Theorems 2.1, 2.3 and 2.4 are satisfied. Here 0 is the common fixed point of $\mathcal{P}$ and $\mathfrak{Q}$.

References

[1] M. Aamri, D.El. Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., **270** (2002), 181-188.

[2] A.H. Ansari and A. Mutlu, *C-class function on coupled fixed point theorem for mixed monotone mappings on partially ordered dislocated quasi metric spaces,* Nonlinear Funct. Anal. Appl., **22**(1) (2017), 99-106.

[3] A. Dahiya, A. Rani and K. Jyoti, *Common fixed point for generalized (ψ, α, β)-weakly contractive mappings in dislocated metric space*, Global J. Pure App. Math., **13**(7) (2017), 3067-3081.

[4] P. Hitzler and A.K. Seda, *Dislocated Topologies*, J. Electr. Engg., **51**(12) (2000), 3-7.

[5] G. Jungck, *Common fixed points for non-continuous non-self maps on non-metric spaces*, Far East J. Math.Sci., **4**(2) (1996), 199-212.

[6] S.M. Kang, M. Kumar, P. Kumar and S. Kumar, *Fixed point theorems for ϕ-weakly expansive mappings in metric space*, Int. J. Pure Appl. Math., **90**(2) (2014), 143-152.

[7] F.A. Khan, A.M. Alanazi, Javid Ali and Dalal J. Alanazi, *Parametric generalized multi-valued nonlinear quasi-variational inclusion problem,* Nonlinear Funct. Anal. Appl., **26**(5) (2021), 917-933.

[8] J.K. Kim, M. Kumar, P. Bhardwaj and M. Imdad, *Common fixed point theorems for generalized $\psi_{\int\varphi}$-weakly contractive mappings in G-metric spaces,* Nonlinear Funct. Anal. Appl., **26**(3) (2021), 565-580.

[9] W. Sintunavarat and P. Kumam, *Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space*, J. Appl. Math., Article ID 637958, **2011** (2011), 14 pages.

[10] S.Z. Wang, B.Y. Li, Z.M. Gao and K. Iseki, *Some fixed point theorems on expansion mappings*, Math. Japon., **29** (1984), 631-636.