



A COMMON FIXED POINT THEOREM IN AN M^* -METRIC SPACE AND AN APPLICATION

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Abstract. In this paper, we introduce the concept of M^* -metric spaces and how much the M^* -metric and the b -metric spaces are related. Moreover, we introduce some ways of generating M^* -metric spaces. Also, we investigate some types of convergence associated with M^* -metric spaces. Some common fixed point for contraction and generalized contraction mappings in M^* -metric spaces. Our work has been supported by many examples and an application.

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1. INTRODUCTION

In 1994, the concept of D^* -metric space is defined by Dhage [12] which is a generalized metric space.

Definition 1.1. ([12]) Let $\mathbb{X} \neq \emptyset$ be a set. A function $D^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ is called a D^* -metric, if the following properties are satisfied for each $x, y, z \in \mathbb{X}$.

- (D^*1) : $D^*(x, y, z) \geq 0$.
- (D^*2) : $D^*(x, y, z) = 0$ iff $x = y = z$.
- (D^*3) : $D^*(x, y, z) = D^*(p(x, y, z))$; for any permutation $p(x, y, z)$ of x, y, z .
- (D^*4) : $D^*(x, y, z) \leq D^*(x, y, \ell_2) + D^*(\ell_2, z, z)$.

A pair (\mathbb{X}, D^*) is called a D^* -metric space.

In the following, the notion of the b -metric space is defined by Bakhtin [6] and Czerwik [11], there are many fixed point theorems in a b -metric space for more information. I refer to the reader to look at [1 – 11], [15 – 39].

Definition 1.2. ([6, 11]) Let $\mathbb{X} \neq \emptyset$ be a set and $S \geq 1$ be a real number. A function $d : \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ is called a b -metric [6, 12], if it satisfies the following properties for each $x, y, z \in \mathbb{X}$.

- (b1) : $d(x, y) = 0$ iff $x = y$;
- (b2) : $d(x, y) = d(y, x)$;
- (b3) : $d(x, z) \leq S [d(x, y) + d(y, z)]$.

Now, we define the notion of the M^* -metric space which is a generalization of a b -metric space and an M^* -metric space the tetrahedral inequality axiom is weaker than for a D^* -metric space.

Definition 1.3. Let \mathbb{X} be a non empty set and $R \geq 1$ be a real number. A function $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ is called a M^* -metric, if the followings are satisfied the properties: for each $x, y, z \in \mathbb{X}$.

- (M^*1) : $M^*(x, y, z) \geq 0$.
- (M^*2) : $M^*(x, y, z) = 0$ iff $x = y = z$.
- (M^*3) : $M^*(x, y, z) = M^*(p(x, y, z))$; for any permutation $p(x, y, z)$ of x, y, z .
- (M^*4) : $M^*(x, y, z) \leq RM^*(x, y, u) + M^*(u, z, z)$.

A pair (\mathbb{X}, M^*) is called an M^* -metric space.

Now, we introduce two examples that satisfy the four axioms for M^* -metric.

Example 1.4. For $x, y, z \in \mathbb{R}$, define

- (1) $M_1^*(x, y, z) = \frac{1}{R} [|x - y| + |y - z| + |z - x|]$.
- (2) $M_\infty^*(x, y, z) = \frac{1}{R} \max \{|x - y|, |y - z|, |z - x|\}$.

Then we can say that (\mathbb{R}, M_1^*) and (\mathbb{R}, M_∞^*) are M^* -metric spaces.

Example 1.5. Define a function M^* on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ by

$$M^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ 1, & \text{otherwise.} \end{cases}$$

Then M^* is the discreet M^* -metric on \mathbb{X} .

Note: In the following, we will present very important characteristics that are always realized in the M^* -metric space, the importance of which lies in the theories presented in this paper. It is worth noting that these characteristics need not be satisfied in MR -metric space defined by Malkawi et. al. [23].

- $(M^*5) : M^*(x, x, y) = M^*(x, y, y).$
- $(M^*6) : M^*(x, y, y) \leq RM^*(y, y, z) + M^*(z, x, x).$

Since

$$\begin{aligned} M^*(x, x, y) &\leq RM^*(x, x, x) + M^*(x, y, y) \\ &= M^*(x, y, y) \end{aligned}$$

and

$$\begin{aligned} M^*(x, y, y) &\leq RM^*(y, y, y) + M^*(y, x, x) \\ &= M^*(x, x, y). \end{aligned}$$

Thus, we have

$$M^*(x, x, y) = M^*(x, y, y).$$

Next, also we have from (M^*5)

$$\begin{aligned} M^*(x, y, y) &= M^*(y, y, x) \\ &\leq RM^*(y, y, z) + M^*(z, x, x) \\ &= RM^*(y, y, z) + M^*(x, z, z). \end{aligned}$$

Remark 1.6. The M^* -metrics in examples 1.4, 1.5 are satisfied the following properties: For all x, y, z, ℓ_1, ℓ_2 in \mathbb{X} , we have

- $(M^*7) : M^*(x, y, y) \leq RM^*(x, y, z).$
- $(M^*8) : M^*(x, y, z) \leq \frac{1}{R} [M^*(x, \ell_1, \ell_1) + M^*(z, \ell_1, \ell_2)].$

We will use the following example to show that (M^*6) does not implies (M^*7) .

Example 1.7. Suppose \mathbb{X} has at least three elements. Define M^* on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ by

$$M^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \frac{1}{2R}, & \text{if } x, y, z \text{ are distinct,} \\ 1, & \text{otherwise.} \end{cases}$$

Then (\mathbb{X}, M^*) is an M^* -metric space but (M^*7) is not satisfied.

By adding some conditions and properties, we will presented some of the interconnections between M^* -metric space and b -metric space.

Proposition 1.8. *If the M^* -metric space (\mathbb{X}, M^*) satisfies (M^*5) and (M^*6) , then $d(x, y) = M^*(x, y, y)$ is a b -metric on \mathbb{X} .*

Proof. Let $x, y \in \mathbb{X}$, we want to show (\mathbb{X}, d) is a b -metric space.

- (i) By (M^*1) , $d(x, y) = M^*(x, y, y) \geq 0$.
- (ii) By (M^*2) , $d(x, y) = M^*(x, y, y) = 0$ iff $x = y$.
- (iii) By (M^*5) , (M^*3) ,

$$d(x, y) = M^*(x, y, y) = M^*(x, x, y) = M^*(y, x, x) = d(y, x).$$

- (iv) By (M^*6) ,

$$\begin{aligned} d(x, y) &= M^*(x, y, y) \leq RM^*(x, z, z) + M^*(z, y, y) \\ &= Rd(x, z) + d(z, y) \\ &\leq R[d(x, z) + d(z, y)]. \end{aligned}$$

Thus, (\mathbb{X}, d) is a b -metric space. □

Example 1.9. Let $\mathbb{X} := l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Define $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}^+$ by

$$M^*(x, y, z) = \begin{cases} 0, & \text{iff } x = y = z, \\ 1, & \text{iff } x, y, z \text{ are distinct,} \\ \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}}, & \text{iff } x \neq y = z \text{ or } x = z \neq y, \\ \left(\sum_{n=1}^{\infty} |y_n - x_n|^p \right)^{\frac{1}{p}}, & \text{iff } y \neq z = x \text{ or } x = y \neq z, \\ \left(\sum_{n=1}^{\infty} |z_n - x_n|^p \right)^{\frac{1}{p}}, & \text{iff } x \neq z = y \text{ or } y = x \neq z, \end{cases}$$

where $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$. Then (\mathbb{X}, M^*) is a M^* -metric space with coefficient $R > 1$.

To show M^* is an M^* -metric, we have to show that only (M^*4) is hold, since (M^*1) , (M^*2) and (M^*3) are obvious.

Case 1: If x, y, z are distinct, then we have two cases:

(1) If $u \notin \{x, y, z\}$, then

$$\begin{aligned} 1 &= M^*(x, y, z) \\ &\leq RM^*(x, y, u) + M^*(u, z, z) \\ &= R \cdot 1 + \left(\sum_{n=1}^{\infty} |z_n - u_n|^p \right)^{\frac{1}{p}}. \end{aligned}$$

(2) If $u = x$, then

$$\begin{aligned} 1 &= M^*(x, y, z) \\ &\leq RM^*(x, y, u) + M^*(u, z, z) \\ &= R \cdot 1 + \left(\sum_{n=1}^{\infty} |u_n - z_n|^p \right)^{\frac{1}{p}}. \end{aligned}$$

It is similar if $u = y$ or $u = z$.

Case 2 : If $x = y \neq z$, then we have three cases:

(1) If $u \notin \{y, z\}$, then

$$\begin{aligned} M^*(x, y, z) &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p \right)^{\frac{1}{p}} \\ &\leq RM^*(x, y, u) + M^*(u, z, z) \\ &= R \left(\sum_{n=1}^{\infty} |y_n - u_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |u_n - z_n|^p \right)^{\frac{1}{p}}. \end{aligned}$$

(2) If $u = x = y \neq z$, then

$$\begin{aligned} M^*(x, y, z) &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p \right)^{\frac{1}{p}} \\ &\leq RM^*(x, y, u) + M^*(u, z, z) \\ &= 0 + \left(\sum_{n=1}^{\infty} |u_n - z_n|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p \right)^{\frac{1}{p}}. \end{aligned}$$

(3) If $x = y \neq z = u$, then

$$\begin{aligned} M^*(x, y, z) &= \left(\sum_{n=1}^{\infty} |y_n - z_n|^p \right)^{\frac{1}{p}} \\ &\leq RM^*(x, y, u) + M^*(u, z, z) \\ &= R \left(\sum_{n=1}^{\infty} |y_n - u_n|^p \right)^{\frac{1}{p}} + 0 \\ &= R \left(\sum_{n=1}^{\infty} |y_n - z_n|^p \right)^{\frac{1}{p}} \end{aligned}$$

and it is similar if $x \neq y = z$.

Moreover, (\mathbb{X}, M^*) is not D^* -metric space.

Let $x = (1, 1, \dots, 1, 0, 0, \dots)$, $y = (-1, -1, \dots, -1, 0, 0, \dots)$ and $u = (1, -1, \dots, -1, 0, 0, \dots)$, where the number of nonzero element of x, y, u is $2n$. So,

$$\begin{aligned} M^*(x, x, y) &= \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{2n} 2^p \right)^{\frac{1}{p}} = 2 \cdot (2n)^{\frac{1}{p}}, \\ M^*(x, x, u) &= \left(\sum_{i=1}^{\infty} |x_i - u_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^n 2^p \right)^{\frac{1}{p}} = 2 \cdot (n)^{\frac{1}{p}}, \\ M^*(u, y, y) &= \left(\sum_{i=1}^{\infty} |x_i - u_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^n 2^p \right)^{\frac{1}{p}} = 2 \cdot (n)^{\frac{1}{p}}. \end{aligned}$$

But,

$$32 = M^*(x, x, y) \not\leq M^*(x, x, u) + M^*(u, y, y) = 8 + 8 = 16,$$

when $n = 2$ and $p = \frac{1}{2}$.

Theorem 1.10. *If (\mathbb{X}, M^*) is an M^* -metric space, then any function $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}^+$ defined by*

- (i) *for $1 \leq q < \infty$, $d(x, y) = \{M^{*q}(x, y, y) + M^{*q}(x, x, y)\}^{\frac{1}{q}}$ is a b-metric on \mathbb{X} .*
- (ii) *$d(x, y) = \max \{M^*(x, y, y), M^*(x, x, y)\}$ for all $x, y \in \mathbb{X}$, is a b-metric on \mathbb{X} .*

Proof. It suffices to prove (i), since (ii) are the same.

Obviously, $d(x, y) \geq 0$ for all $x, y \in \mathbb{X}$ and $d(x, y) = 0$ if and only if $x = y$.

Now, let $x, y, z \in \mathbb{X}$. Then, for $1 \leq q < \infty$,

$$\begin{aligned} d(x, y) &= \{M^{*q}(x, y, y) + M^{*q}(x, x, y)\}^{\frac{1}{q}} \\ &= \{M^{*q}(y, y, x) + M^{*q}(y, x, x)\}^{\frac{1}{q}} \\ &= \{M^{*q}(y, x, x) + M^{*q}(y, y, x)\}^{\frac{1}{q}} \\ &= d(y, x). \end{aligned}$$

$$\begin{aligned} d(x, y) &= \{M^{*q}(x, y, y) + M^{*q}(x, x, y)\}^{\frac{1}{q}} \\ &\leq \{(RM^*(y, y, z) + M^*(z, x, x))^q \\ &\quad + (RM^*(y, y, z) + M^*(z, x, x))^q\}^{\frac{1}{q}} \\ &\leq R\{(M^*(y, y, z) + M^*(z, x, x))^q \\ &\quad + (M^*(y, y, z) + M^*(z, x, x))^q\}^{\frac{1}{q}} \\ &\leq R\{M^{*q}(y, y, z) + M^{*q}(x, x, z)\}^{\frac{1}{q}} \\ &\quad + \{M^{*q}(z, y, y) + M^{*q}(z, z, y)\}^{\frac{1}{q}} \\ &\leq R[d(x, z) + d(z, y)]. \end{aligned}$$

Hence d is a b - M^* -metric on \mathbb{X} . □

Theorem 1.11. *Let $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ be a function satisfying (M^*1) , (M^*2) , (M^*3) , (M^*7) and (M^*8) . Then M^* is an M^* -metric on \mathbb{X} .*

Proof. In order to show that M^* is an M^* -metric on \mathbb{X} it is enough to show that (M^*4) is satisfied. Let $x, y, z \in \mathbb{X}$,

$$\begin{aligned} M^*(x, y, z) &\leq \frac{1}{R}M^*(x, \ell_1, \ell_1) + M^*(z, \ell_1, \ell_2) \\ &\leq \frac{R}{R}M^*(x, \ell_1, y) + M^*(\ell_1, \ell_1, z) \\ &\leq RM^*(x, y, \ell_1) + M^*(\ell_1, z, z). \end{aligned}$$

Thus (M^*4) holds and hence M^* is an M^* -metric on \mathbb{X} . □

2. WAYS OF GENERATING M^* -METRICS

In this section, we present some ways of generating M^* -metric spaces. Let $\aleph = \{(b_1, b_2, b_3) \in (\mathbb{R}^+)^3 : b_1 \leq \frac{1}{R}(b_2 + b_3), b_2 \leq \frac{1}{R}(b_1 + b_3), b_3 \leq \frac{1}{R}(b_1 + b_2)\}$.

Theorem 2.1. *Suppose that the function $\Psi : \aleph \rightarrow \mathbb{R}^+$ satisfies*

- (i) $\Psi(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3) = \Psi(p(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3))$, for any permutation $p(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3)$ of $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3$.

- (ii) $\Psi(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3) = 0$ iff $\mathfrak{S}_1 = \mathfrak{S}_2 = \mathfrak{S}_3 = 0$,
- (iii) $\Psi(t, t, 0) \leq \Psi(\mathfrak{S}, \mathfrak{S}_1, \mathfrak{S}_2)$ for every $(\mathfrak{S}, \mathfrak{S}_1, \mathfrak{S}_2) \in \aleph$,
- (iv) $\Psi(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3) \leq \frac{1}{R} [\Psi(\mathfrak{S}'_1, \mathfrak{S}, \mathfrak{S}''_1) + \Psi(\mathfrak{S}'_2, \mathfrak{S}, \mathfrak{S}''_2)]$

for all $(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3), (\mathfrak{S}'_1, \mathfrak{S}, \mathfrak{S}''_1), (\mathfrak{S}'_2, \mathfrak{S}, \mathfrak{S}''_2)$ and $(\mathfrak{S}_3, \mathfrak{S}, \mathfrak{S}''_3)$ in \aleph , where $(\mathfrak{S}_1, \mathfrak{S}'_1, \mathfrak{S}''_2), (\mathfrak{S}_1, \mathfrak{S}'_1, \mathfrak{S}''_2), (\mathfrak{S}_2, \mathfrak{S}'_2, \mathfrak{S}''_3), (\mathfrak{S}_2, \mathfrak{S}'_2, \mathfrak{S}''_3), (\mathfrak{S}_3, \mathfrak{S}'_3, \mathfrak{S}''_1), (\mathfrak{S}_3, \mathfrak{S}'_3, \mathfrak{S}''_1) \in \aleph$. Let (\mathbb{X}, d) be a b -metric space. Define a function $M^* : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ by

$$M^*(x, y, z) = R\Psi(d(x, y), d(y, z), d(z, x)).$$

Then M^* is an M^* -metric on \mathbb{X} .

Proof. Since M^* satisfies (M^*1) , (M^*2) and (M^*3) , it is enough to show that (M^*7) and (M^*8) are satisfied.

Let $x, y, z, \ell_1, \ell_2 \in \mathbb{X}$. Then it follows from (iii) and (iv) that

$$\begin{aligned} M^*(x, y, y) &= R\Psi(d(x, y), 0, d(y, x)) \\ &\leq R\Psi(d(x, y), d(y, z), d(z, x)) \\ &= RM^*(x, y, z) \end{aligned}$$

and

$$\begin{aligned} M^*(x, y, z) &= R\Psi(d(x, y), d(y, z), d(z, x)) \\ &\leq \Psi(d(x, \ell_1), d(\ell_2, x)) \\ &\quad + \Psi(d(z, \ell_1), d(\ell_1, \ell_2), d(\ell_2, z)) \\ &\leq RM^*(x, \ell_1, \ell_1) + M^*(\ell_1, z, \ell_2). \end{aligned}$$

Thus, the hypothesis of Theorem 1.11 are satisfied for M^* and hence M^* is an M^* -metric on \mathbb{X} . \square

Theorem 2.2. Suppose that $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the following properties:

- (i) $\Phi(b) = 0$ iff $b = 0$,
- (ii) Φ is monotone increasing,
- (iii) $\Phi(s + t) \leq \frac{1}{R} [\Phi(s) + \Phi(t)]$ for all $s, t \in \mathbb{R}^+$.

Then $\Psi(b_1, b_2, b_3) = \Phi(b_1) + \Phi(b_2)$ has all the properties that identified in Theorem 2.1.

Proof. It is clear that Ψ satisfy (i) of Theorem 2.1. Note that

$$\Psi(b_1, b_2, b_3) = 0 \iff \Phi(b_1) + \Phi(b_2) + \Phi(b_3) = 0 \iff b_1 = b_2 = b_3 = 0.$$

Let $(b_1, b_2, b_3) \in \aleph$. Consider the following triples in \aleph :

$$(b_1, b_2, b_3), (b'_1, b, b''_1), (b'_2, b, b''_2), (b'_3, b, b''_3), (b_1, b'_1, b'_2), (b_1, b''_1, b''_2), (b_2, b'_2, b'_3),$$

$(b_2, b_2'', b_3''), (b_3, b_3', b_1') \in \aleph$. Then $(b_3, b_3'', b_1'') \in \aleph$, where $\aleph \subset (\mathbb{R}^+)^3$. So,

$$\begin{aligned} \Psi(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3) &= \Phi(\mathfrak{S}_1) + \Phi(\mathfrak{S}_2) \\ &\leq \Phi(\mathfrak{S}'_1 + \mathfrak{S} + \mathfrak{S}''_2) + \Phi(\mathfrak{S}'_2 + \mathfrak{S} + \mathfrak{S}''_3) \\ &\leq \frac{1}{R}\Phi(\mathfrak{S}'_1) + \frac{1}{R^2}\Phi(\mathfrak{S}) + \frac{1}{R^2}\Phi(\mathfrak{S}''_2) \\ &\quad + \frac{1}{R}\Phi(\mathfrak{S}'_2) + \frac{1}{R^2}\Phi(\mathfrak{S}) + \frac{1}{R^2}\Phi(\mathfrak{S}''_3) \\ &\leq \frac{1}{R}[\Psi(\mathfrak{S}'_1, \mathfrak{S}, \mathfrak{S}''_1) + \Psi(\mathfrak{S}'_2, \mathfrak{S}, \mathfrak{S}''_2)]. \end{aligned}$$

It is clear that $\Psi(b, b, 0) \leq \Psi(b, b_1, b_2)$ for all $(b_1, b, b_2) \in \aleph$. Hence Ψ satisfies all conditions specified in Theorem 2.1. \square

Now, in order to show that the two conditions (ii) and (iii) are independent in previous theorem, we give the following example.

Example 2.3. Define a function $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\Phi(b) = 2b$ for all $b \in \mathbb{R}^+$ satisfies the hypothesis of Theorem 2.2.

Also, the conditions (ii) and (iii) are independent. For example the function $2b^2$, (ii) holds but (iii) does not hold. While $\Phi(b) = 0$ if $b = 0$ and $\Phi(b) = b + \frac{1}{b}$ if $b > 0$ satisfies (iii) but not (ii).

Theorem 2.4. Let (\mathbb{X}, d) be a metric space. Define real functions $M_1^*, M_\infty^*, M_3^*, M_4^*$ on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ by

$$\begin{aligned} M_1^*(x, y, z) &= d(x, y) + d(y, z) + d(z, x), \\ M_\infty^*(x, y, z) &= \max \{d(x, y), d(y, z), d(z, x)\}, \end{aligned}$$

$$M_3^*(x, y, z) = \begin{cases} M_1^*(x, y, z), & \text{if } x, y, z \text{ are distinct,} \\ M_\infty^*(x, y, z), & \text{otherwise,} \end{cases}$$

and

$$M_4^*(x, y, z) = \begin{cases} M_\infty^*(x, y, z), & \text{if } x, y, z \text{ are distinct,} \\ M_1^*(x, y, z), & \text{otherwise.} \end{cases}$$

Then $M_1^*, M_\infty^*, M_3^*, M_4^*$ are M^* -metric on \mathbb{X} .

Proof. It is clear that M_1^* and M_∞^* are M^* -metrics and all the proofs of M_3^* and M_4^* are similar, it is enough to show that M_4^* is an M^* -metric. Also, it is enough to show that tetrahedral inequality is satisfied.

Let $x, y, z, \ell_1 \in \mathbb{X}$.

Case 1: x, y, z are distinct. While preserving the generality, we assume that

$$d(x, y) \leq d(y, z) \leq d(z, x).$$

(1) If $\ell_1 \notin \{x, y, z\}$, then

$$\begin{aligned} M_4^*(x, y, z) &= d(x, z) \\ &\leq R[d(x, \ell_1) + d(\ell_1, z)] \\ &\leq R[M_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z)]. \end{aligned}$$

(2) If $\ell_1 = x$, then

$$\begin{aligned} M_4^*(x, y, z) &= M_4^*(\ell_1, y, z) = d(\ell_1, z) \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

If $\ell_1 = y$ or $\ell_1 = z$, then the proof is similar.

Case 2: Assume $x = y \neq z$.

(1) If $\ell_1 \notin \{y, z\}$, then

$$\begin{aligned} M_4^*(x, y, z) &= d(y, z) + d(z, y) \\ &\leq R[d(y, \ell_1) + d(\ell_1, z) + d(z, \ell_1) + d(\ell_1, y)] \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

(2) If $\ell_1 = y$, then

$$\begin{aligned} M_4^*(x, y, z) &= M_4^*(x, \ell_1, z) \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

(3) If $\ell_1 = z \neq y$, then

$$\begin{aligned} M_4^*(x, y, z) &= M_4^*(x, y, \ell_1) \\ &\leq RM_4^*(x, y, \ell_1) + M_4^*(\ell_1, z, z). \end{aligned}$$

Hence M_4^* is an M^* -metric on \mathbb{X} . □

3. TYPES OF CONVERGENCE ASSOCIATED WITH AN M^* -METRIC

In light of the definition of a D -convergent and a D -Cauchy for a D -metric [13], we define M^* -convergent and M^* -Cauchy for M^* -metric.

Definition 3.1. A sequence $\{x_n\}$ in an M^* -metric space (\mathbb{X}, M^*) is called M^* -convergent if there exists x in \mathbb{X} such that for $\epsilon > 0$, there exists a $N > 0$ integer number such that $M^*(x_n, x_m, x) < \epsilon$ for all $m \geq N, n \geq N$. Then we called that $\{x_n\}$ is M^* -convergent to x and x is a limit of $\{x_n\}$.

Definition 3.2. A sequence $\{x_n\}$ in an M^* -metric space (\mathbb{X}, M^*) is called M^* -Cauchy if for a given $\epsilon > 0$, there exists a positive integer N such that $M^*(x_n, x_m, x_p) < \epsilon$ for all $m, n, p \geq N$.

In the following, we introduce the concept of an M^* -strongly convergent and a very M^* -strongly convergent. Take into can a sequence $\{x_n\}$ in an M^* -metric in the following two definitions.

Definition 3.3. Let (\mathbb{X}, M^*) be an M^* -metric space and $\{x_n\}$ be a sequence in \mathbb{X} , we say that $\{x_n\}$ is M^* -strongly convergent to an element x in \mathbb{X} if

- (i) $M^*(x_n, x_m, x) \rightarrow 0$ as $m, n \rightarrow \infty$,
- (ii) $\{M^*(y, y, x_n)\}$ converges to $M^*(y, y, x)$ for all $y \in \mathbb{X}$.

Definition 3.4. Let (\mathbb{X}, M^*) be an M^* -metric space and $\{x_n\}$ be a sequence in \mathbb{X} , we call that $\{x_n\}$ is very M^* -strongly convergent to an element x in \mathbb{X} if

- (i) $M^*(x_n, x_m, x) \rightarrow 0$ as $m, n \rightarrow \infty$,
- (ii) $\{M^*(y, z, x_n)\}$ converges to $M^*(y, z, x)$ for all $y, z \in \mathbb{X}$.

By using some properties of Remark 1.1, we present some results on M^* -convergence, M^* -Cauchy, M^* -strongly convergent and very M^* -strongly convergent.

Theorem 3.5. Let (\mathbb{X}, M^*) be an M^* -metric space. Then $\{x_n\}$ converges to x in (\mathbb{X}, M^*) strongly if and only if $\{x_n\}$ converges to x in (\mathbb{X}, M^*) and $\lim_{n \rightarrow \infty} M^*(x, x, x_n) = 0$.

Proof. Let $\{x_n\}$ be an M^* -convergent sequence in \mathbb{X} with limit x , that is, $\lim_{n \rightarrow \infty} M^*(x, x, x_n) = 0$ and $\epsilon > 0$. Then there is a positive integer N such that $M^*(x, x, x_n) < \epsilon$ for all $n \geq N$. Let $y \in \mathbb{X}$. Then for $n \geq N$,

$$\begin{aligned} M^*(y, y, x_n) &\leq RM^*(y, y, x) + M^*(x_n, x, x) \\ &\leq R[M^*(y, y, x) + M(x_n, x, x)]. \end{aligned}$$

This produces that

$$|M^*(y, y, x_n) - RM^*(y, y, x)| \leq RM^*(x, x, x_n) < R\epsilon = \epsilon_1 \quad \text{for all } n \geq N.$$

Consequently, we get

$$|M^*(y, y, x_n) - M^*(y, y, x)| \leq RM^*(x, x, x_n) < R\epsilon = \epsilon_1.$$

Hence $\{M^*(y, y, x_n)\}$ converges to $M^*(y, y, x)$ for all $y \in \mathbb{X}$. This means that $\{x_n\}$ converges strongly to x in \mathbb{X} . □

We can easily prove the theorem from the definition of the M^* -metric.

Theorem 3.6. Let (\mathbb{X}, M^*) be an M^* -metric space and let $\{x_n\}$ be a sequence in \mathbb{X} and $x \in \mathbb{X}$. Assume the following implications:

- (1) $M^*(x, x, x_n) \rightarrow 0$ as $n \rightarrow \infty$,
- (2) $M^*(x, x_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$,

- (3) $M^*(x, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$,
- (4) $M^*(y, y, x_n) \rightarrow M^*(y, y, x)$ as $n \rightarrow \infty$ for all $y \in \mathbb{X}$,
- (5) $M^*(y, x_n, x_m) \rightarrow M^*(y, x, x)$ as $n, m \rightarrow \infty$ for all $y \in \mathbb{X}$,
- (6) $M^*(y, x, x_n) \rightarrow M^*(y, x, x)$ as $n \rightarrow \infty$ for all $y \in \mathbb{X}$,
- (7) $M^*(y, z, x_n) \rightarrow M^*(y, z, x)$ as $n \rightarrow \infty$ for all $z, y \in \mathbb{X}$.

Then (7) \implies (6) \implies (1), (7) \implies (4) \implies (1) and (5) \implies (3) \implies (2).

Proof. We can easily prove the theorem from the definition of the M^* -convergent and M^* -metric. \square

Inside the following example, we provide some non implications of Theorem 3.6.

Example 3.7. Either (3) or (4) does not imply (5), (6) or (7).

Let $\mathbb{X} = \mathbb{R}$ with an M^* -metric. Then the function M_3^* is defined in Theorem 2.4 on $\mathbb{X} \times \mathbb{X} \times \mathbb{X}$ reduces to the following.

$$M_3^* = \begin{cases} 0, & \text{if } x = y = z, \\ |x - y| + |y - z| + |z - x|, & \text{if } x, y, z \text{ are distinct,} \\ \max\{|x - y|, |y - z|, |z - x|\}, & \text{otherwise.} \end{cases}$$

Then (\mathbb{X}, M_3^*) is an M^* -metric space in which (3) and (4) are satisfied but (5), (6) and (7) are not satisfied.

Let $x_n = 2^{\frac{1}{n}}$ for $n = 1, 2, 3, \dots$. Then $\{x_n\}$ converges to 1 as $n \rightarrow \infty$ with respect to the M^* -metric. For $m > n$, we have

$$M_3^*(1, 2^{\frac{1}{n}}, 2^{\frac{1}{m}}) = \left|1 - 2^{\frac{1}{n}}\right| + \left|2^{\frac{1}{m}} - 2^{\frac{1}{n}}\right| + \left|2^{\frac{1}{m}} - 1\right| \rightarrow 0, \quad \text{as } n, m \rightarrow \infty.$$

Therefore, $\left\{2^{\frac{1}{n}}\right\}$ convergent to 1 with respect to M_3^* . Since for $y \in \mathbb{X}$,

$$M_3^*(y, y, 2^{\frac{1}{n}}) = \max\left\{\left|y - 2^{\frac{1}{n}}\right|, 0, \left|y - 2^{\frac{1}{n}}\right|\right\} = \left|y - 2^{\frac{1}{n}}\right|,$$

we have $\lim_{n \rightarrow \infty} M_3^*(y, y, 2^{\frac{1}{n}}) = M^*(y, y, 1)$ for all $y \in \mathbb{X}$. Thus (3) and (4) hold.

Let $y = 3$ and $x = 1$. Trivially, $\lim_{n \rightarrow \infty} M_3^*(3, 1, 2^{\frac{1}{n}}) \neq M_3^*(3, 1, 1)$, thus (6) and (7) do not hold. Additionally $\lim_{n, m \rightarrow \infty} M_3^*(3, 2^{\frac{1}{n}}, 2^{\frac{1}{m}}) \neq M_3^*(3, 1, 1)$, thus (5) does not hold.

Theorem 3.8. Let (\mathbb{X}, M^*) be an M^* -metric space satisfying (M_7^*) and (M_8^*) . Then the function d on $\mathbb{X} \times \mathbb{X} \rightarrow [0, \infty)$ is defined by $d(x, y) = M^*(x, y, y)$ is b -metric on \mathbb{X} and the following are equivalent:

- (i) $\lim_{n \rightarrow \infty} x_n = x$ in (\mathbb{X}, d) .
- (ii) $\lim_{n \rightarrow \infty} x_n = x$ in (\mathbb{X}, M^*) .

(iii) $\lim_{n \rightarrow \infty} x_n = x$ strongly in (\mathbb{X}, M^*) .

Proof. By Proposition 1.8, it is clear that (\mathbb{X}, d) is a b -metric space.

Assume (i) holds, then $\lim_{n \rightarrow \infty} x_n = x$ in (\mathbb{X}, d) .

Let $\epsilon > 0$. Then there exists an integer number $N > 0$ such that $d(x, x_n) < \frac{\epsilon}{2}$ for all $n \geq N$. For $n, m \geq N$,

$$\begin{aligned} M^*(x, x_n, x_m) &\leq \frac{1}{R} [M^*(x, x, x_m) + M^*(x, x, x_n)] \\ &= \frac{1}{R} [d(x, x_m) + d(x, x_n)] < \epsilon. \end{aligned}$$

Thus (i) \implies (ii).

Assume (ii) holds, then $\lim_{n \rightarrow \infty} x_n = x$ in (\mathbb{X}, M^*) .

Let $\epsilon > 0$. Then there exists a positive integer N such that $M^*(x_n, x_m, x) < \epsilon$ for all $m, n \geq N$. For $y \in \mathbb{X}$ and $n \geq N$, by (M^*7) ,

$$\begin{aligned} M^*(y, y, x_n) &\leq RM^*(x, y, x_n) \\ &\leq \frac{R}{R} [M^*(x, x, x_n) + M^*(y, x, x)] \\ &\leq M^*(x, x_n, x_n) + M^*(y, y, x). \end{aligned}$$

Then, we get

$$|M^*(y, y, x_n) - M^*(y, y, x)| \leq M^*(x, x_n, x_n) < \frac{\epsilon}{2} \text{ for all } n \geq N.$$

Hence $\{M^*(y, y, x_n)\}$ converges to $M^*(y, y, x)$ for all $y \in \mathbb{X}$. Thus, (ii) \implies (iii) hold.

The implicates (iii) \implies (ii) is trivial.

Now, we need to prove (ii) \implies (i).

Assume (ii) holds, then $\lim_{n \rightarrow \infty} x_n = x$ in (\mathbb{X}, M^*) .

Let $\epsilon > 0$. Then there exists a positive integer N such that $M^*(x_n, x_n, x) < \epsilon$ for all $m, n \geq N$. For $n \geq N$, by (M^*7) ,

$$d(x, x_n) = M^*(x, x, x_n) \leq RM^*(x, x_n, x_n) < R\epsilon.$$

Hence $\lim_{n \rightarrow \infty} x_n = x$ in (\mathbb{X}, d) . Thus (ii) \implies (i). □

4. COMMON FIXED POINT THEOREMS IN M^* -METRIC SPACE

Theorem 4.1. *Let (\mathbb{X}, M^*) be an complete M^* -complete metric space and let $S : \mathbb{X} \rightarrow \mathbb{X}$ be a mapping which satisfies the following condition for all*

$x, y, z \in \mathbb{X}$ with $R \geq 1$,

$$M^*(Sx, Sy, Sz) \leq \frac{1}{R} \max \left\{ \begin{array}{l} aM^*(x, y, z), \\ b[M^*(x, Sx, Sy) + 2M^*(y, Sy, Sy)], \\ b[M^*(x, Sy, Sy) + M^*(y, Sx, Sy)], \\ M^*(z, Ty, Tx) \end{array} \right\}, \quad (4.1)$$

where $0 < a < 1$ and $0 < b < \frac{1}{3}$. Then S has a unique fixed point.

Proof. Let $x_0 \in \mathbb{X}$ be arbitrary, there exists $x_1 \in \mathbb{X}$ such that $Sx_0 = x_1$ and let $\{x_n\}$ in \mathbb{X} be a sequence with $Sx_{n-1} = x_n$. By using (4.1), we have

$$\begin{aligned} & M^*(x_n, x_n, x_{n+1}) \\ &= M^*(Sx_{n-1}, Sx_{n-1}, Sx_n) \\ &\leq \frac{1}{R} \max \left\{ \begin{array}{l} aM^*(x_{n-1}, x_{n-1}, x_n), b[M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1}) \\ + 2M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1})], \\ b[M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1}) + M^*(x_{n-1}, Sx_{n-1}, Sx_{n-1}) \\ + M^*(x_n, Sx_{n-1}, Sx_{n-1})] \end{array} \right\} \\ &= \frac{1}{R} \max \left\{ \begin{array}{l} aM^*(x_{n-1}, x_n, x_n), b[M^*(x_{n-1}, x_n, x_n) \\ + 2M^*(x_{n-1}, x_n, x_n)], \\ b[M^*(x_{n-1}, x_n, x_n) + M^*(x_{n-1}, x_n, x_n) \\ + M^*(x_n, x_n, x_n)] \end{array} \right\} \\ &= \frac{1}{R} \max \left\{ \begin{array}{l} aM^*(x_{n-1}, x_{n-1}, x_n), 3bM^*(x_{n-1}, x_n, x_n), \\ 2bM^*(x_{n-1}, x_n, x_n) \end{array} \right\} \\ &\leq \alpha M^*(x_{n-1}, x_{n-1}, x_n), \end{aligned} \quad (4.2)$$

where $\alpha = \max\{a, 3b\}$ and $0 < \alpha < 1$.

By repeating the application of the above inequality and equality (4.2), we have

$$\begin{aligned} M^*(x_n, x_n, x_m) &\leq RM^*(x_n, x_n, x_{n+1}) + RM^*(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\quad \vdots \\ &\quad + RM^*(x_{m-2}, x_{m-2}, x_{m-1}) + M^*(x_{m-1}, x_{m-1}, x_m) \\ &\leq R[M^*(x_n, x_n, x_{n+1}) + M^*(x_{n+1}, x_{n+1}, x_{n+2})] \\ &\quad \vdots \\ &\quad + M^*(x_{m-2}, x_{m-2}, x_{m-1}) + M^*(x_{m-1}, x_{m-1}, x_m)] \\ &\leq (\alpha^n + \alpha^{n+1} + \dots + \alpha^{m-1})RM^*(x_0, x_0, x_1) \\ &\leq \frac{\alpha^n}{1 - \alpha} RM^*(x_0, x_0, x_1). \end{aligned}$$

Thus, $M^*(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Also, for $n, m, l \in \mathbb{N}$,

$$\begin{aligned} M^*(x_n, x_m, x_l) &\leq RM^*(x_n, x_m, x_m) + M^*(x_m, x_l, x_l) \\ &\leq \frac{\alpha^n}{1 - \alpha} R^2 M^*(x_0, x_0, x_1) + \frac{\alpha^n}{1 - \alpha} RM^*(x_0, x_0, x_1). \end{aligned}$$

Taking $n, m, l \rightarrow \infty$, we get $M^*(x_n, x_m, x_l) \rightarrow 0$, so $\{x_n\}$ is an M^* -Cauchy sequence. Since \mathbb{X} is an M^* -complete, there exists u such that $x_n \rightarrow u$ as $n \rightarrow \infty$.

If $S(\mathbb{X}) \subseteq \mathbb{X}$, we have $u \in \mathbb{X}$. Then there exists $p \in \mathbb{X}$ such that $p = u$. We claim that $Sp = u$. From

$$\begin{aligned} &M^*(Sp, u, u) \\ &= M^*(Sp, Sp, u) \\ &\leq RM^*(Sp, Sp, Sx_n) + M^*(Sx_n, u, u) \\ &\leq \max \left\{ \begin{array}{l} aM^*(u, u, x_n), b[M^*(u, p, x_{n+1}) + 2M^*(u, Sp, Sp)], \\ +b[M^*(u, Sp, Sp) + M^*(u, Sp, Sp) + M^*(x_n, Sp, Sp)] \end{array} \right\}, \end{aligned}$$

as $n \rightarrow \infty$, we get $M^*(Sp, u, u) = 0$ and $Sp = u$, that is, $Sp = p$. Thus, S has a fixed point.

Next, we need to prove that S has a unique fixed point. Assume there exists q in \mathbb{X} such that $q = Sq$. Then, we have

$$\begin{aligned} &M^*(Sp, Sp, Sq) \\ &\leq \frac{1}{R} \max \left\{ \begin{array}{l} aM^*(p, p, q), b[M^*(p, Sp, Sq) + 2M^*(p, Sp, Sp)], \\ b[M^*(p, Sp, Sp) + M^*(p, Sp, Sp) + M^*(q, Sp, Sp)] \end{array} \right\} \\ &= \frac{1}{R} \max \left\{ \begin{array}{l} aM^*(Sp, Sp, Sq), b[M^*(Sp, Sp, Sq) + 2M^*(Sp, Sp, Sp)], \\ b[M^*(Sp, Sp, Sp) + M^*(Sp, Sp, Sp) + M^*(Sq, Sp, Sp)] \end{array} \right\} \\ &= \frac{1}{R} \max \{ aM^*(Sp, Sp, Sq), bM^*(Sq, Sp, Sp) \} \\ &= \frac{\beta}{R} M^*(Sp, Sp, Sq), \text{ where } \beta = \max \{ a, b \}. \end{aligned}$$

Hence, we have

$$M^*(Sp, Sp, Sq) \leq \gamma M^*(Sp, Sp, Sq),$$

where $\gamma = \frac{\beta}{R}$, that is, $(\gamma - 1)M^*(Sp, Sp, Sq) \geq 0$. Since $\gamma \in (0, 1)$,

$$(\gamma - 1)M^*(Sp, Sp, Sq) \leq 0.$$

This means that S has a unique fixed point. This completes the proof. □

5. M^* -CONTRACTION AND AN APPLICATION TO SYSTEM OF
LINEAR EQUATIONS

In this section, we seek to present a solution to a system of linear equations. Therefore, we will be to prove the following theorems.

Definition 5.1. Let M^* be an M^* -metric on a set \mathbb{X} and $T : \mathbb{X} \rightarrow \mathbb{X}$ be a mapping. T is said to be an M^* -contraction if for all $x, \ell \in \mathbb{X}$ there exists $\delta \in [0, 1)$ such that

$$M^*(Tx, Tx, T\ell) \leq \delta M^*(x, x, \ell).$$

Theorem 5.2. Let \mathbb{X} be an M^* -complete metric space and $T : \mathbb{X} \rightarrow \mathbb{X}$ be an M^* -contraction with $\delta \in [0, 1)$ and $R \geq 1$. Assume that there exists $x \in \mathbb{X}$ such that $M^*(x, x, Tx) < \infty$. Then there is $\ell \in \mathbb{X}$ such that $x_n \rightarrow \ell$ and ℓ is a unique fixed point of T .

Proof. Let $x_0 \in \mathbb{X}$ and a sequence $\{x_n\}$ in \mathbb{X} defined by $x_n = Tx_{n-1} = T^n x_0$. Then, we get

$$\begin{aligned} M^*(T^2x_0, T^2x_0, T^2x) &\leq \delta M^*(Tx_0, Tx_0, Tx) \\ &\leq \delta^2 M^*(x_0, x_0, x). \end{aligned}$$

If this process is repeated attain,

$$M^*(T^n x_0, T^n x_0, T^n x) \leq \delta^n M^*(x_0, x_0, x).$$

Now, we have to prove that $\{x_n\}$ is an M^* -Cauchy in \mathbb{X} .

$$\begin{aligned} M^*(x_n, x_n, x_m) &\leq RM^*(x_n, x_n, x_{n+1}) + RM^*(x_{n+1}, x_{n+1}, x_{n+2}) \\ &\vdots \\ &\quad + RM^*(x_{m-1}, x_{m-1}, x_m) \\ &\leq R\delta^n M^*(x_0, x_0, x) + R\delta^{k+1} M^*(x_0, x_0, x) \\ &\vdots \\ &\quad + R\delta^{m-1} M^*(x_0, x_0, x) \\ &= M^*(x_0, x_0, x) R\delta^n [1 + \delta + (\delta)^2 + \dots + (\delta)^{m-n-1}], \quad (5.1) \end{aligned}$$

where $n > m > 0$. Letting $n, m \rightarrow \infty$ in (5.1), we have

$$\lim_{m, n \rightarrow \infty} M^*(x_n, x_n, x_m) = 0.$$

Thus, $\{x_n\}$ is an M^* -Cauchy in \mathbb{X} . Since \mathbb{X} is an M^* -complete, so $\{x_n\}$ is M^* -convergent to some ℓ .

From this inequality,

$$\begin{aligned} M^*(T\ell, T\ell, \ell) &\leq RM^*(\ell, \ell, x_n) + M^*(x_n, x_n, \ell) \\ &\leq R\delta M^*(\ell, \ell, x_{n-1}) + M^*(x_n, x_n, \ell) \\ &= R\delta M^*(x_{n-1}, x_{n-1}, \ell) + M^*(x_n, x_n, \ell) \\ &= 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

we know that ℓ is fixed point of T .

Now, we need to prove that ℓ is unique fixed point of T . Assume ℓ_1 is a fixed point of T such that $\ell \neq \ell_1$. Since

$$M^*(\ell, \ell, \ell_1) = M^*(T\ell, T\ell, T\ell_1) \leq \delta M^*(\ell, \ell, \ell_1),$$

we get

$$(1 - \delta)M^*(\ell, \ell, \ell_1) \leq 0.$$

This implies that $M^*(\ell, \ell, \ell_1) = 0$, that is, $\ell = \ell_1$. This completes the proof. \square

To achieve our purpose in this section, we must prove the following theorem by Theorem 5.2.

Theorem 5.3. *Let $\mathbb{X} = \mathbb{R}^n$ be an M^* -metric space with the M^* -metric:*

$$M^*(\wp, q, \ell) = \sum_{i=1}^n (|\wp_i - q_i| + |q_i - \ell_i| + |\ell_i - \wp_i|).$$

If

$$\sum_{i=1}^n |\alpha_{ij}| \leq \alpha < 1 \quad \text{for all } j = 1, 2, \dots, n,$$

then the linear system

$$\begin{cases} \alpha_{11}\wp_1 + \alpha_{12}\wp_2 + \dots + \alpha_{1n}\wp_n = \gamma_1 \\ \alpha_{21}\wp_1 + \alpha_{22}\wp_2 + \dots + \alpha_{2n}\wp_n = \gamma_2 \\ \vdots \\ \alpha_{n1}\wp_1 + \alpha_{n2}\wp_2 + \dots + \alpha_{nn}\wp_n = \gamma_n \end{cases} \tag{5.2}$$

has a unique solution.

Proof. Since $\mathbb{X} = \mathbb{R}^n$ is an M^* -complete, we have to show that $T : \mathbb{X} \rightarrow \mathbb{X}$ is defined by

$$T(\wp) = A\wp + \gamma,$$

where $\wp = (\wp_1, \wp_2, \dots, \wp_n) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \neq 0,$$

is an M^* -contraction. Since

$$\begin{aligned}
 M^*(T\wp, Tq, T\ell) &= \sum_{i=1}^n \left| \sum_{j=1}^n \alpha_{ij} ((\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j)) \right| \\
 &\leq \sum_{i=1}^n \sum_{j=1}^n |\alpha_{ij}| |(\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j)| \\
 &= \sum_{j=1}^n \sum_{i=1}^n |\alpha_{ij}| |(\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j)| \\
 &\leq \alpha \sum_{j=1}^n |(\wp_j - q_j) + (q_j - \ell_j) + (\ell_j - \wp_j)| \\
 &= \alpha M^*(\wp, q, \ell),
 \end{aligned}$$

T is an M^* -contraction and it is obvious that $M^*(\wp, \wp, \ell) < \infty$. By Theorem 5.2, the linear equation system (5.2) has a unique solution. \square

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