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CONVERGENCE THEOREMS OF THREE STEP RANDOM ITERATION SCHEME WITH ERRORS FOR NONSELF ASYMPTOTICALLY NONEXPANSIVE RANDOM MAPPINGS

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Abstract. In this paper, we introduce a three step random iteration scheme with errors and prove that the iteration scheme converges to a random fixed point of nonself asymptotically nonexpansive random mappings in uniformly convex separable Banach spaces. The results presented in this paper extend and improve the recent ones announced by Zhou and Wang [21] and many others.

1. Introduction

Let K be a nonempty closed convex subset of real normed linear space E. A self-mapping $T\colon K \to K$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$. A self-mapping $T\colon K \to K$ is called asymptotically nonexpansive if there exists sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\| \tag{1.1}$$

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for all $x, y \in K$ and each $n \geq 1$. A self-mapping $T\colon K \to K$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L\,\|x - y\| \tag{1.2}$$

for all $x, y \in K$ and each $n \geq 1$.

Being an important generalization of the class of asymptotically nonexpansive self-mappings, the concept of deterministic non-self asymptotically nonexpansive mappings was introduced by Chidume et al. [4] in 2003. The non-self asymptotically nonexpansive mapping is defined as follows:

Definition 1.1. ([4]) Let K be a nonempty subset of a real normed linear space E. Let $P\colon E \to K$ be the nonexpansive retraction of E onto K. A non-self mapping $T\colon K \to E$ is called asymptotically nonexpansive if there exists sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \quad \leq \quad k_n\,\|x - y\| \tag{1.3}$$

for all $x, y \in K$ and each $n \geq 1$. A mapping $T\colon K \to E$ is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \quad \leq \quad L\,\|x - y\| \tag{1.4}$$

for all $x, y \in K$ and each $n \geq 1$.

In 2003, Chidume et al. [4] studied the following iteration process:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{cases} \tag{1.5}$$

in the framework of uniformly convex Banach space, where K is a closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retract. They got some strong convergence theorems for non-self asymptotically nonexpansive mapping.

In 2006, Wang [19] further generalized the iteration scheme as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \quad n \geq 1 \end{cases} \tag{1.6}$$

and got some new results.

Remark 1.1. If T is a self mapping, then P becomes identity mapping. Thus (1.3) and (1.4) reduce to (1.1) and (1.2), respectively.

Remark 1.2. If we take $\beta_n = 0$ for all $n \geq 1$, the iteration scheme (1.6) reduces to iteration scheme (1.5).

The theory of random operator is an important branch of probabilistic analysis which plays a key role in many applied areas. The study of random fixed point theory was initiated by the Prague school of Probabilities in the 1950s [7, 8, 18]. The machinery of random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see e.g. [12]) and references mentioned therein. The survey article by Bharucha-Reid attracted the attention of several mathematicians and gave wings to the theory. A lot of efforts have been devoted to random fixed point theory and applications (see e.g. [1, 2, 5, 6, 9, 10, 11, 15, 16, 20]).

In recent years, many results about deterministic nonexpansive self-mappings and asymptotically nonexpansive self-mappings have been randomized by some authors ([2] and therein). The purpose of this paper is to construct a random three-step iteration scheme to approximate random fixed points of non-self asymptotically nonexpansive random mappings and to prove some convergence theorems for such mappings in uniformly convex separable Banach spaces.

2. Preliminaries

Let (Ω, Σ) be a measurable space (Σ- sigma algebra) and K a nonempty subset of a real Banach space E. A mapping $\xi\colon \Omega \to K$ is said to be measurable if $\xi^{-1}(U \cap K) \in \Sigma$ for every Borel subset U of E. A mapping $T\colon \Omega \times K \to K$ is said to be a random mapping if for each fixed $x \in K$, the mapping $T(., x)\colon \Omega \to K$ is measurable. A measurable mapping $\xi^*\colon \Omega \to K$ is called a random fixed point of the random mapping $T\colon \Omega \times K \to K$ if $T(\omega, \xi^*(\omega)) = \xi^*(\omega)$, for each $\omega \in \Omega$.

Throughout of this paper, we denote the set of all random fixed points of a random mapping T by $RF(T)$.

Definition 2.1. A subset K of E is said to be retract if there exists continuous mapping $P\colon E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P\colon E \to E$ is said to be a retraction if $P^2 = P$.

Note: If mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P.

Definition 2.2. Let K be a nonempty closed convex subset of a real uniformly convex separable Banach space E and $T\colon \Omega \times K \to E$ be a non-self random mapping. Then the random mapping T is said to be a

(1) nonself asymptotically nonexpansive random mapping [21] if there exists a measurable mapping sequence $k_n\colon \Omega \to [0, \infty)$ with $\lim_{n\to\infty} k_n(\omega) = 0$ for each $\omega \in \Omega$, such that for arbitrary $x, y \in K$ and each $\omega \in \Omega$,

$$\left\|T(PT)^{n-1}(\omega, x) - T(PT)^{n-1}(\omega, y)\right\| \leq (1 + k_n(\omega)) \|x - y\|, \quad (2.1)$$

where, $n = 1, 2, \cdots$.

(2) uniformly L-Lipschitzian random mapping [21] if there exists a constant $L > 0$ such that for arbitrary $x, y \in K$ and each $\omega \in \Omega$,

$$\left\|T(PT)^{n-1}(\omega, x) - T(PT)^{n-1}(\omega, y)\right\| \leq L\,\|x - y\|, \tag{2.2}$$

where, $n = 1, 2, \cdots$.

(3) semi-compact random mapping if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to K, with $\lim_{n\to\infty}\|\xi_n(\omega) - T(\omega, \xi_n(\omega)\| = 0$, for every $\omega \in \Omega$, one has a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi\colon \Omega \to K$ such that $\{\xi_{n_k}\}$ converges pointwisely to ξ as $k \to \infty$.

(4) completely continuous random mapping if the sequence $\{x_n\}$ in K converges weakly to x_0 implies that $\{T(\omega, x_n)\}$ converges strongly to $T(\omega, x_0)$ for each $\omega \in \Omega$.

(5) demicompact random mapping if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to K, with $\lim_{n\to\infty}\|\xi_n(\omega) - T(\omega, \xi_n(\omega)\| = 0$ for each $\omega \in \Omega$, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi_{n_k}(\omega) \to \xi(\omega)$ as $k \to \infty$, for each $\omega \in \Omega$, where ξ is a measurable mapping from $\Omega \to K$.

Remark 2.1. As a matter of fact, every non-self asymptotically nonexpansive random mapping is uniformly L-Lipschitzian, where $L = 1 + \sup_{\omega\in\Omega, n\geq 1} k_n(\omega)$.

In 2007, Zhou and Wang [21] studied the following random iteration scheme for convergence of random fixed point for non-self asymptotically nonexpansive random mappings in uniformly convex separable Banach space.

Let $T\colon \Omega \times K \to E$ be a non-self random mapping, where K is a nonempty convex subset of a separable real uniformly convex Banach space E. The random iteration scheme is defined as follows:

$$\begin{cases} \xi_{n+1}(\omega) = P((1-\alpha_n)\xi_n(\omega) + \alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega))), \\ \eta_n(\omega) = P((1-\beta_n)\xi_n(\omega) + \beta_n T(PT)^{n-1}(\omega, \xi_n(\omega))), \quad n \geq 1 \end{cases} \tag{2.3}$$

where $0 \leq \alpha_n, \beta_n < 1$ and $\xi_1\colon \Omega \to K$ is an arbitrary given measurable mapping from Ω to K, P is a nonexpansive retraction from E to K.

If we take $\beta_n = 0$ for any $n \geq 1$, the iteration scheme (2.3) reduces to the following random iteration scheme:

$$\xi_{n+1}(\omega) = P((1-\alpha_n)\xi_n(\omega) + \alpha_n T(PT)^{n-1}(\omega, \xi_n(\omega))), \quad n \geq 1. \tag{2.4}$$

Obviously, the sequences $\{\xi_n\}$ and $\{\eta_n\}$ are two measurable sequences from Ω to K.

Motivated and inspired by Zhou and Wang [21] and some others we define the following random iteration scheme:

Definition 2.3. Let $T\colon \Omega \times K \to E$ be a non-self random mapping, where K is a nonempty convex subset of a separable real uniformly convex Banach space E. Let $\xi_0\colon \Omega \to K$ be a measurable mapping from Ω to K, let $\{f_n\}$,

$\{f'_n\}$, $\{f''_n\}$ be bounded sequences of measurable functions from Ω to K. Define sequences of functions $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$, as given below:

$$\begin{cases} \zeta_n(\omega) = P(\alpha''_n T(PT)^{n-1}(\omega, \xi_n(\omega) + \beta''_n \xi_n(\omega) + \gamma''_n f''_n(\omega)), \\ \eta_n(\omega) = P(\alpha'_n T(PT)^{n-1}(\omega, \zeta_n(\omega) + \beta'_n \xi_n(\omega) + \gamma'_n f'_n(\omega)), \\ \xi_{n+1}(\omega) = P(\alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega)), \end{cases} \tag{2.5}$$

for each $\omega \in \Omega$, $n = 0, 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ and $\{\gamma''_n\}$ are sequences of real numbers in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$, P is a nonexpansive retraction from E to K.

If we take $\gamma_n = \gamma'_n = \gamma''_n = 0$, $\alpha''_n = 0$ and $\alpha'_n = \beta_n$, the random iteration scheme (2.5) reduces to the random iteration scheme (2.3) of Zhou and Wang [21].

In the sequel, we will need the following lemmas.

Lemma 2.1. (Tan and Xu [17]) *Let $\{a_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + r_n)a_n + \beta_n, \quad \forall n \in N.$$

If $\sum_{n=1}^\infty r_n < \infty$, $\sum_{n=1}^\infty \beta_n < \infty$. Then
(i) $\lim_{n\to\infty} a_n$ exists.
(ii) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (Schu [14]) *Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r,$$
$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r,$$

for some $r \geq 0$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Lemma 2.3. (Chidume et al. [4]) *Let E be a real uniformly convex Banach space, K a nonempty closed subset of E, and let $T\colon K \to E$ be a non-self asymptotically nonexpansive mapping with a sequence $\{k_n\} \in [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

3. Main result

In this section, we investigate the convergence of three-step random iterative process with errors for non-self asymptotically nonexpansive random mappings to obtain the random solution of the random fixed point. This random iteration process extends the random iteration process of Zhou and Wang [21].

Lemma 3.1. *Let E be a real uniformly convex separable Banach space, and let K be a nonempty closed and convex subset which is also a nonexpansive retract of E. Let $T\colon \Omega \times K \to K$ be non-self asymptotically nonexpansive random mapping with sequence of measurable mapping $k_n\colon \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega) < \infty$, for each $\omega \in \Omega$. Assume that $RF(T) \neq \phi$ and let $\{\xi_n\}$ be the sequence as defined by (2.5) with $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$ and $\sum_{n=1}^{\infty} \gamma_n'' < \infty$. Then $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ exists for all $\xi(\omega) \in RF(T)$ and for each $\omega \in \Omega$.*

Proof. The existence of random fixed point of T follows from Bharucha-Reid's stochastic analogue (see [3]) of well-known Schauder's fixed point theorem. Let $\xi\colon \Omega \to K$ be the random fixed point of T. Since $\{f_n\}$, $\{f_n'\}$ and $\{f_n''\}$ are bounded sequences of measurable functions from Ω to K, we can put

$$M(\omega) = \sup_{n\geq 1} \|f_n(\omega) - \xi(\omega)\| \vee \sup_{n\geq 1} \left\|f_n'(\omega) - \xi(\omega)\right\| \vee \sup_{n\geq 1} \left\|f_n''(\omega) - \xi(\omega)\right\|.$$

Then $M(\omega)$ is a finite number for each $\omega \in \Omega$. For $n \geq 1$, we have

$$\begin{aligned} & \|\xi_{n+1}(\omega) - \xi(\omega)\| \\ = \;& \left\|P(\alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega)) - P(\xi(\omega))\right\| \\ = \;& \left\|\alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega) - \xi(\omega)\right\| \\ \leq \;& \alpha_n \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega)\right\| + \beta_n \|\xi_n(\omega) - \xi(\omega)\| \\ & + \gamma_n \|f_n(\omega) - \xi(\omega)\| \\ \leq \;& \alpha_n (1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\| + \beta_n \|\xi_n(\omega) - \xi(\omega)\| \\ & + \gamma_n M \end{aligned} \tag{3.1}$$

Similarly, we have

$$\begin{aligned} & \|\eta_n(\omega) - \xi(\omega)\| \\ \leq \;& \alpha_n' (1 + k_n(\omega)) \|\zeta_n(\omega) - \xi(\omega)\| + \beta_n' \|\xi_n(\omega) - \xi(\omega)\| \\ & + \gamma_n' \left\|f_n'(\omega) - \xi(\omega)\right\| \\ \leq \;& \alpha_n' (1 + k_n(\omega)) \|\zeta_n(\omega) - \xi(\omega)\| + \beta_n' \|\xi_n(\omega) - \xi(\omega)\| \\ & + \gamma_n' M \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\|\zeta_n(\omega)-\xi(\omega)\| \\
\leq\;& \alpha''_n(1+k_n(\omega))\,\|\xi_n(\omega)-\xi(\omega)\|+\beta''_n\,\|\xi_n(\omega)-\xi(\omega)\| \\
&+\gamma''_n\,\|f''_n(\omega)-\xi(\omega)\| \\
\leq\;& \alpha''_n(1+k_n(\omega))\,\|\xi_n(\omega)-\xi(\omega)\|+\beta''_n\,\|\xi_n(\omega)-\xi(\omega)\| \\
&+\gamma''_nM \qquad (3.3)
\end{aligned}$$

Substituting (3.3) in (3.2), we get

$$\begin{aligned}
&\|\eta_n(\omega)-\xi(\omega)\| \\
\leq\;& \alpha'_n\alpha''_n(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\|+\alpha'_n\beta''_n(1+k_n(\omega))\,\|\xi_n(\omega)-\xi(\omega)\| \\
&+\alpha'_n\gamma''_n(1+k_n(\omega))M+\beta'_n\,\|\xi_n(\omega)-\xi(\omega)\|+\gamma'_nM \\
=\;& (1-\beta'_n-\gamma'_n)\alpha''_n(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n\,\|\xi_n(\omega)-\xi(\omega)\| \\
&+(1-\beta'_n-\gamma'_n)\beta''_n\,\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \\
\leq\;& (1-\beta'_n)\alpha''_n(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\| \\
&+(1-\beta'_n)\beta''_n(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \\
\leq\;& (1-\beta'_n)(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\|+\beta'_n(1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\| \\
&+m_n(\omega) \\
=\;& (1+k_n(\omega))^2\,\|\xi_n(\omega)-\xi(\omega)\|+m_n(\omega) \qquad (3.4)
\end{aligned}$$

where $m_n(\omega)=\alpha'_n\gamma''_n(1+k_n(\omega))M+\gamma'_nM$.

Note that $\sum_{n=1}^{\infty}m_n(\omega)<\infty$. Now substituting (3.4) in (3.1), we get

$$\begin{aligned}
&\|\xi_{n+1}(\omega)-\xi(\omega)\| \\
\leq\;& \alpha_n(1+k_n(\omega))^3\,\|\xi_n(\omega)-\xi(\omega)\|+\alpha_n(1+k_n(\omega))m_n(\omega) \\
&+\beta_n\,\|\xi_n(\omega)-\xi(\omega)\|+\gamma_nM \\
\leq\;& (\alpha_n+\beta_n)(1+k_n(\omega))^3\,\|\xi_n(\omega)-\xi(\omega)\|+A_n(\omega) \\
\leq\;& (1+k_n(\omega))^3\,\|\xi_n(\omega)-\xi(\omega)\|+A_n(\omega) \qquad (3.5)
\end{aligned}$$

where $A_n(\omega)=\alpha_n(1+k_n(\omega))m_n(\omega)+\gamma_nM$ with $\sum_{n=1}^{\infty}A_n(\omega)<\infty$. Since $\sum_{n=1}^{\infty}k_n(\omega)<\infty$ and $\sum_{n=1}^{\infty}A_n(\omega)<\infty$, it follows from Lemma 2 [17] that $\lim_{n\to\infty}\|\xi_n(\omega)-\xi(\omega)\|$ exists for all $\omega\in\Omega$. This completes the proof. □

Lemma 3.2. *Let E be a real uniformly convex separable Banach space, and let K be a nonempty closed and convex subset which is also a nonexpansive retract of E. Let $T\colon\Omega\times K\to K$ be non-self asymptotically nonexpansive random mapping with sequence of measurable mapping $k_n\colon\Omega\to[0,\infty)$ satisfying $\sum_{n=1}^{\infty}k_n(\omega)<\infty$, for each $\omega\in\Omega$. Assume that $RF(T)\neq\phi$ and let $\{\xi_n\}$ be the sequence as defined by (2.5) with the following restrictions:*

(i) $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=\alpha''_n+\beta''_n+\gamma''_n=1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $\exists n_0 \in N$ *such that* $0 < \alpha \le \alpha_n, \alpha'_n, \alpha''_n < 1 - \alpha$, *for some* $\alpha \in (0, 1)$ *and for all* $n \ge n_0$.

Then

$$\begin{aligned}\lim_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi_n(\omega)\right\| &= \lim_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \zeta_n(\omega)) - \xi_n(\omega)\right\| \\ &= \lim_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \xi_n(\omega)) - \xi_n(\omega)\right\| \\ &= 0.\end{aligned}$$

Proof. For any $\xi(\omega) \in RF(T)$, it follows from Lemma 3.1, we have $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\|$ exists for all $\omega \in \Omega$. Let $\lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = a$ for some $a \ge 0$. From (3.4), we have

$$\|\eta_n(\omega) - \xi(\omega)\| \quad \le \quad (1 + k_n(\omega))^2 \|\xi_n(\omega) - \xi(\omega)\| + m_n(\omega).$$

Taking $\limsup_{n\to\infty}$ on both sides, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| &\le \limsup_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \\ &= \lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| = a.\end{aligned}$$

Note that

$$\begin{aligned}\limsup_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega)\right\| &\le \limsup_{n\to\infty} (1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\| \\ &\le a.\end{aligned}$$

Next consider

$$\begin{aligned}&\left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\right\| \\ &\le \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega)\right\| + \gamma_n \|f_n(\omega) - \xi_n(\omega)\|.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\right\| \le a.$$

Also,

$$\begin{aligned}&\|\xi_n(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\| \\ &\le \|\xi_n(\omega) - \xi(\omega)\| + \gamma_n \|f_n(\omega) - \xi_n(\omega)\|\end{aligned}$$

gives that

$$\limsup_{n\to\infty} \|\xi_n(\omega) - \xi(\omega) + \gamma_n(f_n(\omega) - \xi_n(\omega))\| \le a.$$

Moreover, we note that

$$\begin{aligned}
a &= \lim_{n\to\infty} \|\xi_{n+1}(\omega) - \xi(\omega)\| \\
&= \lim_{n\to\infty} \|\alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega) - \xi(\omega)\| \\
&= \lim_{n\to\infty} \|\alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega)) + \beta_n \xi_n(\omega) + \gamma_n f_n(\omega) \\
&\qquad - (1-\alpha_n)\xi(\omega) - \alpha_n \xi(\omega)\| \\
&= \lim_{n\to\infty} \|\alpha_n [T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega) + \gamma_n (f_n(\omega) - \xi_n(\omega))] \\
&\qquad + (1-\alpha_n)[\xi_n(\omega) - \xi(\omega) + \gamma_n (f_n(\omega) - \xi_n(\omega))]\|.
\end{aligned}$$

By Lemma 2 (Schu [14]), we have

$$\lim_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi_n(\omega)\right\| = 0.$$

Next we prove that $\lim_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \zeta_n(\omega)) - \xi_n(\omega)\right\| = 0$. For each $n \geq 1$, we have

$$\begin{aligned}
&\|\xi_n(\omega) - \xi(\omega)\| \\
&\leq \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi_n(\omega)\right\| + \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi(\omega)\right\| \\
&\leq \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi_n(\omega)\right\| + (1 + k_n(\omega)) \|\eta_n(\omega) - \xi(\omega)\|.
\end{aligned}$$

Since, $\lim_{n\to\infty} \left\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi_n(\omega)\right\| = 0 = \lim_{n\to\infty} k_n(\omega)$, we obtain that

$$a = \lim_{n\to\infty} \|\xi_n(\omega) - \xi(\omega)\| \leq \liminf_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\|.$$

It follows that

$$a \leq \liminf_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| \leq \limsup_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| \leq a.$$

This implies that

$$\lim_{n\to\infty} \|\eta_n(\omega) - \xi(\omega)\| = a.$$

On the other hand, we note that

$$\begin{aligned}
&\|\zeta_n(\omega) - \xi(\omega)\| \\
&= \left\|\alpha''_n T(PT)^{n-1}(\omega, \xi_n(\omega)) + \beta''_n \xi_n(\omega) + \gamma''_n f''_n(\omega) - \xi(\omega)\right\| \\
&\leq \alpha''_n \left\|T(PT)^{n-1}(\omega, \xi_n(\omega)) - \xi(\omega)\right\| + \beta''_n \|\xi_n(\omega) - \xi(\omega)\| \\
&\quad + \gamma''_n \left\|f''_n(\omega) - \xi(\omega)\right\|
\end{aligned}$$

$$\le \alpha_n''(1+k_n(\omega))\left\|\xi_n(\omega)-\xi(\omega)\right\| + \beta_n''\left\|\xi_n(\omega)-\xi(\omega)\right\| + \gamma_n''\left\|f_n''(\omega)-\xi(\omega)\right\|$$
$$\le \alpha_n''(1+k_n(\omega))\left\|\xi_n(\omega)-\xi(\omega)\right\| + (1-\alpha_n'')(1+k_n(\omega))\left\|\xi_n(\omega)-\xi(\omega)\right\| + \gamma_n''\left\|f_n''(\omega)-\xi(\omega)\right\|$$
$$\le (1+k_n(\omega))\left\|\xi_n(\omega)-\xi(\omega)\right\| + \gamma_n''\left\|f_n''(\omega)-\xi(\omega)\right\|.$$

By boundedness of $\{f_n''(\omega)\}$ and $\lim_{n\to\infty} k_n(\omega) = 0 = \lim_{n\to\infty}\gamma_n''$, we have

$$\limsup_{n\to\infty}\left\|\zeta_n(\omega)-\xi(\omega)\right\| \le \limsup_{n\to\infty}\left\|\xi_n(\omega)-\xi(\omega)\right\| \le a,$$

and

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi(\omega)\right\| \le \limsup_{n\to\infty}(1+k_n(\omega))\left\|\zeta_n(\omega)-\xi(\omega)\right\| \le a.$$

Next, we consider

$$\left\|T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\right\|$$
$$\le \left\|T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi(\omega)\right\| + \gamma_n'\left\|f_n'(\omega)-\xi_n(\omega)\right\|.$$

Taking $\limsup_{n\to\infty}$ on both sides, we have

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\right\| \le a.$$

Also,

$$\left\|\xi_n(\omega)-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\right\| \le \left\|\xi_n(\omega)-\xi(\omega)\right\| + \gamma_n'\left\|f_n'(\omega)-\xi_n(\omega)\right\|$$

gives that

$$\limsup_{n\to\infty}\left\|\xi_n(\omega)-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))\right\| \le a.$$

Since $\lim_{n\to\infty}\left\|\eta_n(\omega)-\xi(\omega)\right\| = a$, we obtain

$$\begin{aligned} a &= \lim_{n\to\infty}\left\|\eta_n(\omega)-\xi(\omega)\right\| \\ &= \lim_{n\to\infty}\left\|\alpha_n' T^n(\omega,\zeta_n(\omega)) + \beta_n'\xi_n(\omega) + \gamma_n' f_n'(\omega) - \xi(\omega)\right\| \\ &= \lim_{n\to\infty}\|\alpha_n'[T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))] \\ &\qquad + (1-\alpha_n')[\xi_n(\omega)-\xi(\omega)+\gamma_n'(f_n'(\omega)-\xi_n(\omega))]\|. \end{aligned}$$

By Lemma 2 (Schu [14]), we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi_n(\omega)\right\| = 0.$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\xi_n(\omega))-\xi_n(\omega)\right\| = 0,$$

for all $\omega \in \Omega$. This completes the proof. □

Lemma 3.3. *Let E be a real uniformly convex separable Banach space, and let K be a nonempty closed and convex subset which is also a nonexpansive retract of E. Let $T\colon \Omega\times K\to K$ be non-self asymptotically nonexpansive random mapping with sequence of measurable mapping $k_n\colon \Omega\to[0,\infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega)<\infty$, for each $\omega\in\Omega$. Assume that $RF(T)\neq\phi$ and let $\{\xi_n\}$ be the sequence as defined by (2.5) with the following restrictions:*

(i) $\alpha_n+\beta_n+\gamma_n = \alpha'_n+\beta'_n+\gamma'_n= \alpha''_n+\beta''_n+\gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\gamma'_n<\infty$, $\sum_{n=1}^{\infty}\gamma''_n<\infty$.

(iii) $\exists n_0\in N$ such that $0<\alpha\le\alpha_n,\alpha'_n,\alpha''_n<1-\alpha$, for some $\alpha\in(0,1)$ and for all $n\ge n_0$.

Then

$$\lim_{n\to\infty}\|\xi_n(\omega)-T(\omega,\xi_n(\omega))\|=0.$$

Proof. It follows from Lemma 3.2 that

$$\begin{aligned}\lim_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\eta_n(\omega))-\xi_n(\omega)\right\| &= \lim_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\zeta_n(\omega))-\xi_n(\omega)\right\|\\ &= \lim_{n\to\infty}\left\|T(PT)^{n-1}(\omega,\xi_n(\omega))-\xi_n(\omega)\right\|\\ &= 0,\end{aligned}$$

and this implies that

$$\begin{aligned}&\|\xi_{n+1}(\omega)-\xi_n(\omega)\|\\ &\le \alpha_n\left\|T(PT)^{n-1}(\omega,\eta_n(\omega))-\xi_n(\omega)\right\|+\gamma_n\|f_n(\omega)-\xi_n(\omega)\|\\ &\to 0\end{aligned}$$

as $n\to\infty$ and for each $\omega\in\Omega$.

We now to show that $\lim_{n\to\infty}\|\xi_n(\omega)-T(\omega,\xi_n(\omega))\| = 0$. Since T is a non-self asymptotically nonexpansive random mapping, so it is uniformly L-Lipschitzian for some constant $L>0$. Hence observe that

$$\begin{aligned}&\left\|\xi_n(\omega)-T(PT)^{n-2}(\omega,\xi_n(\omega))\right\|\\ &\le \|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\left\|\xi_{n-1}(\omega)-T(PT)^{n-2}(\omega,\xi_{n-1}(\omega))\right\|\\ &\quad+\left\|T(PT)^{n-2}(\omega,\xi_{n-1}(\omega))-T(PT)^{n-2}(\omega,\xi_n(\omega))\right\|\\ &\le \|\xi_n(\omega)-\xi_{n-1}(\omega)\|+\left\|\xi_{n-1}(\omega)-T(PT)^{n-2}(\omega,\xi_{n-1}(\omega))\right\|\\ &\quad+L\|\xi_{n-1}(\omega)-\xi_n(\omega)\|\to 0\end{aligned}$$

as $n\to\infty$ and for each $\omega\in\Omega$.

Thus by the above inequality, we have

$$\begin{aligned}\|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| &\le \left\|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\right\| \\ &\quad + \left\|T(PT)^{n-1}(\omega, \xi_n(\omega)) - T(\omega, \xi_n(\omega))\right\| \\ &\le \left\|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\right\| \\ &\quad + L\left\|T(PT)^{n-2}(\omega, \xi_n(\omega)) - \xi_n(\omega)\right\| \\ &\to 0\end{aligned}$$

as $n \to \infty$ and for each $\omega \in \Omega$.

It implies that

$$\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0.$$

This completes the proof. □

Theorem 3.4. *Let E be a real uniformly convex separable Banach space, and let K be a nonempty closed and convex subset which is also a nonexpansive retract of E. Let $T\colon \Omega \times K \to K$ be non-self asymptotically nonexpansive random mapping with sequence of measurable mapping $k_n\colon \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega) < \infty$, for each $\omega \in \Omega$. Assume that $RF(T) \neq \phi$ and let $\{\xi_n\}$ be the sequence as defined by (2.5) with the following restrictions:*

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $0 < \alpha \le \alpha_n, \alpha'_n, \alpha''_n < 1 - \alpha$, for some $\alpha \in (0,1)$ and for all $n \ge n_0$, $\exists n_0 \in N$. If T is completely continuous, then sequences $\{\xi_n\}$, $\{\eta_n\}$ and $\{\zeta_n\}$ converges to a random fixed point of T.

Proof. From Lemma 3.3, we have

$$\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0. \tag{3.6}$$

It follows from Lemma 3.1 that sequence $\{\xi_n\}$ is bounded. By assumption T is completely continuous there exists a subsequence $\{T\xi_{n_k}\}$ of $\{T\xi_n\}$ and a measurable mapping $p\colon \Omega \to K$ such that for each $\omega \in \Omega$, $T\xi_{n_k} \to p \in RF(T)$ as $n_k \to \infty$. Moreover, by (3.6), we have $\|T(\omega, \xi_{n_k}(\omega)) - \xi_{n_k}(\omega)\| \to 0$ which implies that $\xi_{n_k}(\omega) \to p(\omega)$ as $n_k \to \infty$. By (3.6) again, we have

$$\|p(\omega) - T(\omega, p(\omega))\| = \lim_{n_k\to\infty} \|\xi_{n_k}(\omega) - T(\omega, \xi_{n_k}(\omega))\| = 0.$$

It shows that p is a random fixed point of T i.e. $p \in RF(T)$. Furthermore, since $\lim_{n\to\infty} \|\xi_n(\omega) - p(\omega)\|$ exists. Therefore $\lim_{n\to\infty} \|\xi_n(\omega) - p(\omega)\| = 0$, that is, $\{\xi_n\}$ converges to a random fixed point of T.

Similarly, we can show that $\{\eta_n\}$ and $\{\zeta_n\}$ converges to a random fixed point of T.

Since $\{\xi_n\}$ converges to a random fixed point p of T, that is, $p \in RF(T)$, so that it is also bounded. Thus by Lemma 3.2 and 3.3, we obtain that

$$\begin{aligned}\|\eta_n(\omega) - \xi_n(\omega)\| &\leq \alpha'_n \left\|T(PT)^{n-1}(\omega, \zeta_n(\omega)) - \xi_n(\omega)\right\| + \gamma'_n \left\|f'_n(\omega) - \xi_n(\omega)\right\| \\ &\to 0,\end{aligned}$$

as $n \to \infty$ and for each $\omega \in \Omega$, and

$$\begin{aligned}\|\zeta_n(\omega) - \xi_n(\omega)\| &\leq \alpha''_n \left\|T(PT)^{n-1}(\omega, \xi_n(\omega)) - \xi_n(\omega)\right\| + \gamma''_n \left\|f''_n(\omega) - \xi_n(\omega)\right\| \\ &\to 0,\end{aligned}$$

as $n \to \infty$ and for each $\omega \in \Omega$. Therefore

$$\lim_{n\to\infty} \eta_n(\omega) = p(\omega) = \lim_{n\to\infty} \zeta_n(\omega).$$

Thus $\{\eta_n\}$ and $\{\zeta_n\}$ also converges to a random fixed point of T. This completes the proof. □

Theorem 3.5. *Let E be a real uniformly convex separable Banach space, and let K be a nonempty closed and convex subset which is also a nonexpansive retract of E. Let $T\colon \Omega \times K \to K$ be non-self asymptotically nonexpansive random mapping with sequence of measurable mapping $k_n\colon \Omega \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega) < \infty$, for each $\omega \in \Omega$. Assume that $RF(T) \neq \phi$ and let $\{\xi_n\}$ be the sequence as defined by* (2.5) *with the following restrictions:*

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $\exists n_0 \in N$ such that $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n < 1 - \alpha$, for some $\alpha \in (0, 1)$ and for all $n \geq n_0$. If T is semi-compact, then sequence $\{\xi_n\}$ converges to a random fixed point of T.

Proof. Since T is semi-compact random mapping and by Lemma 3.3,

$$\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$$

for each $\omega \in \Omega$, then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ and a measurable mapping $\xi_0\colon \Omega \to K$ such that ξ_{n_k} converges pointwisely to ξ_0. The mapping $\xi_0\colon \Omega \to K$, being a pointwise limit of measurable mappings, $\{\xi_{n_k}\}$ is measurable. Now,

$$\lim_{k\to\infty} \|\xi_{n_k}(\omega) - T(\omega, \xi_{n_k}(\omega))\| = \|\xi_0(\omega) - T(\omega, \xi_0(\omega))\| = 0$$

for each $\omega \in \Omega$. Hence, $\xi_0(\omega)$ is a random fixed point of T, that is, $\xi_0 \in RF(T)$. Thus $\{\xi_n\}$ converges to a random fixed point of T. This completes the proof. □

Theorem 3.6. *Let E be a real uniformly convex separable Banach space, and let K be a nonempty closed and convex subset which is also a nonexpansive* *retract of E. Let $T\colon \Omega \times K \to K$ be non-self asymptotically nonexpansive random mapping with sequence of measurable mapping $k_n\colon \Omega \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} k_n(\omega) < \infty$, for each $\omega \in \Omega$. Assume that $RF(T) \neq \phi$ and let $\{\xi_n\}$ be the sequence as defined by (2.5) with the following restrictions:*

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$.

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$, $\sum_{n=1}^{\infty} \gamma''_n < \infty$.

(iii) $\exists n_0 \in N$ such that $0 < \alpha \leq \alpha_n, \alpha'_n, \alpha''_n < 1 - \alpha$, for some $\alpha \in (0,1)$ and for all $n \geq n_0$. If T is demicompact, then sequence $\{\xi_n\}$ converges to a random fixed point of T.

Proof. Since T is demicompact, by Lemma 3.1 and 3.3, $\{\xi_n(\omega)\}$ is bounded and $\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$, then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\{\xi_{n_k}(\omega)\}$ converges strongly to $\xi^*(\omega)$ for each $\omega \in \Omega$. It follows from Lemma 2 that $\xi^*(\omega) = T(\omega, \xi^*(\omega))$. Since $\{\xi_{n_k}\}$ is a measurable mapping sequence, ξ^* is a measurable mapping too. Therefore, $\xi^*(\omega) \in RF(T)$. Thus $\lim_{n\to\infty} \|\xi_n(\omega) - \xi^*(\omega)\|$ exists by Lemma 3.1. Since the subsequence $\{\xi_{n_k}(\omega)\}$ of $\{\xi_n(\omega)\}$ such that $\{\xi_{n_k}(\omega)\}$ converges to $\xi^*(\omega)$, then $\{\xi_n(\omega)\}$ converges to a random fixed point $\xi^*(\omega) \in RF(T)$. This completes the proof. □

Remark 3.1. Theorem 3.4 and 3.6 extend the corresponding results of Zhou and Wang [21] to the case of three step random iteration scheme with errors.

Remark 3.2. Theorem 3.4 also extends the corresponding result of Plubteing et al. [13] to the case of non-self maps with $T_i = T$ for $i = 1, 2, 3$.

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